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The Exterior Tricomi and Frankl Problems for Quaterelliptic-Quaterhyperbolic Equations With Eight Parabolic Lines

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Abstract. The famous Tricomi equation was established in 1923 by F. G. Tricomi who is the pioneer of parabolic elliptic and hyperbolic boundary value problems and related problems of variable type. In 1945 F. I. Frankl established a generalization of these problems for the well-known Chaplygin equation subject to a certain Frankl condition. In 1953 and 1955 M. H. Protter generalized these problems even further by improving the Frankl condition. In 1977 we generalized these results in several n-dimensional simply connected domains. In 1990 we proposed the exterior Tricomi problem in a doubly connected domain. In 2002 we considered uniqueness of quasi-regular solutions for a bi-parabolic elliptic bi-hyperbolic Tricomi problem. In 2006 G. C. Wen investigated the exterior Tricomi problem for general mixed type equations. In this paper we establish uniqueness of quasi-regular solutions for the exterior Tricomi and Frankl problems for quaterelliptic - quaterhyperbolic mixed type partial differential equations of second order with eight parabolic degenerate lines and propose certain open problems. These mixed type boundary value problems are very important in fluid mechanics.

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Key Words and Phrases: Quasi-regular solution, Tricomi equation, Chaplygin equation, quaterelliptic equation, quaterhyperbolic equation, Tricomi problem.

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1. Introduction

In 1904 S. A. Chaplygin [11] pointed out that the nonlinear equation of an adiabatic potential perfect gas:

$$(\rho^2\alpha^2 - {\psi_y}^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} + (\rho^2\alpha^2 - {\psi_x}^2)\psi_{yy} = 0,$$

is closely connected with the study of the linear mixed type equation

$$K(y)u_{xx} + u_{yy} = 0$$

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named Chaplygin equation, where $\psi = \psi(x, y)$ is the stream function, $\alpha :=$ local velocity of sound and $\rho :=$ density of gas.

In 1923 F. G. Tricomi [19] initiated the work on boundary value problems for linear partial differential mixed type equations of second order and related equations of variable type. The well-known mixed type partial differential equation was called *Tricomi equation*:

$$yu_{xx} + u_{yy} = 0$$

after F. G. Tricomi, who introduced this equation, for functions $u = u(x, y)$ in a real $(x, y)-$region. It plays a central role in the mathematical analysis of the transonic flows, as it is of *elliptic* and *hyperbolic type* where the coefficient y of the second partial derivative of the involved function $u = u(x, y)$ with respect to x, changes sign. Besides, this equation is of *parabolic type* where y vanishes.

In 1945 F. I. Frankl [3] drew attention to the fact that the Tricomi problem was closely connected to the study of gas flow with nearly sonic speeds. In 1953 and 1955 M. H. Protter [7] generalized and improved the afore-mentioned results in the euclidean plane. In 1977 we [8] generalized these results in R^n $(n > 2)$. In 1982 we [9] established a maximum principle of the Cauchy problem for hyperbolic equations in R^{n+1} $(n \geq 2)$. In 1983 we [10] solved the Tricomi problem with two parabolic lines of degeneracy and, in 1992, we [12] established the well-posedness of the Tricomi problem in euclidean regions. Interesting results for the Tricomi problem were achieved by G. Baranchev [1] in 1986, and M. Kracht and E. Kreyszig [4] in 1986, as well. Related information was reported by G. Fichera [2] in 1985, and E. Kreyszig [5-6] in 1989 and 1994. Our [11,14-15] work, in 1990 and 1999, was in analogous areas of mixed type equations. In 1990-2009, G. C. Wen et al. [17,20-28] have applied the complex analytic method and achieved fundamental uniqueness and existence results for solutions of the Tricomi and Frankl problems for classical mixed type partial differential equations with boundary conditions. In 1993 R.I. Semerdjieva [18] introduced the hyperbolic equation $K_1(y)u_{xx} + (K_2(y)u_y)_y + ru = f$ in the lower half-plane. In 1997 we [13] considered the more general case of the above hyperbolic equation, so that it was elliptic in the upper half-plane and parabolic on the line $y = 0$. In 2002, we [16] considered the more general Tricomi problem with partial differential equation the new *bi-parabolic elliptic bi-hyperbolic equation*

$$Lu \equiv K_1(y)(M_2(x)u_x)_x + M_1(x)(K_2(y)u_y)_y + r(x, y)u = f(x, y), \tag{1}$$

which is parabolic on both segments $x = 0,\ 0 < y \leq 1;\ y = 0,\ 0 < x \leq 1$, elliptic in the Euclidean region $G_e = \{(x, y) \in G(\subset R^2) : x > 0,\ y > 0\}$ and hyperbolic in both regions $G_{h_1} = \{(x, y) \in G(\subset R^2) : x > 0,\ y < 0\}$; $G_{h_2} = \{(x, y) \in G(\subset R^2) : x < 0,\ y > 0\}$, with G the mixed domain of (1). In 1999 we [15] proved existence of weak solutions for a particular Tricomi problem. Then we established uniqueness of quasi-regular solutions [8,10-13,16] for the Tricomi problem. However, the question about the uniqueness of quasi-regular solutions and the existence of weak solutions for the Tricomi and Frankl problems associated to the said mixed type equation (1) for even more general doubly connected mixed domain is still *open.*

