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A Complete Classification of Liénard Equation

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Abstract. We consider scalar Liénard equations

$$\ddot{x}(t) = f(x(t))\dot{x}(t) + g(x(t)), x(t) \in \mathbb{R} \tag{1}$$

and the diffeomorphisms $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ in the form

$$\varphi(x,t) = (\beta(x), a.t + \alpha(x)) \tag{2}$$

where the derivative of the function β is non zero and where the real number a is non zero. The aim result of this paper is to study the symmetries in the form given by (2) for the equation (1).

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1. Introduction

Van der pol equation is an example of the long-standing interaction between differential equations and the physical and biological sciences. During the development of radio and vacuum tubes, Liénard equations were intensely studied as they can be used to model oscillating circuits. In 1920 the Dutch physicist Balthasar van der Pol, when he was an engineer working for Philips Company, studied the differential equation

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0$$

that describes the circuit of a vacuum tube and where ε is positive parameter. A few years after, [9] modeled the electric activity of the heart rate. POLES (in CHAOS 2007), developed their own modified van der Pol oscillator reproducing irregular heart rate, asystole, certain kinds of heart block, and others. In the sixties, Fitzhugh [4] and Nagumo [8] extended the

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van der Pol equation in a planar field as a model for action potentials of neurons. Recently, The equation has also been utilised in seismology to model the two plates in a geological fault.

The french engineer, Liénard propose the following generalization

$$\ddot{x} = f(x)\dot{x} + g(x) \tag{3}$$

where f and g are two real-valued analytic functions.

An other example of Liénard equation is given by the Duffing's equation, The Duffing equation, named after Georg Duffing, is a non-linear second-order differential equation used to model certain damped and driven oscillators. The equation is given by

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$$

where the (unknown) function $x = x(t)$ is the displacement at time t, $\dot{x}$ is the first derivative of x with respect to time, i.e. velocity, and $\ddot{x}$ is the second time-derivative of x, i.e. acceleration. The numbers δ, α, β, γ and ω are given constants.

The equation describes the motion of a damped oscillator with a more complicated potential than in simple harmonic motion (which corresponds to the case $\beta = \delta = 0$); in physical terms, it models, for example, a spring pendulum whose spring's stiffness does not exactly obey Hooke's law. The Duffing equation is an example of a dynamical system that exhibits chaotic behavior. Moreover the Duffing system presents in the frequency response the jump resonance phenomenon that is a sort of frequency hysteresis behaviour. The forced Duffing's equation, which is one of the classical oscillators first published by Duffing in 1918 .It is the simplest oscillator displaying catastrophic jumps of amplitude and phase when the frequency of the forcing term is taken as a gradually changing parameter. The main applications have been in electronics, mechanic, in biology. For example, the brain is full of oscillators at micro level, and at macro level displays jumps in sensory perception, in psychological perception, in regulation, in switches of mood, memory, and behaviour, to say nothing of falling asleep and waking up.

In addition, the two-dimensional autonomous dynamical system is defined by two coupled first order differential equations of the form

$$\dot{x} = P(x, y), \dot{y} = Q(x, y) \tag{4}$$

where P and Q are two functions of the variables x and y and the overdots denote a time derivative. Such a dynamical system appears very often within several branches of science, such as biology, chemistry, astrophysics, mechanics, electronics, fluid mechanics. One of the most difficult problems connected with the study of system (4) is the question of the number of limit cycles. A limit cycle is an isolated closed trajectory. Isolated means that the neighboring trajectories are not closed; they spiral either toward or away from the limit cycle. If all neighboring trajectories approach the limit cycle, we say that the limit cycle is stable or attracting. Otherwise the limit cycle is unstable or, in exceptional cases, half stable. Stable limit cycles are very important in science. They model systems that exhibit self-sustained oscillations. For more details we can see [5]

In this paper we consider the transformations $\varphi \in \mathrm{Diff}^{\mathrm{loc}}(\mathbb{R}^2)$ of the form

$$(t,x) \to (at + \alpha(x),\ \beta(x))$$

where α and β are two real-valued function such that $(\eta_y \neq 0)$. We shall see that such transformations form a Lie pseudogroup and have the important feature of preserving periodic solutions. Strongly aided by the computer algebraic package DIFFALG, written by François Boulier [3], we give a complete symmetry classification of the Liénard. We shall see how ROSENFELD-GRÖBNER allows us to discuss the structure of the symmetry Lie algebra of the Liénard equation w.r.t f and g.

