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Local Compactness in L-Fuzzy Spaces

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Abstract. In an L-topological space we present good definitions for local compactness, weak local compactness and relative local compactness. We obtain the equivalence of these properties in a Hausdorff space and we also obtain a one point compactification theorem.

Key words: Fuzzy lattice, L-topology, local compactness, weak local compactness, relative local compactness

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1. Introduction

In general topology there are three usual ways to define local compactness, which ones we call here local compactness, weak local compactness and relative local compactness.

Definition 1.1. *Let $\langle X, \delta \rangle$ be a topological space. We say that $\langle X, \delta \rangle$ is:*

(i) *locally compact if and only if for each $x \in X$ and $V \in \delta$ with $x \in V$ there exist $U \in \delta$ and a compact subset K of X with $x \in U$ and $U \subset K \subset V$.*

(ii) *weakly locally compact if and only if for each $x \in X$ there exist $U \in \delta$ and a compact subset K of X with $x \in U$ and $U \subset K$.*

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(iii) *relatively locally compact if and only if for each $x \in X$ there exist $U \in \delta$ with $x \in U$ and $\overline{U}$ compact.*

In this paper we present a generalization for an L topological spaces of these three properties. We show the goodness of the proposed definitions, the equivalence in Hausdorff spaces and present a one point compactification theorem.

2. Preliminaries

Throughout this paper X and Y are assumed nonempty ordinary sets, and $L = L\left\langle \leq, \vee, \wedge, ' \right\rangle$ always will denote a fuzzy lattice with its Scott topology, i.e., a complete completely distributive lattice with a smallest element **0** and a greatest element **1** $(0 \neq 1)$, with an order reversing involution $a \to a'$, and the topology is generated by the sets $\left\{x \in L\,;\, x \nleq p\right\}$ where $p \in pr(L)$ is a prime element of L, details in [1]. If $A \subset X$ we denote by χ_A the characteristic function of A in X.

We denote by L^X the set of functions $f : X \to L$ called L-sets. An L-point in X is an L-set $x_p : X \to L$, where $x \in X$ and $p \in pr(L)$, defined by: $x_p(y) = p$ if $y = x$, and $x_p(y) = 1$ otherwise. We say that $x_p \in f$ if and only if $f(x) \nleq p$, see [6].

Let $\langle X, \delta \rangle$ be a topological space. In [7], Warner proved that the set $\omega(\delta)$ formed by the continuous functions $f : X \to L$ is an L-topology. The base for the space $\omega(\delta)$ is formed by the functions

$$f(x) = \begin{cases} b & \text{if } x \in V \in \delta \\ 0 & \text{if } x \notin V \end{cases}$$

This provides a "goodness of extension" criterion for L-topological spaces.

Definition 2.1. *[4, Pu and Liu] Let $\langle X, T \rangle$ be an L topological space and let $f \in L^X$. The closure of f, $cl(f)$ or $\overline{f}$, is the L-set defined by:*

$$cl(f) = \wedge \left\{ g \in L^X \,;\, f \leq g,\ g' \in T \right\}$$

Definition 2.2. *[2, Kudri] Let $\langle X, T\rangle$ be an L-topological space and let $g \in L^X$. We say that g is compact if and only if for each $p \in pr(L)$ and each family $\left\{f_j\right\}_{j\in J}$ of open L-sets such that $\left(\vee_{j\in J} f_j\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, there exist a finite set J_1 of J such that $\left(\vee_{j\in J_1} f_j\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$.*

Theorem 2.1. *[5, Warner and McLean] Let $\langle X, \delta\rangle$ be a topological space. Then: $\langle X, \delta\rangle$ is compact if and only if $\langle X, \omega(\delta)\rangle$ is compact.*

Proposition 2.1. *[2, Kudri] Let $\langle X, T\rangle$ be an Hausdorff L-topological space and $F \subset X$. If χ_F is compact in $\langle X, T\rangle$ then χ_F is closed.*

Proposition 2.2. *[2, Kudri] Let $\langle X, T\rangle$ be an L-topological space. If $g \in L^X$ is a compact L-set, then for each closed L-set $h \in L^X$, $h \wedge g$ is a compact L-set.*

Proposition 2.3. *[2, Kudri] Let $\left\langle X, T_X\right\rangle$ and $\left\langle Y, T_Y\right\rangle$ be L-topological spaces and let $f : X \to Y$ be a continuous mapping. If $g \in L^X$ is a compact L-set, then $f(g) \in L^Y$ is a compact L-set.*

Proposition 2.4. *[2, Kudri] Let $\left\{\left\langle X_j, T_j\right\rangle\right\}_{j\in J}$ be a family of L-topological spaces and $g_j \in L^{X_j}$ be a compact L-set for each $j \in J$. Then the product set $g = \wedge_{j\in J}\pi_j^{-1}(g_j)$ is a compact L-set in the product space.*