In particular via this paper we propose and investigate the exterior Tricomi and Frankl problems for quaterelliptic and quaterhyperbolic equations with eight parabolic lines of degeneracy and establish uniqueness of quasi-regular solutions. Also we propose new open problems.

These results are interesting in Aerodynamics and Hydrodynamics. The Mixed type partial differential equations are encountered in the theory of transonic flow and they give rise to special boundary value problems, called the Tricomi and Frankl problems. The Transonic flows involve a transition from the subsonic to the supersonic region through the sonic.

Definition 1. *The Tricomi problem or Problem T consists of finding a function u which satisfies the afore-mentioned Tricomi equation in a mixed domain D : a simply connected and bounded $(x, y)-$ region by a rectifiable Jordan (non-self-intersecting) elliptic arc σ (for $y > 0$) with endpoints $O = (0,0)$ and $A = (1,0)$ and by two real hyperbolic characteristics Γ, γ of the Tricomi equation satisfying the pertinent characteristic equation such that these characteristics Γ, γ meet at a point P (for $y < 0$) with Γ emanating from A and γ from O,*

$$\Gamma : \; x + \frac{2}{3}(-y)^{3/2} = 1 \text{ and } \gamma : \; x - \frac{2}{3}(-y)^{3/2} = 0$$

and u assumes prescribed continuous boundary values on both arcs σ and γ. The portion of D lying in the upper half-plane, above the x-axis, is the elliptic region; portion of D lying in the lower half-plane, below the x-axis, is the hyperolic region; and the segment OA is parabolic.

Definition 2. *A function $u = u(x, y)$ is a regular solution of Problem T in the sense of F. G. Tricomi if:*

1) *u is continuous in the closure of D which is the union of D with its boundary consisting of the three curves σ, Γ, γ;*
2) *The first order partial derivatives of u are continuous in the closure of D except points O,A, where they may have poles of order less than $2/3$;*
3) *The second order partial derivatives of u are continuous in D except possibly on OA where they may not exist;*
4) *u satisfies Tricomi equation at all points of D except OA;*
5) *u assumes prescribed continuous boundary values on both arcs σ, γ.*

2. The Exterior Tricomi Problem

Consider the quaterelliptic - quaterhyperbolic equation (1) with eight parabolic lines of degeneracy in a bounded doubly connected mixed domain D with a piecewise smooth boundary ∂D, where $f = f(x, y)$ is continuous in D, $r = r(x, y)$ is once-continuously differentiable in D, $K_i = K_i(y)$ $(i = 1, 2)$ are once-continuously differentiable for $y \in [-k_1, k_2]$ with $-k_1 = inf\{y : (x, y) \in D\}$ and $k_2 = sup\{y : (x, y) \in D\}$, and $M_i = M_i(x)$ $(i = 1, 2)$ are once-continuously differentiable for $x \in [-m_1, m_2]$ with $-m_1 = inf\{x : (x, y) \in D\}$ and $m_2 = sup\{x : (x, y) \in D\}$.