The paper is organized around three sections. The second section gives a brief description of concept of symmetry. The aim of the third section is to present the symmetry classication of Léinard equation and somes examples.

2. Concept of Symmetry

To define the notion of symmetry in any general information we give a group Φ operative on the set E Now, in this context the definition of a symmetry is

Definition 1. *A symmetry of the Pfaffian system $\mathscr{E}_f = (M, \Delta_f)$ is a local diffeormorphism $\varphi \in Diff^{loc}(M)$ which preserves the contact structure of $\mathscr{E}_f$ i.e.*

$$\varphi_*(\Delta_f) = \Delta_f.$$

Symmetries in this definition are internal [1]. The set of all symmetries of a given Pfaffian system $\mathscr{E}_f$ is a Lie pseudogroup denoted by $Aut^{loc}(\mathscr{E}_f) \subset Diff^{loc}M$. Since the distribution Δ_f is involutive, $Aut^{loc}(\mathscr{E}_f)$ is the symmetry pseudogroup of a foliation. Such a pseudogroup is infinite dimensional. And this why in practice (in order to classify), we restrict ourselves to symmetries belonging to a certain Lie pseudogroup $\Phi \subset Diff^{loc}M$ of local diffeomorphisms of interest. Let $\mathscr{S}_f = Aut^{loc}(\mathscr{E}_f) \cap \Phi$ denotes the Lie pseudogroup of such symmetries. Its defining equations are given by the *non linear* PDE's system

$$\varphi_*(\Delta_f) = \Delta_f \text{ et } \varphi \in \Phi \tag{5}$$

where the second constraint means that φ fulfills the Lie defining equations of the Lie pseudogroup Φ.

The non linear PDE system (5) simplifies to a *linear* system if we switch to the calculation of *infinitesimal generators* of the Lie pseudogroup $\mathscr{S}_f$. Now we present briefly this technique due to S. Lie. A good reference is the book [7] but also [5] and [2].

Let G be one-dimensional Lie group (in practice G is the additive group $(\mathbb{R},+)$). Recall that a one-parameter transformations group on manifold M is a a map $(\epsilon, p) \in G \times M \to \varphi_\epsilon(p) \in M$ satisfying $\varphi_{\epsilon+\tau}(p) = \varphi_\epsilon \circ \phi_\tau(p)$ and if e is the identity element of G, φ_e is the identity transformation. Each one-parameter transformations group φ_ϵ induces a vector field X in the following manner. For each $p \in M$, X_p is the tangent vector of the curve $\gamma(\epsilon) = \varphi_\epsilon(p)$ at the point $p = \varphi_0(p)$ i.e. $\frac{d\varphi_\epsilon(p)}{d\epsilon}|_{\epsilon=0} = X_p$. The vector field X is called *infinitesimal generator* associated to the one-parameter group φ_ϵ. Conversely, to each vector field X we can associate a "local" one-parameter transformations group. The diffeomorphism $M \ni p \to \varphi_\epsilon(p) \in M$ is called the *flow* or the dynamic generated by X. [If we can take $\epsilon = \infty$, for each p , X is said to be complete. If M is compact, every X is complete]. The operator

$$\mathscr{L}_X : \Gamma\left(\otimes^r TM \otimes^s T^*M\right) \to \Gamma\left(\otimes^r TM \otimes^s T^*M\right)$$

defined by

$$\mathscr{L}_X = \lim_{\epsilon \to 0} \frac{\varphi_\epsilon^* - Id}{\epsilon}$$

is called the *Lie derivative* in the direction X and we have

$$\varphi_\epsilon^* = Id + \epsilon \mathscr{L}_X + O(\epsilon^2). \tag{6}$$

In particular, if Y is a vector field ($Y \in \Gamma(TM)$) then we have $\mathscr{L}_X(Y) = [X, Y]$.