Proposition 2.5. *[2, Kudri] Let $\mathbb{S}$ be a subbase for the L-topology T in X and let $g \in L^X$. If for each $p \in pr(L)$ and each family $\left\{f_j\right\}_{j\in J}$ of sub basis open L-sets with $\left(\vee_{j\in J} f_j\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$ there exists a finite subset F of J with $\left(\vee_{j\in F} f_j\right)(x) \nleq p$ for all $x \in X$ with $g(x) \geq p'$, then, g is compact in $\langle X, T\rangle$.*

Definition 2.3. *[5, Warner and McLean] An L-topological space $\langle X, T\rangle$ is Hausdorff if and only if for every $p, q \in pr(L)$ and every $x \neq y$ in X there exist $f, g \in T$ such that $f(x) \nleq p$, $g(y) \nleq q$ and, $f(z) = 0$ or $g(z) = 0$ for all $z \in X$.*

Theorem 2.2. *[5, Warner and McLean] Let $\langle X, \delta\rangle$ be a topological space. Then: $\langle X, \delta\rangle$ is Hausdorff if and only if $\langle X, \omega(\delta)\rangle$ is Hausdorff.*

Theorem 2.3. *[5, Warner and McLean] If $\langle X, T\rangle$ is a compact Hausdorff fully stratified L-topological space then it's topological, thats it, there is a topology $\delta \in X$ such that $T = \omega(\delta)$.*

Definition 2.4. *[5, Warner and McLean] An L-topological space $\langle X, T\rangle$ is regular if and only if for every $p \in pr(L)$, for each $x \in X$ and each closed L-set f such that there is $y \in X$ with $f(y) \geq p'$ and $f(x) = 0$, there are $u, v \in T$ with $u(x) \nleq p$, $v(z) \nleq p$ for each $z \in X$ with $f(z) \geq p'$, and, $u(z) = 0$ or $v(z) = 0$ for each $z \in X$.*

Theorem 2.4. *[2, Kudri] If $\langle X, \delta\rangle$ is a compact Hausdorff L-topological space then $\langle X, T\rangle$ is regular.*

3. Proposed definitions and their goodness theorems

Definition 3.1. *An L-topological space $\langle X, T\rangle$ is locally compact if and only if for each $x \in X$, $p \in pr(L)$ and $f \in T$ with $f(x) \nleq p$ there exist $g \in T$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $f(x) \nleq p$ and $g \leq k \leq f$.*

Theorem 3.1. *(The goodness of local compactness) Let $\langle X, \delta\rangle$ be an topological space. Then: $\langle X, \delta\rangle$ is locally compact if and only if $\langle X, \omega(\delta)\rangle$ is locally compact.*

Proof. Necessity: Let $x \in X$, let $p \in pr(L)$ and let $f \in \omega(\delta)$ such that $f(x) \nleq p$. Let $h \in \omega(\delta)$ be an basic open L-set with $h(x) \nleq p$ and $h \leq f$ defined by

$$h(y) = \begin{cases} e & \text{if } y \in V \in \delta \\ 0 & \text{if } y \notin V \end{cases}$$

Since $\langle X, \delta\rangle$ is locally compact, there exist $U \in \delta$ and a compact subset J of X such that $x \in U$ and $U \subset J \subset V$.

Let $g \in \omega(\delta)$ and $K \in L^X$ defined by:

$$g(y) = \begin{cases} e & \text{if } y \in U \in \delta \\ 0 & \text{if } y \notin U \end{cases} \qquad k(y) = \begin{cases} e & \text{if } y \in J \in \delta \\ 0 & \text{if } y \notin J \end{cases}$$

Then $g(x) \nleq p$, $g \leq k \leq h \leq f$ and $\chi_{supp(k)} = \chi_J$ is compact since J is compact. Hence, $\langle X, \omega(\delta)\rangle$ is locally compact.

Suficiency: Let $x \in X$ and $V \in \delta$ such that $x \in V$. Fix $p \in pr(L)$. Since $\langle X, \omega(\delta)\rangle$ is locally compact, for $f = \chi_V$, there exist $g \in \omega(\delta)$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $g(x) \nleq p$ and $g \leq k \leq f$.

Let $U = g^{-1}\{t \in L\,;\, t \nleq p\}$ and let $K = supp(k)$, then, $U \in \delta$, $x \in U$, K is a compact subset of X since $\chi_{supp(k)}$ is compact and $U \subset K \subset V$. Hence, $\langle X, \delta\rangle$ is locally compact.