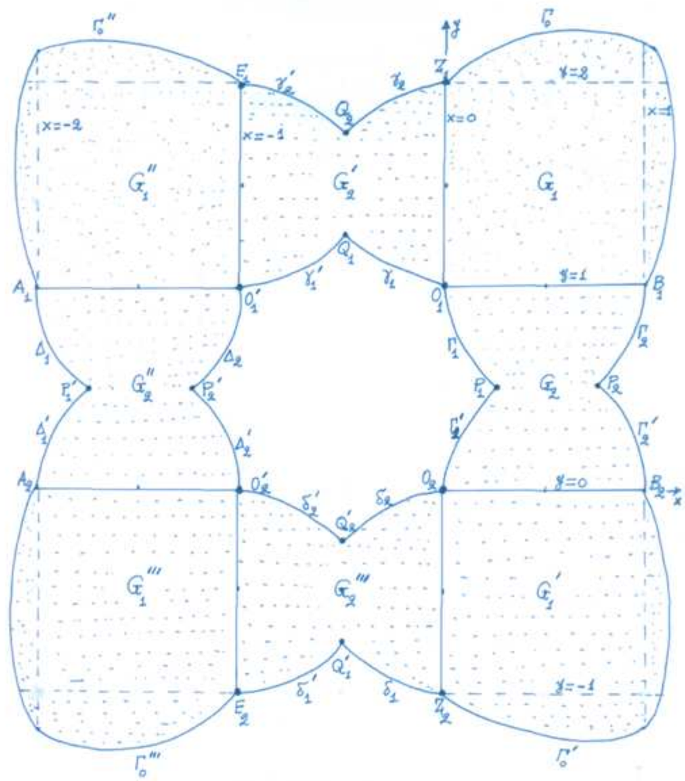

Figure 1

Besides,

$$K_1(y)\begin{cases} >0 & \text{for} \quad \{y<0\}\cup\{y>1\} \\ =0 & \text{for} \quad \{y=0\}\cup\{y=1\} \\ <0 & \text{for} \quad \{0<y<1\} \end{cases};$$

$$M_1(x)\begin{cases} >0 & \text{for} \quad \{x<-1\}\cup\{x>0\} \\ =0 & \text{for} \quad \{x=0\}\cup\{x=-1\}, \\ <0 & \text{for} \quad \{-1<x<0\} \end{cases}$$

as well as $K_2 = K_2(y) > 0$, $M_2 = M_2(x) > 0$, everywhere in D, so that

$$K = K(y) = \frac{K_1(y)}{K_2(y)}\begin{cases} >0 & \text{for} \quad \{y<0\}\cup\{y>1\} \\ =0 & \text{for} \quad \{y=0\}\cup\{y=1\} \\ <0 & \text{for} \quad \{0<y<1\} \end{cases};$$

$$M = M(x) = \frac{M_1(x)}{M_2(x)} \begin{cases} > 0 & \text{for} \quad \{x < -1\} \cup \{x > 0\} \\ = 0 & \text{for} \quad \{x = 0\} \cup \{x = -1\}. \\ < 0 & \text{for} \quad \{-1 < x < 0\} \end{cases}$$

The boundary $\partial D = Ext(D) \cup Int(D)$ of the doubly connected domain D is formed by the following two exterior and interior boundaries

$$Ext(D) = (\Gamma_0 \cup \Gamma_0{}' \cup \Gamma_0{}'' \cup \Gamma_0{}''') \cup (\Gamma_2 \cup \Gamma_2{}') \cup (\gamma_2 \cup \gamma_2{}') \cup (\Delta_1 \cup \Delta_1{}') \cup (\delta_1 \cup \delta_1{}'),$$
$$Int(D) = (\Gamma_1 \cup \Gamma_1{}') \cup (\gamma_1 \cup \gamma_1{}') \cup (\Delta_2 \cup \Delta_2{}') \cup (\delta_2 \cup \delta_2{}'),$$

respectively: In the right hyperbolic domain $G_2 = \{(x, y) \in D : 0 < x < 1, 0 < y < 1\}$ with boundary $\partial G_2 = (O_1B_1) \cup (O_2B_2) \cup (\Gamma_1 \cup \Gamma_1{}') \cup (\Gamma_2 \cup \Gamma_2{}')$, where O_1B_1, O_2B_2 are two parabolic lines with end points $O_1 = (0, 1)$, $B_1 = (1, 1)$ and $O_2 = (0, 0)$, $B_2 = (1, 0)$ and $\Gamma_1, \Gamma_1{}', \Gamma_2, \Gamma_2{}'$ are four characteristics, so that:

$$\Gamma_1 : \int_0^x \sqrt{M(t)}dt = -\int_1^y \sqrt{-K(t)}dt \, : 0 < x < 1, \frac{1}{2} < y < 1 \text{ emanating from } O_1 = (0, 1),$$
$$\Gamma_1{}' : \int_0^x \sqrt{M(t)}dt = \int_0^y \sqrt{-K(t)}dt \, : 0 < x < 1 \, , 0 < y < \frac{1}{2}, \text{ emanating from } O_2 = (0, 0),$$
$$\Gamma_2 : \int_1^x \sqrt{M(t)}dt = \int_1^y \sqrt{-K(t)}dt \, : 0 < x < 1 \, , \frac{1}{2} < y < 1, \text{emanating from } B_1 = (1, 1),$$
$$\Gamma_2{}' : \int_1^x \sqrt{M(t)}dt = -\int_0^y \sqrt{-K(t)}dt : 0 < x < 1, 0 < y < \frac{1}{2}, \text{emanating from } B_2 = (1, 0),$$