Let us go back to our symmetries: now we are looking for local one-parameter symmetry groups. We know, such symmetries are of the form $\varphi_\epsilon(p) = p + \epsilon X(p) + O(\epsilon^2)$ for all $p \in M$ and for a certain $X \in \Gamma(TM)$ [Of course one needs to combine this with the fact that they also of the form $(t, x) \to (at + \alpha(x), \beta(x))$ which is explained in Section 3]. Applying (6) to $\varphi_\epsilon^*(\Delta) = \Delta$, shows that a transformation φ_ϵ is a symmetry of the Pfaffian system $\mathscr{E}_f = (M, \Delta_f)$ if and only if

$$\mathscr{L}_X \Delta = 0 \mod \Delta. \tag{7}$$

The components of the vector field X (called the *infinitesimals*) are now solutions of a *linear* PDE's system. The fluxes (the φ_ϵ) are recovered by solving the system of ordinary differential equations $\frac{d\varphi_\epsilon(p)}{d\epsilon}|_{\epsilon=0} = X_p$ with the initial condition $p = \varphi_0(p)$.

Example 1. *Let be M variety of local co-ordinates $(x, y^1, \ldots, y^n)$. Any multi-sheet on M, of codimension n, is localment redressable in (C_i being arbitrary constants)*

$$y^1 = C_1, \ldots, y^n = C_n$$

fiber_preserving transformation $\varphi \in Diff^{loc}M$

$$(x, y^1, \ldots, y^n) \to (\varphi^0(x, y^1, \ldots, y^n), \varphi^1(x, y^1, \ldots, y^n), \ldots, \varphi^n(x, y^1, \ldots, y^n)),$$

where φ^i are arbitrary functions.

Remark 1 (Lie's Classical Method)**.** *A symmetry of a differential equation is a transformation mapping an arbitrary solution to another solution of the differential equation. The classical Lie groups of point invariance transformations depend on continuous parameters and act on the system's graph space that is co-ordinatised by the independent and dependent variables. As these symmetries can be determined by an explicit computational algorithm If a partial differential equation (PDE) is invariant under a point symmetry, one can often find similarity solutions or invariant solutions which are invariant under some subgroup of the full group admitted by the PDE. These solutions result from solving a reduced equation in fewer variables.*

3. Symmetry Classification of the Liénard Equation

As announced, we consider the pseudogroup of transformations $\varphi \in Diff^{loc}(\mathbb{R}^2)$ of the form

$$\begin{cases} \bar{t} = at + \alpha(x), & a \neq 0 \\ \bar{x} = \beta(x), & \beta_x \neq 0 \end{cases}. \tag{8}$$

where $a \in \mathbb{R}$ is an arbitrary constant and α, β are two arbitrary functions.

Proposition 1. *Any transformation of the form* (8) *maps a periodic function $x(t)$ of period T to another periodic solution of period equals to aT.*

Proof. See [10].

3.1. Generation of Lie Equations

Let us first determine the infinitesimal generators X with fluxes of the form (8). Let make the substitution

$$\bar{t} = t + \epsilon A(x,t) + O(\epsilon^2),\ \bar{x} = x + \epsilon B(x,t) + O(\epsilon^2),$$

in the defining equations of the Lie pseudogroup (8):

$$\frac{\partial^2 \bar{t}}{\partial t^2} = 0,\ \frac{\partial^2 \bar{t}}{\partial x \partial t} = 0,\ \frac{\partial \bar{x}}{\partial t} = 0,\ \frac{\partial \bar{t}}{\partial t}\frac{\partial \bar{x}}{\partial x} \neq 0.$$

We obtain

$$\frac{\partial^2 A(x,t)}{\partial t^2} = 0,\ \frac{\partial^2 A(x,t)}{\partial x \partial t} = 0,\ \frac{\partial B(x,t)}{\partial t} = 0.$$