Definition 3.2. *An L-topological space $\langle X, T\rangle$ is weakly locally compact if and only if for each $x \in X$ and $p \in pr(L)$ there exist $f \in T$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $f(x) \nleq p$ and $f \leq k$.*

Theorem 3.2. *(The goodness of weak local compactness) Let $\langle X, \delta\rangle$ be an topological space. Then: $\langle X, \delta\rangle$ is weakly locally compact if and only if $\langle X, \omega(\delta)\rangle$ is weakly locally compact.*

Proof. Necessity: Let $x \in X$ and let $p \in pr(L)$. Since $\langle X, \delta\rangle$ is weakly locally compact, there exist $U \in \delta$ and a compact subset J of X such that $x \in U$ and $U \subset J$.

Let $g = \chi_U$ and let $K = \chi_J$, then $g \in \omega(\delta)$, $g(x) \nleq p$, $g \leq k$ and $\chi_{supp(k)} = \chi_J$ is compact since J is compact. Hence, $\langle X, \omega(\delta)\rangle$ is weakly locally compact.

Suficiency: Let $x \in X$ and fix $p \in pr(L)$. Since $\langle X, \omega(\delta)\rangle$ is locally compact there exist $g \in \omega(\delta)$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $g(x) \nleq p$ and $g \leq k$.

Let $V = g^{-1}\{t \in L\,;\, t \nleq p\}$ and let $K = supp(k)$, then, $V \in \delta$, $x \in V$, K is a compact subset of X since $\chi_{supp(k)}$ is compact and $V \subset K$. Hence, $\langle X, \delta\rangle$ is weakly locally compact.

Definition 3.3. *An L-topological space $\langle X, T\rangle$ is relatively locally compact if and only if for each $x \in X$ and $p \in pr(L)$ there exists $f \in T$, with $\chi_{supp(\overline{f})}$ compact, such that $f(x) \nleq p$.*

Theorem 3.3. *(The goodness of relative local compactness) Let $\langle X, \delta\rangle$ be an topological space. Then: $\langle X, \delta\rangle$ is relatively locally compact if and only if $\langle X, \omega(\delta)\rangle$ is relatively locally compact.*

Proof. Necessity: Let $x \in X$ an let $p \in pr(L)$. Since $\langle X, \delta\rangle$ is relatively locally compact there is $V \in \delta$ with $x \in V$ and $\overline{V}$ compact. Let $f = \chi_V$, then $f(x) = 1 \nleq p$. We also have that $\overline{f} = \chi_{\overline{V}}$, hence $supp(\overline{f}) = \overline{V}$ is compact, therefore $\chi_{supp(\overline{f})}$ is compact.

Suficiency: Let $x \in X$ and let $p \in pr(L)$ be fixed. Since $\langle X, \omega(\delta)\rangle$ is relatively locally compact there is $f \in \omega(\delta)$, with $\chi_{supp(\overline{f})}$ compact, such that $f(x) \nleq p$, hence $supp(\overline{f})$ is compact. Let $g \in L^X$ a basic open L-set, $g(x) \nleq p$ and $g \leq f$, defined by

$$g(y) = \begin{cases} e & \text{if } y \in V \in \delta \\ 0 & \text{if } y \notin V \end{cases}$$

Since $g \leq f$ and

$$g(y) = \begin{cases} e & \text{se } y \in V \in \delta \\ 0 & \text{se } y \notin V \end{cases}$$

we have $\overline{g} \leq \overline{f}$ and $\overline{V} = supp(\overline{g}) \subset supp(\overline{f})$, thus, $\overline{V}$ is compact since it is closed and $supp(\overline{f})$ is compact.

4. Some properties and Comparison

Theorem 4.1. *Let $\langle X, T_X\rangle$ be a locally compact L-topological space and let $\langle Y, T_Y\rangle$ be an L-topological space. If $h: X \to Y$ is a continuous open surjection then $\langle Y, T_Y\rangle$ is locally compact.*

Proof. Let $y \in Y$ with $y = h(x)$, let $p \in pr(L)$ and $f \in T_Y$ with $f(y) \nleq p$. Let $j = h^{-1}(f)$, then $j \in T_X$ since h is continuous and $j(x) = f(y) \nleq p$. Since $\langle X, T_X\rangle$ is locally compact there exist $i \in T_X$ and $c \in L^X$, with$\chi_{supp(c)}$ compact, such that $i(x) \nleq p$ and $i \nleq c \nleq j$.

Let $g = h(j)$ and let $k = h(c)$. Then $g \in T_Y$ since h is open and $g \leq k \leq f$ since $i \nleq c \nleq j$. Since h is continuous and $\chi_{supp(c)}$ is compact we have $h(\chi_{supp(c)})$, but:

$$h(\chi_{supp(c)}) = \chi_{h(supp(c))} = \chi_{supp(h(c))} = \chi_{supp(k)}$$

Hence, $\langle Y, T_Y\rangle$ is locally compact.