where $M = M(x) > 0$, $0 < x < 1$ and $K = K(y) < 0$, $0 < y < 1$. In the upper hyperbolic domain $G_2{}' = \{(x, y) \in D : -1 < x < 0, 1 < y < 2\}$ with boundary $\partial G_2{}' = (O_1Z_1) \cup (O_1{}'E_1) \cup (\gamma_1 \cup \gamma_1{}') \cup (\gamma_2 \cup \gamma_2{}')$, where O_1Z_1, $O_1{}'E_1$ are two parabolic lines with end points $O_1 = (0, 1)$, $Z_1 = (0, 2)$ and $O_1{}' = (-1, 1)$, $E_1 = (-1, 2)$ and $\gamma_1, \gamma_1{}', \gamma_2, \gamma_2{}'$ are four characteristics, so that:

$$\gamma_1 : \int_0^x \sqrt{-M(t)}dt = -\int_1^y \sqrt{K(t)}dt \, : -\frac{1}{2} < x < 0 \, , \ 1 < y < 2, \text{ emanating from } O_1 = (0, 1),$$
$$\gamma_1{}' : \int_{-1}^x \sqrt{-M(t)}dt = \int_1^y \sqrt{K(t)}dt \, : -1 < x < -\frac{1}{2}, 1 < y < 2, \text{ emanating from } O_1{}' = (-1, 1),$$
$$\gamma_2 : \int_0^x \sqrt{-M(t)}dt = \int_2^y \sqrt{K(t)}dt \, : \quad -\frac{1}{2} < x < 0, 1 < y < 2, \text{ emanating from } Z_1 = (0, 2),$$
$$\gamma_2{}' : \int_{-1}^x \sqrt{-M(t)}dt = -\int_2^y \sqrt{K(t)}dt : -1 < x < -\frac{1}{2}, 1 < y < 2, \text{ emanating from } E_1 = (-1, 2),$$

where $M = M(x) < 0, -1 < x < 0$ and $K = K(y) > 0, 1 < y < 2$. In the left hyperbolic domain $G_2'' = \{(x, y) \in D : -2 < x < -1, 0 < y < 1\}$ with boundary $\partial G_2'' = (O_1'A_1) \cup (O_2'A_2) \cup (\Delta_1 \cup \Delta_1') \cup (\Delta_2 \cup \Delta_2')$, where $O_1'A_1$, $O_2'A_2$ are two parabolic lines with end points $O_1' = (-1, 1), A_1 = (-2, 1)$ and $O_2' = (-1, 0), A_2 = (-2, 0)$ and $\Delta_1, \Delta_1', \Delta_2, \Delta_2'$ are four characteristics, so that:

$$\Delta_1 : \int_{-2}^{x} \sqrt{M(t)}dt = -\int_{1}^{y} \sqrt{-K(t)}dt : -2 < x < -1, \frac{1}{2} < y < 1, \text{ emanating from } A_1 = (-2, 1),$$
$$\Delta_1' : \int_{-2}^{x} \sqrt{M(t)}dt = \int_{0}^{y} \sqrt{-K(t)}dt : -2 < x < -1, 0 < y < \frac{1}{2}, \text{ emanating from } A_2 = (-2, 0),$$
$$\Delta_2 : \int_{-1}^{x} \sqrt{M(t)}dt = \int_{1}^{y} \sqrt{-K(t)}dt : -2 < x < -1, \frac{1}{2} < y < 1, \text{ emanating from } O_1' = (-1, 1),$$
$$\Delta_2' : \int_{-1}^{x} \sqrt{M(t)}dt = -\int_{0}^{y} \sqrt{-K(t)}dt : -2 < x < -1, 0 < y < \frac{1}{2}, \text{ emanating from } O_2' = (-1, 0),$$

where $M = M(x) > 0, -2 < x < -1$ and $K = K(y) < 0, 0 < y < 1$. In the lower hyperbolic domain $G_2''' = \{(x, y) \in D : -1 < x < 0, -1 < y < 0\}$ with boundary $\partial G_2''' = (O_2Z_2) \cup (O_2'E_2) \cup (\delta_1 \cup \delta_1') \cup (\delta_2 \cup \delta_2')$, where O_2Z_2, $O_2'E_2$ are two parabolic lines with end points $O_2 = (0, 0)$, $Z_2 = (0, -1)$ and $O_2' = (-1, 0)$, $E_2 = (-1, -1)$ and $\delta_1, \delta_1', \delta_2, \delta_2'$ are four characteristics, so that:

$$\delta_1 : \int_{0}^{x} \sqrt{-M(t)}dt = -\int_{-1}^{y} \sqrt{K(t)}dt : -\frac{1}{2} < x < 0, -1 < y < 0, \text{ emanating from } Z_2 = (0, -1),$$
$$\delta_1' : \int_{-1}^{x} \sqrt{-M(t)}dt = \int_{-1}^{y} \sqrt{K(t)}dt : -1 < x < -\frac{1}{2}, -1 < y < 0, \text{ emanating from } E_2 = (-1, -1),$$
$$\delta_2 : \int_{0}^{x} \sqrt{-M(t)}dt = \int_{0}^{y} \sqrt{K(t)}dt : -\frac{1}{2} < x < 0, \quad -1 < y < 0, \text{ emanating from } O_2 = (0, 0),$$
$$\delta_2' : \int_{-1}^{x} \sqrt{-M(t)}dt = -\int_{0}^{y} \sqrt{K(t)}dt : -1 < x < -\frac{1}{2}, -1 < y < 0, \text{ starting from } O_2' = (-1, 0),$$

where $M = M(x) < 0, -1 < x < 0$ and $K = K(y) > 0, 1 < y < 2$. In the upper right elliptic domain $G_1 = \{(x, y) \in D : x > 0, y > 1\}$ with boundary $\partial G_1 = (O_1B_1) \cup (O_1Z_1) \cup \Gamma_0$, where O_1B_1, O_1Z_1 are two parabolic lines with end points $O_1 = (0, 1)$, $B_1 = (1, 1)$ and $O_1 = (0, 1)$, $Z_1 = (0, 2)$ and Γ_0 is the upper right elliptic arc connecting points $B_1 = (1, 1)$ and $Z_1 = (0, 2)$. In the lower right elliptic domain $G_1' = \{(x, y) \in D : x > 0, y < 0\}$ with boundary $\partial G_1' = (O_2B_2) \cup (O_2Z_2) \cup \Gamma_0'$, where O_2B_2, O_2Z_2 are two parabolic lines with end points $O_2 = (0, 0)$, $B_2 = (1, 0)$ and $O_2 = (0, 0)$, $Z_2 = (0, -1)$ and Γ_0' is the lower right elliptic arc connecting points $B_2 = (1, 0)$ and $Z_2 = (0, -1)$. In the upper left elliptic domain $G_1'' = \{(x, y) \in D : x < -1, y > 1\}$ with boundary $\partial G_1'' = (O_1'E_1) \cup (O_1'A_1) \cup \Gamma_0''$,

where $O_1{}'E_1$, $O_1{}'A_1$ are two parabolic lines with end points $O_1{}' = (-1,1)$, $E_1 = (-1,2)$ and $O_1{}' = (-1,1)$, $A_1 = (-2,1)$ and $\Gamma_0{}''$ is the upper left elliptic arc connecting points $A_1 = (-2,1)$ and $E_1 = (-1,2)$. In the lower left elliptic domain $G_1{}''' = \{(x,y) \in D : x < -1, y < 0\}$ with boundary $\partial G_1{}''' = (O_2{}'A_2) \cup (O_2{}'E_2) \cup \Gamma_0{}'''$, where $O_2{}'E_2$, $O_2{}'A_2$ are two parabolic lines with end points $O_2{}' = (-1,0), E_2 = (-1,-1)$ and $O_2{}' = (-1,0), A_2 = (-2,0)$ and $\Gamma_0{}'''$ is the lower left elliptic arc connecting points $A_2 = (-2,0)$ and $E_2 = (-1,-1)$.
Let us consider the intersection points of the hyperbolic characteristics:
$\Gamma_1 \cap \Gamma_1{}' = \{P_1\}$, where $P_1 = (x_1, \frac{1}{2})$, $\quad 0 < x_1 < 1$; $\Gamma_2 \cap \Gamma_2{}' = \{P_2\}$, where $P_2 = (x_2, \frac{1}{2})$, $0 < x_1 < \frac{1}{2} < x_2 < 1$; $\Delta_1 \cap \Delta_1{}' = \{P_1{}'\}$, where $P_1{}' = (x_1{}', \frac{1}{2})$, $-2 < x_1{}' < -1$; $\Delta_2 \cap \Delta_2{}' = \{P_2{}'\}$, where $P_2{}' = (x_2{}', \frac{1}{2})$, $-2 < x_1{}' < -\frac{3}{2} < x_2{}' < -1$; $\gamma_1 \cap \gamma_1{}' = \{Q_1\}$, where $Q_1 = (-\frac{1}{2}, y_1)$, $1 < y_1 < 2$; $\gamma_2 \cap \gamma_2{}' = \{Q_2\}$, where $Q_2 = (-\frac{1}{2}, y_2)$, $\quad 1 < y_1 < \frac{3}{2} < y_2 < 2$; $\delta_1 \cap \delta_1{}' = \{Q_1{}'\}$, where $Q_1{}' = (-\frac{1}{2}, y_1{}')$, $-1 < y_1{}' < 0$; $\delta_2 \cap \delta_2{}' = \{Q_2{}'\}$, where $Q_2{}' = (-\frac{1}{2}, y_2{}')$, $-1 < y_1{}' < -\frac{1}{2} < y_2{}' < 0$. If we denote $\Theta = \Theta(x) = \sqrt{|M(x)|}$, $H = H(y) = \sqrt{|K(y)|}$, we set

