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On Properties of the Dual Quaternions

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Abstract. In this paper, Euler's and De Moivre's formulas for complex numbers and quaternions are generalized for the dual quaternions. Also, the matrix representation of dual quaternions is expressed.

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1. Introduction

A *dual number* z is an ordered pair of real numbers (x, y) associated with a real unit $+1$ and the *dual unit*, or operator ε, where $\varepsilon^2 = \varepsilon^3 = \ldots = 0$. A dual number is usually denoted in the form

$$z = x + \varepsilon y. \tag{1}$$

Thus, the dual numbers are elements of the 2-dimensional real algebra

$$\mathbb{D} = \mathbb{R}[\varepsilon] = \{z = x + \varepsilon y | x, y \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}$$

generated by 1 and ε, [7].
Addition and multiplication of the dual numbers are defined by

$$(x + \varepsilon y) + (a + \varepsilon b) = (x + a) + \varepsilon(y + b), \tag{2}$$

$$(x + \varepsilon y).(a + \varepsilon b) = (xa) + \varepsilon(xb + ya). \tag{3}$$

This multiplication is commutative, associative and distributes over addition. The *conjugate dual number* $\bar{z}$ of $z = x + \varepsilon y$ is defined by $\bar{z} = x - \varepsilon y$ and we obtain

$$z\bar{z} = x^2. \tag{4}$$

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The algebra of dual numbers has been originally conceived by W.K Clifford [4], but its first application to mechanics are due to E. Study [8]. Because of conciseness of notation, dual algebra has been often used for the search of closed form solutions in the field of displacement analysis, kinematic synthesis and dynamic analysis of spatial mechanisms.

Dual numbers can be represented as follows:

1. Gaussian representation: $z = x + \varepsilon y$
2. Polar representation: $z = \rho(1 + \varepsilon\varphi)$
3. Exponential representation: $z = \rho e^{\varepsilon\varphi}$, where $\rho = x (x \neq 0), \varphi = y/x$ and $e^{\varepsilon\varphi} = 1 + \varepsilon\varphi$.

As the complex number,$|z| = |x| = \rho$ is called the *modulus* of the dual number z and $\varphi = y/x$ is called the *parameter*, [2,9]. If $|z| = 1$, then $z = x + \varepsilon y$ is called *unit dual number* and the set that satisfy $|z| = 1$ (or $x = \mp 1$) is called *Galilean unit circle* on the Dual plane. For any real φ, the *Galilean cosine* of φ (abbreviated $cosg$) and the *Galilean sine* of φ (abbreviated $sing$) are the $x-$ and $y-$ coordinates of the point $P = (x, y)$ on the Galilean unit circle, respectively. Furthermore, the Galilean cosine, $cosg\varphi$, and the Galilean sine, $sing\varphi$, are defined by

$$cosg\varphi = x/x = 1, \quad sing\varphi = y/x = \varphi \tag{5}$$

for all real φ , (see, [5]). The following formulas can be checked algebraically by using the definition of $cosg$ and $sing$ and the laws of exponents:

$$\begin{gathered} cosg(x + y) = cosgx\, cosgy - \varepsilon^2 singx\, singy, \\ sing(x + y) = singx\, cosgy + cosgx\, singy \\ cosg^2 x + \varepsilon^2 sing^2 x = 1. \end{gathered} \tag{6}$$

The dual number has a geometrical meaning which is discussed detail in [5,7].

2. Dual Quaternions

A *dual quaternion* Q is a linear combination $Q = a1 + bi + cj + dk$, where a, b, c, d are real numbers and $1 = (1, 0, 0, 0), i = (0, 1, 0, 0), j = (0, 0, 1, 0), k = (0, 0, 0, 1)$. The sum of quaternions is the usual componentwise sum and the multiplication is defined so that $(1, 0, 0, 0)$ is the identity and i, j and k satisfy

$$i^2 = j^2 = k^2 = ijk = 0. \tag{7}$$

It follows from (7) that $ij = -ji = jk = -kj = ki = -ik = 0$.
The set of dual quaternions denoted by

$$H_{\mathbb{D}} = \{Q = a + bi + cj + dk| \quad a, b, c, d \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = 0\}.$$

We can also write $Q = a + \mathbf{w}$, where a is the *real part* of Q and $\mathbf{w} = bi + cj + dk \in \mathbb{R}^3$, called the *pure dual quaternion part* of Q . The *conjugate* of Q is $\bar{Q} = a - \mathbf{w}$. A simple computation shows

$$\mathbf{w}_1\mathbf{w}_2 = 0 \tag{8}$$

where $\mathbf{w}_1$ and $\mathbf{w}_2$ are the pure dual quaternions. Let a_1, a_2 be real numbers. Then,

$$Q_1Q_2 = (a_1 + \mathbf{w}_1)(a_2 + \mathbf{w}_2) = a_1a_2 + a_1\mathbf{w}_2 + a_2\mathbf{w}_1 = Q_2Q_1. \tag{9}$$

It follows form (8) that $\overline{Q_1Q_2} = \overline{Q_1}\ \overline{Q_2} = \overline{Q_2}\ \overline{Q_1}$ for two dual quaternions . The *norm* of a dual quaternion $Q = a + bi + cj + dk = a + \mathbf{w}$ is defined as $||Q|| = \sqrt{Q\bar{Q}} = \sqrt{a^2} = |a|$. If $||Q|| = 1$, then Q is called *unit dual quaternion*.