This allows one to deduce that the infinitesimal generators X must be of the form

$$X = (\lambda t + A(x))\frac{\partial}{\partial t} + B(x)\frac{\partial}{\partial x}.$$

Now Lie equations are obtained by writing that the Lie derivative $[X, D_t]$ is zero modulo $D_t = \frac{\partial}{\partial t} + p\frac{\partial}{\partial x} + (f(x)p + g(x))\frac{\partial}{\partial p}$ where $p = \dot{x}$. We obtain the ODE system

$$\begin{cases} Bg_x + B_x g - 2\lambda g = 0, \\ B_{x,x} - 2A_x f = 0, \\ A_{x,x} = 0, \\ Bf_x + 3A_x g + \lambda f = 0, \\ \lambda_x = 0. \end{cases} \tag{9}$$

The system (9) depends on two arbitrary functions f and g and linear in the differential unknowns A, B and λ.

Theorem 1. *The classification below is complete.*

Proof. See [10].

In the following paragraphs, we give the characteristic representations of the ideals $\sqrt{I_i}$.

The generic Case

The first characteristic set is

$$A_x = 0,\ B = 0, \lambda = 0.$$

This is the generic case, there is no constraint on the functions f and g. The dimension of the corresponding Lie algebra is equals to the number of points under the three stairs associated to the unknowns A, B and λ. Hence equals to one. The integration shows that the infinitesimal generator is $X_1 = \frac{\partial}{\partial t}$ and the corresponding fluxes form the one-parameter Lie group of temporal translations. Van der Pol equation $\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$, belongs to this class.

Case II.

It has four subcases where the number of points under the stairs corresponding to the unknowns A, B and λ (i.e. the dimension of the symmetry Lie algebra) is equal to two.
If $X_2 = (\lambda t + A(x))\frac{\partial}{\partial t} + B(x)\frac{\partial}{\partial x}$ is another vector field (different from $X_1 = \frac{\partial}{\partial t}$) then $[X_2, X_1] = \lambda X_1$. The symmetry Lie algebra is consequently the affine algebra $\mathfrak{a}(1, \mathbb{R})$ if $\lambda \neq 0$ or the abelian algebra otherwise. In both situations, it solvable and Liénard equation can be reduced to a quadrature [6, 2].

Subcase II-1. $3g_{xx} + 2ff_x \neq 0$ **and** $g \neq 0$ The first of the four characteristic sets is

$$\begin{cases} \lambda_x = 0, \\ A_x = -\frac{\lambda\left(f g_{xx} - 2f_x g_x\right)}{g\left(3g_{xx} + 2ff_x\right)}, \\ B = -2\frac{\lambda\left(f^2 + 3g_x\right)}{3g_{xx} + 2ff_x}, \\ g_{xxx} = \frac{5g_{xx} g f f_x + 6g g_{xx}{}^2 - 2g f_x{}^2 g_x - 3g_{xx} g_x{}^2 - 2ff_x g_x{}^2}{g\left(f^2 + 3g_x\right)} \\ f_{xx} = \frac{9g_{xx} g f_x - 3g_{xx} g_x f + 6f f_x{}^2 g - 2f^2 f_x g_x}{2g\left(f^2 + 3g_x\right)} \end{cases}$$

The two last equations (in addition to the inequalities) constrain the function f and g. They characterize the differential ideal $\sqrt{I_2} \cap \{f, g\}$. The other equations give the functions A, B and λ. In particular, one sees that λ furnishes a non zero structure constant. This proves:

Proposition 2. *The symmetry Lie algebra in this case is isomorphic to $\mathfrak{a}(1, \mathbb{R})$.*

Example 2. *We can consider the equation $\ddot{x} = x\dot{x} + x^3$. The infinitesimal generators are $X_1 = \frac{\partial}{\partial t}$, $X_2 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}$ and the fluxes generated by X_1 and X_2 form the two-dimensional Lie group if special affine transformations $(t, x) \to (\lambda t + \mu, \frac{x}{\lambda})$. The Jacobian $\begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ of such transformation has determinant equals to 1.*