$$
\begin{aligned}
D_1(x) &= \int_0^x \Theta(t)dt, \quad D_2(x) = \int_1^x \Theta(t)dt,\\
D_3(x) &= \int_{-1}^x \Theta(t)dt, \quad D_4(x) = \int_{-2}^x \Theta(t)dt,\\
G_1(y) &= \int_0^y H(t)dt, \quad G_2(y) = \int_1^y H(t)dt,\\
G_3(y) &= \int_{-1}^y H(t)dt, \quad G_4(y) = \int_2^y H(t)dt.
\end{aligned}
$$

Domains G_1, G_2 differ in notation from functions $G_1(y), G_2(y)$. Thus, we have the following equations for the hyperbolic characteristics

$$
\Gamma_1 \cup \gamma_1 : D_1(x) = -G_2(y), \quad \Gamma_1{}' \cup \gamma_1{}' : G_2(y) = \begin{cases} D_1(x) \text{ on } & \Gamma_1{}' \\ D_3(x) \text{ on } & \gamma_1{}' \end{cases},
$$

$$
\Gamma_2 \cup \gamma_2 : \begin{cases} D_2(x) = G_2(y) \text{ on } \Gamma_2 \\ D_1(x) = G_4(y) \text{ on } \gamma_2 \end{cases}, \Gamma_2{}' \cup \gamma_2{}' : \begin{cases} D_2(x) = -G_1(y) \text{ on } \Gamma_2{}' \\ D_3(x) = -G_4(y) \text{ on } \gamma_2{}' \end{cases}
$$

$$
\Delta_1 \cup \delta_1 : \begin{cases} D_4(x) = -G_2(y) \text{ on } \Delta_1 \\ D_1(x) = -G_3(y) \text{ on } \delta_1 \end{cases}, \Delta_1{}' \cup \delta_1{}' : \begin{cases} D_4(x) = G_1(y) \text{ on } \Delta_1{}' \\ D_3(x) = G_3(y) \text{ on } \delta_1{}' \end{cases}
$$

$$
\Delta_2 \cup \delta_2 : \begin{cases} D_3(x) = G_2(y) \text{ on } \Delta_2 \\ D_1(x) = G_1(y) \text{ on } \delta_2 \end{cases}, \Delta_2{}' \cup \delta_2{}' : D_3(x) = |G_1(y)|.
$$

Note that:

1) The boundary ∂D is assumed to be a piecewise continuously differentiable arc. The elliptic arcs are "star-shaped" (counterclockwise).

2) We consider continuous solutions u of the quaterelliptic-quaterhyperbolic equation (1) with eight parabolic lines, which have the property that u_x, u_y are continuous in the closure $\bar{D} = D \cup \partial D$. These continuity conditions may be weakened at the following eight points $A_1, A_2, B_1, B_2, O_1, O_2, O_1{}', O_2{}'$, by considering u_x, u_y continuous on the boundary ∂D except at these points. By "quaterelliptic" and "quaterhyperbolic" we mean that equation (1) is elliptic in four different subdomains and hyperbolic in four other subdomains of the whole domain D. In fact, equation (1) is elliptic and hyperbolic in $G_1 \cup G_1{}' \cup G_1{}'' \cup G_1{}'''$ and $G_2 \cup G_2{}' \cup G_2{}'' \cup G_2{}'''$, respectively.