Theorem 4.2. *Let $\langle X, T_X\rangle$ be a weakly locally compact L-topological space and let $\langle Y, T_Y\rangle$ be an L-topological space. If $h: X \to Y$ is a continuous open surjection then $\langle Y, T_Y\rangle$ is weakly locally compact.*

Proof. Let $y \in Y$ with $y = h(x)$ and let $p \in pr(L)$. Since $\langle X, T_X \rangle$ is weakly locally compact there exist $i \in T_X$ and $c \in L^X$, with $\chi_{supp(c)}$ compact, such that $i(x) \nleq p$ and $i \nleq c$.

Let $g = h(j)$ and let $k = h(c)$. Then $g \in T_Y$ since h is open and $g \leq k$ since $i \nleq c$. Since h is continuous and $\chi_{supp(c)}$ is compact we have $h(\chi_{supp(c)})$, but:

$$h(\chi_{supp(c)}) = \chi_{h(supp(c))} = \chi_{supp(h(c))} = \chi_{supp(k)}$$

Hence, $\langle Y, T_Y \rangle$ is weakly locally compact.

Theorem 4.3. *Let $\langle X, T_X \rangle$ be a relatively locally compact L-topological space and let $\langle Y, T_Y \rangle$ be an L-topological space. If $h : X \to Y$ is a continuous open surjection with $\overline{h(g)} \leq h(\overline{g})$ for every $g \in L^X$, then, $\langle Y, T_Y \rangle$ is relatively locally compact.*

Proof. Let $y \in Y$ with $y = h(x)$ and let $p \in pr(L)$. Since $\langle X, T_X \rangle$ is relatively locally compact there is $g \in T_X$, with $\chi_{supp(\overline{g})}$ compact, such that $g(x) \leq p$. Let $f = h(g)$, then: $f(x) \nleq p$, $f \in T_Y$ since h is open, and $h(\chi_{supp(\overline{g})})$ is a compact L-set in L^Y since h ï£¡ continuous. But:

$$h(\chi_{supp(\overline{g})}) = \chi_{h(supp(\overline{g}))} = \chi_{supp(h(\overline{g}))} = \chi_{supp(\overline{h(g)})} = \chi_{supp(\overline{f})}$$

where the last equality is due to the continuity of h and the condition mention in theorem.

Hence, $\langle Y, T_Y \rangle$ is relatively locally compact.

Theorem 4.4. *Let $\langle X, T \rangle$ be a locally compact L-topological space, then $\langle X, T \rangle$ is weakly locally compact.*

Proof. Let $x \in X$ and let $p \in pr(L)$. Since $\langle X, T \rangle$ is locally compact, for $f = X$, there exist $g \in T$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $g(x) \nleq p$ and $g \leq k \leq f$. So $\langle X, T \rangle$ is weakly locally compact.

Theorem 4.5. *If $\langle X, T \rangle$ is a compact Hausdorff fully stratified L-topological space then $\langle X, T \rangle$ is locally compact.*

Proof. Since $\langle X, T \rangle$ a compact Hausdorff fully stratified L-topological space there is a topology δ in X such that $T = \omega(\delta)$. By theorems 2.1 and 2.2 we have that $\langle X, \delta \rangle$ is a compact Hausdorff topological space, hence it's locally compact. by theorem 3.1, $\langle X, T\rangle$ is locally compact.

Theorem 4.6. *Let $\langle X, T\rangle$ be a weakly locally compact Hausdorff fully stratified L-topological space, then $\langle X, T\rangle$ is locally compact.*

Proof. Let $x \in X$, let $p \in pr(L)$ and let $f \in T$ such that $f(x) \not\leq p$. We must show that there exist $g \in T$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $g(x) \not\leq p$ and $g \leq k \leq f$.

Since $\langle X, T\rangle$ is weakly locally compact there exist $i \in T$ and $j \in L^X$, with $\chi_{supp(j)}$ compact, such that $i(x) \not\leq p$ and $i \leq j$. Let $D = supp(j)$. Since $\chi_{supp(j)}$ is compact and $\langle X, T\rangle$ is Hausdorff fully stratified, the subspace $\langle D, T_D\rangle$ is a compact Hausdorff fully stratified L-topological space, then, by theorem 4.5 it's locally compact, hence for $f_D = f|_D$ there exist $h_D \in T_D$ and $c \in L^D$, with $\chi_{supp(c)}$ compact, such that $h_D \leq c_D \leq f_D$ and $h_D(x) \not\leq p$.

Let $h \in T$ such that $h|_D = h_D$ and define $k \in L^X$ by

$$k(y) = \begin{cases} c_D(y) & \text{if } y \in D \\ 0 & \text{if } y \notin D \end{cases}$$

then, $h(x) \not\leq p$ and $\chi_{supp(k)}$ is compact since $supp(k) = supp(c_D)$.

Let $g = h \wedge j$, then $g \in T$ and $g(x) \not\leq p$. We proof now that $g \leq k \leq f$, in fact, if $y \in D$ then $g(y) \leq h(y) \leq k(y) \leq f(y)$ since $h_D \leq c_D \leq f_D$, and if $y \notin D$ then $j(y) = 0$ and $k(y) = 0$, so $g(y) = 0 = k(y) \leq f(y)$.