Subcase II.2. $3g_{xx} = -2ff_x, f_{xx} = 0$ **and** $g \neq 0$ We have the characteristic set

$$\begin{cases} \lambda = 0, \\ A_x = -\frac{Bf_x}{3g}, \\ B_x = \frac{Bg_x}{g}, \\ f_{xx} = 0, \\ g_{xx} = -\frac{2}{3}ff_x. \end{cases}$$

The first equation immediately proves

Proposition 3. *The symmetry Lie algebra is the two-dimensional abelian algebra.*

The integration of the two last equations, which characterize $\sqrt{I_3} \cap \{f, g\}$, yields (the a_i are arbitrary constants)

$$\begin{cases} f(x) = a_1x + a_2, \\ g(x) = -\frac{1}{9}a_1^2x^3 - \frac{1}{3}a_1a_2x^2 + a_3x + a_4. \end{cases}$$

And this in turn yields

$$\begin{cases} X_1 = \frac{\partial}{\partial t}, \\ X_2 = -x\frac{a_1}{3a_4}\frac{\partial}{\partial t} + \left(1 + x\frac{a_3}{a_4} - x^2\frac{a_2a_1}{3a_4} - x^3\frac{{a_1}^2}{9a_4}\right)\frac{\partial}{\partial x}. \end{cases}$$

Subcase II.3. $3g_{xx} - 2g_x^2 \neq 0, f_{xx} \neq 0$ **and** $g \neq 0$ Here the function A, B and λ satisfy

$$\begin{cases} \lambda_x = 0, \\ A_x = \frac{\lambda\left(-3gff_{x,x} + 9{f_x}^2g + f^3f_x\right)}{9g^2f_{x,x}}, \\ B = -\frac{\lambda\left(9gf_x + f^3\right)}{3gf_{x,x}}. \end{cases}$$

The first equation proves

Proposition 4. *The symmetry Lie algebra is isomorphic to* $\mathfrak{a}(1, \mathbb{R})$.

The functions f and g satisfy the ODE system ($\sqrt{I_4} \cap \{f, g\}$)

$$\begin{cases} f^2 = -3g_x, \\ g_{xxxx} = -\frac{1}{2{g_x}^2g\left(3gg_{xx} - 2{g_x}^2\right)}(2g^2{g_{xx}}^4 + g_x{g_{xx}}^2g^2g_{xxx} - 10{g_x}^2g^2{g_{xxx}}^2 \\ -9{g_x}^2{g_{xx}}^3g + 18{g_x}^3g_{xx}gg_{xxx} + 4{g_x}^4{g_{xx}}^2 - 8g_{xxx}{g_x}^5). \end{cases}$$

Example 3. *In this class, we can take* $\ddot{x} = x^2\dot{x} - \frac{1}{15}x^5$ *for which we have* $X_1 = \frac{\partial}{\partial t}$ *and* $X_2 = t\frac{\partial}{\partial t} - \frac{x}{2}\frac{\partial}{\partial x}$. *The corresponding fluxes form the two-parameter group of affine transformations* $(t, x) \to (\lambda t + \mu, \frac{x}{\sqrt{\lambda}})$.

Subcase II.4. $g = 0, f f_x \neq 0$ The last of the four subcases is

$$\begin{cases} \lambda_x = 0, \\ A_x = \dfrac{\lambda \left({f_x}^2 f_{xx} + f f_x f_{xxx} - 2 f {f_{xx}}^2 \right)}{2{f_x}^3 f}, \\ B = -\dfrac{\lambda f}{f_x}, \\ f_{xxxx} = \dfrac{-f {f_x}^3 f_{xxx} - 6 {f_{xx}}^3 f^2 + 6 f^2 f_x f_{xx} f_{xxx} + {f_{xx}}^2 f {f_x}^2 + {f_x}^4 f_{xx}}{f^2 {f_x}^2} \\ g = 0. \end{cases}$$

We deduce that the symmetry Lie algebra can not be abelian and hence:

Proposition 5. *The symmetry Lie algebra is isomorphic to* $\mathfrak{a}(1, \mathbb{R})$.