Definition 3. *A function $u = u(x, y)$ is a quasi-regular solution [7,8,10-16] of Problem (ET) if*

i) $u \in C^2(D) \cap C(\overline{D})$, $\overline{D} = D \cup \partial D$;

ii) the Green's theorem (of the integral calculus) is applicable to the integrals

$$\iint_D u_x Lu dxdy, \quad \iint_D u_y Lu dxdy;$$

iii) the boundary and region integrals, which arise, exist; and

iv) u satisfies the mixed type equation (1) in D and the following boundary condition on the exterior boundary $Ext(D)$:

$$u = \begin{cases} \varphi_1(s) \text{ on } \Gamma_0; & \varphi_2(s) \text{ on } \Gamma_0{}' \\ \varphi_3(s) \text{ on } \Gamma_0{}''; & \varphi_4(s) \text{ on } \Gamma_0{}''' \\ \psi_1(x) \text{ on } \Gamma_2; & \psi_2(x) \text{ on } \Gamma_2{}' \\ \psi_3(x) \text{ on } \gamma_2; & \psi_4(x) \text{ on } \gamma_2{}' \\ \psi_5(x) \text{ on } \Delta_1; & \psi_6(x) \text{ on } \Delta_1{}' \\ \psi_7(x) \text{ on } \delta_1; & \psi_8(x) \text{ on } \delta_1{}' \end{cases} \tag{2}$$

with continuous prescribed values.

The Exterior Tricomi Problem or Problem (ET): consists of finding a solution u of the quaterelliptic -quaterhyperbolic equation (1) with eight parabolic lines in D and which assumes continuous prescribed values (2).

Uniqueness Theorem 1. *Consider the quaterelliptic - quaterhyperbolic equation (1) with eight parabolic lines and the boundary condition (2). Assume the above mixed doubly connected domain D and the following conditions:*

(R_1) $r \leq 0$ on the interior boundary $Int(D)$,

$$(R_2)\quad \begin{cases} x\,dy-(y-1)dx\ge 0 \text{ on } \Gamma_0 \\ x\,dy-y\,dx\ge 0 \text{ on } {\Gamma_0}' \\ (x+1)dy-(y-1)dx\ge 0 \text{ on } {\Gamma_0}'' \\ (x+1)dy-y\,dx\ge 0 \text{ on } {\Gamma_0}''' \end{cases},$$

$$(R_3)\quad \begin{cases} 2r+xr_x+(y-1)r_y\le 0 \text{ in } G_1 \\ 2r+xr_x+yr_y\le 0 \text{ in } {G_1}' \\ 2r+(x+1)r_x+(y-1)r_y\le 0 \text{ in } {G_1}'' \\ 2r+(x+1)r_x+yr_y\le 0 \text{ in } {G_1}''' \\ r+xr_x\le 0 \text{ in } G_2 \\ r+(y-1)r_y\le 0 \text{ in } {G_2}' \\ r+(x+1)r_x\le 0 \text{ in } {G_2}'' \\ r+yr_y\le 0 \text{ in } {G_2}''' \end{cases},$$

(R_4) $K_i>0$, $M_i>0$ $(i=1,2)$, in $G_1\cup {G_1}'\cup {G_1}''\cup {G_1}'''$,

$$(R_5)\quad \begin{cases} K_1<0, & M_1>0 \text{ in } G_2\cup {G_2}'' \\ K_1>0, & M_1<0 \text{ in } {G_2}'\cup {G_2}''' \end{cases},$$

$$(R_6)\quad \begin{cases} \dot M_1\ge 0,\ \dot M_2\ge 0 & ; \quad {K_1}'\ge 0,\ {K_2}'\ge 0 \text{ in } G_1 \\ \dot M_1\ge 0,\ \dot M_2\ge 0 & ; \quad {K_1}'\le 0,\ {K_2}'\le 0 \text{ in } {G_1}' \\ \dot M_1\le 0,\ \dot M_2\le 0 & ; \quad {K_1}'\ge 0,\ {K_2}'\ge 0 \text{ in } {G_1}'' \\ \dot M_1\le 0,\ \dot M_2\le 0 & ; \quad {K_1}'\le 0,\ {K_2}'\le 0 \text{ in } {G_1}''' \end{cases},$$

(R_7) $K_2>0$, $M_2>0$ in D,

$$(R_8)\quad \begin{cases} \dot M_1\ge 0, & \dot M_2\le 0 \text{ in } G_2 \\ {K_1}'\ge 0, & {K_2}'\le 0 \text{ in } {G_2}' \\ \dot M_1\le 0, & \dot M_2\ge 0 \text{ in } {G_2}'' \\ {K_1}'\le 0, & {K_2}'\ge 0 \text{ in } {G_2}''' \end{cases}.$$

Let $()_x=\partial()/\partial x$, $()^{\cdot}=d()/dx$, $()_y=\partial()/\partial y$, $()'=d()/dy$, where $f=f(x,y)$ is continuous in D, $r=r(x,y)$ is once-continuously differentiable in D, $K_i=K_i(y)(i=1,2)$ are once-continuously differentiable for $y\in[-k_1,k_2]$ with $-k_1=inf\{y:(x,y)\in D\}$ and $k_2=sup\{y:(x,y)\in D\}$, and $M_i=M_i(x)(i=1,2)$ are once-continuously differentiable for $x\in[-m_1,m_2]$ with $-m_1=inf\{x:(x,y)\in D\}$ and $m_2=sup\{x:(x,y)\in D\}$. Then the Problem (ET) has at most one quasi-regular solution in D.