Theorem 4.7. *Let $\langle X, T\rangle$ be a relatively locally compact L-topological space, then $\langle X, T\rangle$ is weakly locally compact.*

Proof. Let $x \in X$ and let $p \in pr(L)$. Since $\langle X, T\rangle$ is relatively locally compact there exists $g \in T$, with $\chi_{supp(\overline{g})}$ compact, such that $g(x) \not\leq p$. Since $g \leq \overline{g}$, $\langle X, T\rangle$ is weakly locally compact.

Theorem 4.8. *Let $\langle X, T\rangle$ be a weakly locally compact Hausdorff fully stratified L-topological space such that $\chi_{supp(\overline{f})} = \overline{\chi_{supp(f)}}$, then it's relatively locally compact.*

Proof. Let $x \in X$ and let $p \in pr(L)$. Since $\langle X, T\rangle$ is weakly locally compact there exist $f \in T$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $f(x) \nleq p$ and $f \leq k$. Since $\chi_{supp(k)}$ is a compact L-set in a Hausdorff space, it's closed, by proposition 2.1, so, $\chi_{supp(k)} = \overline{\chi_{supp(k)}}$. Since $f \leq k$, $\chi_{supp(f)} \leq \chi_{supp(k)}$ then $\overline{\chi_{supp(f)}} \leq \chi_{supp(k)}$, hence $\overline{\chi_{supp(f)}}$ is a compact L-set since it's closed and $\chi_{supp(k)}$ is compact, by proposition 2.2. But $\overline{\chi_{supp(f)}} = \chi_{supp(\overline{f})}$, then $\chi_{supp(\overline{f})}$ is a compact L-set. Therefore $\langle X, T\rangle$ is relatively locally compact.

Theorem 4.9. *Let $\{X_\lambda\}_{\lambda\in J}$ be a family of nonempty fully stratified L-topological spaces. Then: The product L-topological space $\prod_{\lambda\in J} X_\lambda$ is locally compact if and only if each X_λ is locally compact and all but finitely many X_λ are compact.*

Proof. *Necessity:* Since the λth projection, $\pi_\lambda : \prod_{\lambda\in J} X_\lambda \to X_\lambda$, is a continuous open surjection and $\prod_{\lambda\in J} X_\lambda$ is locally compact, by theorem 4.1, X_λ is locally compact for each $\lambda \in J$. Now, let $p \in pr(L)$, $x \in \prod_{\lambda\in J} X_\lambda$ and let F be an open L-set in $\prod_{\lambda\in J} X_\lambda$ with $f(x) \nleq p$. Then by the local compactness of $\prod_{\lambda\in J} X_\lambda$, there are an open L-set g in $\prod_{\lambda\in J} X_\lambda$ with $g(x) \nleq p$ and an L-set k in $\prod_{\lambda\in J} X_\lambda$ with $\chi_{supp(k)}$ compact such that $g \leq k \leq f$.

Let $\wedge_{i=1}^m \pi_{\lambda_i}^{-1}(g_{\lambda_i})$ be a basic open L-set such that $\wedge_{i=1}^m \pi_{\lambda_i}^{-1}(g_{\lambda_i}) \leq g \leq k \leq f$. Then

$$\begin{aligned}\chi_{supp(k)} &\geq \chi_{supp(\wedge_{i=1}^m \pi_{\lambda_i}^{-1}(g_{\lambda_i}))} = \chi_{\cap_{i=1}^m supp(\pi_{\lambda_i}^{-1}(g_{\lambda_i}))} = \wedge_{i=1}^m \chi_{supp(\pi_{\lambda_i}^{-1}(g_{\lambda_i}))}\\ &= \wedge_{i=1}^m \pi_{\lambda_i}^{-1}(\chi_{supp(g_{\lambda_i})})\end{aligned}$$

Thus $\pi_\lambda(supp(k)) \geq \pi_\lambda(\wedge_{i=1}^m \pi_{\lambda_i}^{-1}(\chi_{supp(g_{\lambda_i})})) = X_\lambda$ for all $\lambda \notin \{\lambda_1, \cdots, \lambda_m\}$. Since π_λ is continuous, $\chi_{supp(k)}$ is compact in $\prod_{\lambda\in J} X_\lambda$ and $\pi_\lambda(\chi_{supp(k)}) = X_\lambda$, we have by proposition 2.3 that X_λ is compact for each λ except possibly $\lambda \in \{\lambda_1, \cdots, \lambda_m\}$.