The set $H_{\mathbb{D}}$ forms a commutative ring under the dual quaternion multiplication and also it is a vector space of dimensions four on $\mathbb{R}$ and its basis is the set $\{1, i, j, k\}$. The interesting property of dual quaternions is that by their means one can express the Galilean transformation in one quaternion equation. Since the multiplication and ratio of two dual quaternions Q_1 and Q_2 is again a dual quaternion, the set of dual quaternions form a division algebra under addition and multiplication.
For more details on dual quaternions, we refer the reader to [1], [6].

3. Euler'S Formula and De Moivre'S Formula for Dual Quaternions

Let $Q = a+bi+cj+dk$ be a unit dual quaternion. We can express any unit dual quaternion Q as

$$Q = cosg\theta + \mathbf{w}sing\theta \tag{10}$$

where $w = \frac{bi+cj+dk}{\sqrt{b^2+c^2+d^2}}$ and $sing\theta = \theta = \sqrt{b^2 + c^2 + d^2}$. This is similar to the polar coordinated expression of a complex number $z = \cos\varphi + i\sin\varphi$. Since $\mathbf{w}^2 = 0$ for any pure dual quaternion $\mathbf{w}$, we have a natural generalization of Euler's formula for dual quaternion

$$e^{\mathbf{w}\theta} = 1 + \mathbf{w}\theta + \frac{(\mathbf{w}\theta)^2}{2!} + \frac{(\mathbf{w}\theta)^3}{3!} + \ldots = 1 + \mathbf{w}\theta = cosg\theta + \mathbf{w}sing\theta = Q$$

for any real Q . For more on Euler's formula for complex numbers and real quaternion, we refer the reader to [2,3].

We can give the following Lemma using equation (6) according to the addition formula for Galilean cosine and Galilean sine:

Lemma 1. *For any pure dual quaternion* $\mathbf{w}$*, we have,*

1. $e^{\mathbf{w}\theta_1}e^{\mathbf{w}\theta_2} = e^{\mathbf{w}(\theta_1+\theta_2)}$
2. $\frac{1}{e^{\mathbf{w}\theta}} = e^{\mathbf{w}(-\theta)}$
3. $\frac{e^{\mathbf{w}\theta_1}}{e^{\mathbf{w}\theta_2}} = e^{\mathbf{w}(\theta_1-\theta_2)}$

Proposition 1 (De Moivre's formula). *Let $Q = e^{\mathbf{w}\theta} = cosg\theta + \mathbf{w}sing\theta$ be a unit dual quaternion. Then,*

$$Q^n = (e^{\mathbf{w}\theta})^n = (cosg\theta + \mathbf{w}sing\theta)^n = cosg(n\theta) + \mathbf{w}sing(n\theta)$$

for every integer.

Proof. The proof will be a induction on nonnegative integers n using equation (6):

$$\begin{aligned} Q^{n+1} &= (cosg\theta + \mathbf{w}sing\theta)^{n+1} \\ &= (cosg(n\theta) + \mathbf{w}sing(n\theta))(cosg\theta + \mathbf{w}sing\theta) \\ &= cosg(n+1)\theta + \mathbf{w}sing(n+1)\theta. \end{aligned}$$

The formulas holds for all integers n since

$$Q^{-1} = cosg\theta - \mathbf{w}sing\theta$$

and

$$Q^{-n} = cosg(n\theta) - \mathbf{w}sing(n\theta) = cosg(-n\theta) + \mathbf{w}sing(-n\theta).$$

4. The Matrix Representation of the Dual Quaternions

Let $Q = a + bi + cj + dk$ be a dual quaternion. We will define the linear map L_Q as $L_Q : H_{\mathbb{D}} \to H_{\mathbb{D}}$ such that $L_Q(Q_1) = QQ_1$. Using our newly defined operator and the basis $\{1, i, j, k\}$ of the vector space $H_{\mathbb{D}}$, we can write

$$L_Q(1) = Q \times 1 = a1 + bi + cj + dk$$

$$L_Q(i) = Q \times i = 01 + ai + 0j + 0k$$

$$L_Q(j) = Q \times j = 01 + 0i + aj + 0k$$

$$L_Q(k) = Q \times k = 01 + 0i + 0j + ak.$$

Then, we find the following real matrix representation

$$L_Q = \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & 0 & a & 0 \\ d & 0 & 0 & a \end{pmatrix}.$$

It is interesting to note that $\sqrt{detL_Q} = a^2 = ||Q||^2$.

References

[1] B. Artmann. The concept of Number: From Quaternions to Modads and topological Fields, Ellis Horwood, Chicherster, 1988.

[2] P. P. Boas. Invitation to Complex Analysis, Random House, New York, 1987.

[3] E. Cho. De Moivre's formula for quaternions. *Appl. Math. Lett.*, 11.6: 33-35. 1998.

[4] W. K. Clifford. Preliminary sketch of bi-quaternions. *Proc. London Math. Soc.* 4. 381-395. 1873.

[5] G. Helzer. Special Relativity with acceleration. *Amer. Math. Monthy*, 107.3: 215-237. 2000.

[6] V. Majernik. Quaternion formulation of the Galilean space-time transformation. *Acta Phy. Slovaca*, 56.1: 9-14. 2006.

[7] E. Pennestrì and R. Stefanelli. Linear algebra and numerical algorithms using dual numbers. *Multibody Syst. Dyn.* 18.3: 323-344. 2007.

[8] E. Study. Geometrie der Dynamen, Leipzig, Germany, 1903.

[9] I. M. Yaglom. Complex Numbers in Geometry, New York, Academic, 1968.