Remark 2. *Liénard equation such that* $g = 0$ *has the first integral* $t - \int^x \frac{1}{\int f(s)+C}$. *Hence, such equations can not have limit cycles.*

Example 4. *In this case, we can take* $\ddot{x} = x\dot{x}$ *for which we have* $X_1 = \frac{\partial}{\partial t}$ *and* $X_2 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x}$. *The corresponding fluxes form the two-parameter group of affine transformations* $(t, x) \to (\lambda t + \mu, \frac{x}{\lambda})$.

3.1.1. Third Case $g \neq 0$

The characteristic presentation is

$$\begin{cases} \lambda_x = 0, \\ A_x = -\dfrac{f\left(9\lambda g - B f^2\right)}{27 g^2}, \\ B_x = \dfrac{6\lambda g - B f^2}{3g}, \\ f_x = -\dfrac{f^3}{9g}, \\ g_x = \dfrac{f^2}{3} \end{cases}$$

The number of points under the stairs of λ, A and B shows that the Lie algebra is three-dimensional.

Proposition 6. *In this case the Lie algebra is 3-dimensional and generated by*

$$\begin{cases} X_1 = \frac{\partial}{\partial t}, \\ X_2 = (x\frac{f(0)}{3g(0)} + t)\frac{\partial}{\partial t} + \left(2x - x^2\frac{f(0)^2}{3g(0)} + \frac{1}{81}x^3\frac{f(0)^4}{g(0)^2}\right)\frac{\partial}{\partial x} \\ X_3 = x\frac{f(0)^3}{27g(0)^2}\frac{\partial}{\partial t} + \left(1 - x\frac{f(0)^2}{3g(0)} + x^2\frac{f(0)^4}{27g(0)^2} - x^3\frac{1}{729}\frac{f(0)^6}{g(0)^3}\right)\frac{\partial}{\partial x}, \end{cases}$$

where $f(0)$, $g(0)$ *denote the values of* f *et* g *at* $x = 0$.

Proof. See [10].

3.1.2. Fourth Case

This case corresponds to the characteristic set

$$\begin{cases} \lambda = 0, \\ A_{x,x} = 0, \\ B_{x,x} = 2A_x f, \\ f_x = 0, \\ g = 0. \end{cases}$$

We deduce:

Proposition 7. *The symmetry algebra is four-dimensional. Moreover, Liénard equation is necessarily of the form*

$$\ddot{x} = a\dot{x},$$

where $a \in \mathbb{R}$. The infinitesimal generators are

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = x\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x},$$

generating $gl(2, \mathbb{R})$. The corresponding fluxes are

$$(t, x) \to (t - \ln(1 - \varepsilon x) + \mu,\ \sigma\frac{x}{1 - \varepsilon x} + \nu).$$

where ϵ, μ, σ and ν are the group parameters

3.1.3. Fifth Case

This case completes the classification. We have the characteristic set

$$\begin{cases} \lambda_x = 0, \\ A_{xx} = 0, \\ B_{xx} = 0, \\ f = 0, \\ g = 0. \end{cases}$$

The last two equations implies that the last differential ideal $\sqrt{I_8} \cap \mathbb{Q}\{f, g\}$ is generated by $\{f, g\}$. We have immediately

Proposition 8. *Liénard equation is reduced to*

$$\ddot{x} = 0.$$

The symmetry Lie algebra is generated by the vector fields

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = x\frac{\partial}{\partial x},\ X_4 = x\frac{\partial}{\partial t},\ X_5 = t\frac{\partial}{\partial t}.$$

The corresponding fluxes form a five-parameter transformations group

$$(t, x) \to (\lambda t + \mu + \epsilon x, \rho x + \sigma).$$

$(\lambda, \mu, \epsilon, \rho, \sigma)$ *are the group parameters.*

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