Sufficiency: Let $p \in pr(L)$, $x \in \prod_{\lambda\in J} X_\lambda$ and $f \wedge_{\lambda=1}^m \pi_{\lambda_i}^{-1}(f_{\lambda_i})$ be a basic open L-set in the product L-topological space $\prod_{\lambda\in J} X_\lambda$ such that $f(x) \nleq p$ where f_{λ_i} is an open L-set in X_{λ_i}. We assume that $\{\lambda_1, \cdots, \lambda_m\}$ is expanded to include all λ for which X_λ is not compact.

We have that $f(x) \nleq p$ implies $f_{\lambda_i}(x_{\lambda_i}) \nleq p$ for all $i \in \{1, \cdots, m\}$. From the local compactness of each X_{λ_i}, there are an open L-set g_{λ_i} in X_{λ_i} and an L-set k_{λ_i} in X_{λ_i}, with $\chi_{supp(k_{\lambda_i})}$ compact, such that $g_{\lambda_i}(x_{\lambda_i}) \nleq p$ and $g_{\lambda_i} \leq k_{\lambda_i} \leq f_{\lambda_i}$.

Let $g = \wedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(g_{\lambda_i})$ and $k = \wedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(k_{\lambda_i})$, then, g is an open L-set in $\prod_{\lambda\in J} X_\lambda$, $g \leq k \leq f$ and

$$g(x) = \wedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(g_{\lambda_i})(x) = \wedge_{i=1}^{m} g_{\lambda_i}(x_{\lambda_i}) \nleq p$$

We also have

$$\chi_{supp(k)} = \chi_{supp(\wedge_{i=1}^{m}\pi_{\lambda_i}^{-1}(k_{\lambda_i}))} = \wedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(\chi_{supp(k_{\lambda_i})}) = \wedge_{\lambda\in J} w_\lambda$$

where $w_{\lambda_i} = \pi_{\lambda_i}^{-1}(\chi_{supp(k_{\lambda_i})})$ for $i \in \{1,\cdots,m\}$ and $w_\lambda = X_\lambda$ for $\lambda \notin \{1,\cdots,m\}$. Then $\chi_{supp(k)}$ is a compact L-set in $\prod_{\lambda ß J} X_\lambda$ by proposition 2.4 since $\chi_{supp(k_{\lambda_i})}$ is compact for each $i \in \{1,\cdots,m\}$ and X_λ is compact for each $\lambda \notin \{1,\cdots,m\}$.

Theorem 4.10. *Let $\{X_\lambda\}_{\lambda\in J}$ be a family of nonempty fully stratified L-topological spaces. Then: The product L-topological space $\prod_{\lambda\in J} X_\lambda$ is weakly locally compact if and only if each X_λ is weakly locally compact and all but finitely many X_λ are compact.*

Proof. The proof is analogous to the theorem 4.9, so we just give the outline for the proof.

Necessity: The weak local compactness of X_j is by theorem 4.2. For the rest, use the weak local compactness to obtain an open L-set g in $\prod_{j\in J} X_j$ and an L-set k in $\prod_{j\in J} X_j$, with $\chi_{supp(k)}$ compact, such that $g(x) \nleq p$ and $g \leq k$.

Sufficiency: For $p \in pr(L)$ and $x \in \prod_{j\in J} X_j$ use the weak local compactness of X_j, $j \notin \{j_1,\cdots j_m\}$ where the set is the index where X_j is not compact, to obtain g_{j_i} in $\prod_{j\in J} X_j$ and an L-set k_{j_i} in $\prod_{j\in J} X_j$, with $\chi_{supp(k_{j_i})}$ compact, such that $g_{j_i}(x) \nleq p$ and $g_{j_i} \leq k_{j_i}$. For the rest just take $g = \wedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(g_{\lambda_i})$ and $k = \wedge_{i=1}^{m} \pi_{\lambda_i}^{-1}(k_{\lambda_i})$.

Theorem 4.11. *If $\langle X, T\rangle$ is a Hausdorff weakly locally compact L-topological space then $\langle X, T\rangle$ is regular.*

Proof. let $x \in X$, let $p \in pr(L)$ and let h be a closed L-set such that $h(x) = 0$ and there exists $y_0 \in X$ with $h(y_0) \geq p'$.

Let's show that there are $u, v \in T$ such that $u(x) \nleq p$, $v(y) \nleq p$ for each $y \in X$ with $h(y) \geq p'$, and, $u(z) = 0$ or $v(z) = 0$ for each $z \in X$.

Since $\langle X, T\rangle$ is weakly locally compact there are $f \in T$ and $h \in L^X$, with $\chi_{supp(\overline{k})}$ compact, such that $f(x) \nleq p$ and $f \leq k$. let $D = supp(\overline{k})$, then:

(1) $x \in D$ since $f(x) \nleq p$ and $f \leq k$.

(2) $f(z) = 0$ for each $z \in D^c$ since $f \leq k$ and $k(z) = 0$ for each $z \in D^c$.

(3) Since χ_D is a compact L-set and $\langle X, T\rangle$ is Hausdorff we have that χ_D is a closed L-set, then $\chi'_D = \chi_{D^c} \in T$ is an open L-set.

since χ_D is compact and $\langle X, T\rangle$ is Hausdorff, the L-topological space $\langle D, T_D\rangle$ is compact and Hausdorff, hence $\langle D, T_D\rangle$ is regular by theorem 2.4.

Case 1: There exist $y \in D$ such that $h(y) \geq p'$.

In this case, by (1) and by regularity of $\langle D, T_D\rangle$, there are $u_D, v_D \in T_D$ such that $u_D(x) \nleq p$, $v_D(z) \nleq p$ for each $z \in D$ with $h(z) \leq p$, and, $u_D(z) = 0$ or $v_D(z) = 0$ for each $z \in D$. Let $u^*, v^* \in T$ such that $u^*|_D = u_D$ and $v^*|_D = v_D$, and define $u = u^* \vee f$ and $v = v^* \wedge \chi_{D^c}$. Then we have:

(a) $u \in T$ since $u^*, f \in T$, and $v \in T$ since $v^* \in T$ an by (3).

(b) Since $x \in D$, $u^*(x) = u_D(x) \nleq p$, then $u(x) \nleq p$ since $f(x) \nleq p$ and $p \in pr(L)$.

(c) Let $z \in X$ such that $h(z) \geq p'$.

$$z \in D \quad \Rightarrow \quad v(z) = v^*(z) = v_D(z) \nleq p$$
$$z \in D^c \quad \Rightarrow \quad \chi_{D^c}(z) = 1 \nleq p \Rightarrow v(z) = \chi_{D^c}(z) \nleq p$$

(d) Let $z \in X$ such that $u(z) \neq 0$, then, $u^*(z) \neq 0$ and $f(z) \neq 0$. By (2), $z \in D$, then $u_D(z) = u^*(z) \neq 0$, hence $v^*(z) = v_D(z) = 0$, so

$$v(z) = v^*(z) \wedge \chi_{D^c}(z) = 0 \wedge 0 = 0$$

Case 2: There is not $y \in D$ such that $h(y) \geq p'$.

Let $u = f$ and $v = \chi_{D^c}$, then $u, v \in T$ and:

(a) $u(x) = f(x) \nleq p$

(b) let $z \in X$ such that $h(z) \geq p'$, then $z \in D^c$ since in this case there is not $z \in D$ with $h(z) \geq p'$, hence $\chi_{D^c}(z) = 1$, so $v(z) = 1 \nleq p$.

(c) Let $z \in X$ such that $u(z) = f(z) \neq 0$, then by (2), $z \in D$, hence $v(z) = \chi_{D^c}(z) = 0$.

Therefore $\langle X, T \rangle$ is regular.

It's immediate that locally compact and relatively locally compact Hausdorff spaces are regular since these spaces are weakly locally compact by theorems 4.4 and 4.7.

5. The one point compactification

The following is based on [3].

Let $\langle X, T_X \rangle$ be a Hausdorff L topologiacal space which is not compact, but weakly locally compact. Let $Y = X \cup \{\infty\}$ with L topology T_Y generated by the subbase $\mathbb{S} = \left\{ f_1 \in L^Y ; f \in T_X \right\} \cup \left\{ \chi_{B_\infty} \in L^Y ; \chi_B \in \mathbb{C} \right\}$, where:

(i) $f_1 \in L^X$ is defined by:

$$f_1(y) = \begin{cases} f(y) & \text{if } y \in X \\ 0 & \text{if } y = \infty \end{cases}$$

(ii) $\mathbb{C} = \left\{ \chi_B \in L^X ; B \subset X, \chi_B \text{ compacto} \right\}$

(iii) For $\chi_B \in \mathbb{C}$, define $B_\infty = \{\infty\} \cup (X - B)$ and:

$$\chi_{B_\infty}(y) = \begin{cases} 1 & \text{if } y \in B_\infty \\ 0 & \text{if } y \in B. \end{cases}$$

The L topological space $\langle Y, T_Y \rangle$ is called the one point compactification of $\langle X, T_X \rangle$.

Theorem 5.1. *Let $\langle X, T_X \rangle$ be a weakly locally compact Hausdorff L-topological space which is not compact, and let $\langle Y, T_Y \rangle$ be their one point compactification. Then, $\langle Y, T_Y \rangle$ is a compact Hausdorff L-topological space, $cl(X) = Y$ and $\langle X, T_X \rangle$ is a subspace of $\langle Y, T_Y \rangle$.*

Proof.

(i) $\langle X, T_X \rangle$ is a subspace of $\langle Y, T_Y \rangle$. In fact, given $g \in T_Y$, $g|_X \in T_X$.

(ii) $cl(X) = Y$. In fact, if $cl(X) \neq Y$, then $cl(X)$ is an L-set of the form

$$cl(X)(y) = \begin{cases} 1 & \text{if } y \in X \\ l \neq 1 & \text{if } y = \infty \end{cases}$$

The complement of $cl(X)$ is the open L-set

$$cl(X)'(y) = \begin{cases} 0 & \text{if } y \in X \\ l' \neq 0 & \text{if } y = \infty \end{cases}$$

Let $f = \chi_{B_{1\infty}} \wedge \cdots \wedge \chi_{B_{n\infty}}$ be a basic open L-set such that $f \leq cl(X)'$ where $B_1, \cdots, B_n$ are subsets of X with the L-sets $\chi_{B_1}, \cdots, \chi_{B_1}$ compacts. We have for $y = \infty$ that $f(y) = 1 \leq l'$, then $l' = 1$, so $l = 0$.

We also have:

$$\begin{aligned} f \leq cl(X)' \quad &\Rightarrow \quad \chi_{B_{1\infty}} \wedge \cdots \wedge \chi_{B_{n\infty}} \leq cl(X)' \\ &\Rightarrow \quad cl(X) \leq \chi'_{B_{1\infty}} \vee \cdots \vee \chi'_{B_{n\infty}}. \end{aligned}$$

Since $X \leq cl(X)$ and $\chi'_{B_{i\infty}}|_X = \chi_{B_i}$ we have $X \leq \chi_{B_1} \vee \cdots \vee \chi_{B_n}$, thus, $X = \chi_{B_1} \vee \cdots \vee \chi_{B_n}$. Hence, X is compact which leads to a contradiction .

(iii) $\langle Y, T_Y \rangle$ is compact. In fact, let $p \in pr(L)$ and $\mathbb{B} = \{f_j\}_{j \in J}$ be a family of subbasis open L-sets with $\left(\vee_{j \in J} f_j\right)(y) \nleq p$ for each $y \in Y$. Then there is $j \in J$ such that $f_j = \chi_{B_\infty}$ with $B \subset X$ and χ_B compact, since in the other side, $\left(\vee_{j \in J} f_j\right)(\infty) = 0 \leq p$.

Let $\mathbb{B}_1 = \{f_j|_X\}_{j \in J_1}$ where $J_1 = \{j \in J \; ; \; f_j \neq \chi_{B_\infty}\}$, then $\mathbb{B}_1$ is such that $\left(\vee_{j \in J_1} f_j|_X\right)(x) \nleq p$ for each $x \in X$ with $\chi_B(x) \geq p'$.

Since χ_B is compact there is a finite subset J_2 of J_1 such that $\left(\vee_{j \in J_2} f_j|_X\right)(x) \nleq p$ for each $x \in X$ with $\chi_B(x) \geq p'$. Then, $\left(\chi_{B_\infty} \vee \vee_{j \in J_2} f_j\right)(y) \nleq p$ for each $y \in Y$. Hence by proposition 2.5 $\langle Y, T_Y \rangle$ is compact.

(iv) $\langle Y, T_Y \rangle$ ï£i Hausdorff. In fact, let $x \not\leq y$ in Y and $p,q \in pr(L)$. If $x,y \in X$, since X is Hausdorff, there exist $f,g \in T_X$ with $f(x) \not\leq p$, $g(y) \not\leq q$, and, $f(z) = 0$ or $g(z) = 0$ for each $z \in X$. Then $f_1 \in T_Y$, $g_1 \in T_Y$, $f_1(x) = f(x) \not\leq p$, $g_1(y) = g(y) \not\leq q$, and, $f_1(z) = 0$ or $g_1(z) = 0$ for each $z \in Y$.

If $x \in X$ and $y = \infty$, since X is weakly locally compact there are $f \in T_X$ and $k \in L^X$, with $\chi_{supp(k)}$ compact, such that $f(x) \not\leq p$ and $f \leq k$. Let $B = supp(\overline{k})$ and

$$f_1(y) = \begin{cases} f(y) & \text{if } y \in X \\ 0 & \text{if } y = \infty \end{cases}$$

then $f_1 \in T_Y$ and $\chi_{B_\infty} \in T_Y$.

It follow that $f_1(x) = f(x) \not\leq p$ and $\chi_{B_\infty}(y) = 1 \not\leq q$. Also:

(a) $z \in B \Rightarrow \chi_{B_\infty}(z) = 0$

(b) $z \in X - B = X - supp(\overline{f}) \Rightarrow \overline{f}(z) = 0 \Rightarrow f_1(z) = f(z) = 0$

(c) $z = \infty \Rightarrow f_1(z) = 0$

hence for each $z \in Y$, $f_1(z) = 0$ or $\chi_{B_\infty}(z) = 0$.

These conditions proof the theorem.

By an analogous way we can obtain one point compactification theorems for locally compact an relatively locally compact spaces since by theorems 4.4 and 4.7 these space are weakly locally compact.

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