Proof. We apply the well-known *a-b-c* energy integral method with $a=0$, and use the above mixed type equation (1) as well as the boundary condition (2). First, we assume two quasi-regular solutions u_1,u_2 of the Problem (ET). Then we claim that $u=u_1-u_2=0$ holds in the domain D. In fact, we investigate

$$0=J=2<lu,Lu>_0=\iint_D 2luLu\,dx\,dy \tag{3}$$

where $lu = b(x)u_x + c(y)u_y$, and $Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0$ in D, with choices

$$b = b(x) \quad = \quad \begin{cases} x & \text{in} G_1 \cup {G_1}' \cup G_2 \\ x+1 & \text{in} {G_1}'' \cup {G_1}''' \cup {G_2}'' \\ 0 & \text{in} {G_2}' \cup {G_2}''' \end{cases} ,$$

$$c = c(y) \quad = \quad \begin{cases} y & \text{in } {G_1}' \cup {G_1}''' \cup {G_2}''' \\ y-1 & \text{in } G_1 \cup {G_1}'' \cup {G_2}' \\ 0 & \text{in } G_2 \cup {G_2}'' \end{cases} . \tag{4}$$

We consider the new differential identities

$$\begin{gathered} 2bK_1M_2u_xu_{xx} = \left(bK_1M_2{u_x}^2\right)_x - (bM_2)_xK_1{u_x}^2, \\ 2bK_2M_1u_xu_{yy} = \left(2bK_2M_1u_xu_y\right)_y - 2bM_1{K_2}'u_xu_y - \left(bK_2M_1{u_y}^2\right)_x + (bM_1)_xK_2{u_y}^2, \\ 2cK_1M_2u_yu_{xx} = \left(2cK_1M_2u_xu_y\right)_x - 2cK_1\dot{M}_2u_xu_y - \left(cK_1M_2{u_x}^2\right)_y + (cK_1)'M_2{u_x}^2, \\ 2cK_2M_1u_yu_{yy} = \left(cK_2M_1{u_y}^2\right)_y - (cK_2)'M_1{u_y}^2, \\ 2bruu_x = (bru^2)_x - (br)_xu^2, \quad 2cruu_y = (cru^2)_y - (cr)_yu^2, \end{gathered}$$

as well as t_1 is the coefficient of u_x in Lu, or

$$t_1 = t_1(x,y) = K_1(y)\dot{M}_2(x), \tag{5}$$

and t_2 is the coefficient of u_y in Lu, or

$$t_2 = t_2(x,y) = {K_2}'(y)M_1(x). \tag{6}$$

Employing these identities and the classical Green's theorem of the integral calculus we obtain from (1),(3),(5), and (6) that

$$\begin{aligned} 0 \quad = J = \quad & \iint\limits_D 2(bu_x + cu_y)\left[K_1(M_2u_x)_x + M_1(K_2u_y)_y + ru\right]dxdy \\ = \quad & \iint\limits_D 2(bu_x + cu_y)\left[K_1M_2u_{xx} + K_2M_1u_{yy} + t_1u_x + t_2u_y + ru\right]dxdy, \\ = \quad & I_D + I_{\partial D}, \end{aligned} \tag{7}$$

where $I_D = \iint\limits_D Q(u_x,u_y)dxdy = \iint\limits_D (A{u_x}^2 + B{u_y}^2 + \Gamma u^2 + 2\Delta u_xu_y)dxdy$;

$$I_{\partial D} = \int\limits_{\partial D} \tilde{Q}(u_x,u_y)ds = \int\limits_{\partial D} (\tilde{A}{u_x}^2 + \tilde{B}{u_y}^2 + \tilde{\Gamma}u^2 + 2\tilde{\Delta}u_xu_y)ds,$$

with $A=-K_1(bM_2)_x+(cK_1)'M_2+2bt_1$, $B=K_2(bM_1)_x-(cK_2)'M_1+2ct_2$,

$$\Gamma=-\left[(br)_x+(cr)_y\right],$$

$$\Delta=-\left[bK_2{}'M_1+cK_1\dot{M}_2-bt_2-ct_1\right]=-\left[b(K_2{}'M_1-t_2)+c(K_1\dot{M}_2-t_1)\right]=0$$

(because of (5) and (6)) in D, and $\tilde{A}=(bv_1-cv_2)K_1M_2$, $\tilde{B}=(-bv_1+cv_2)K_2M_1$, $\tilde{\Gamma}=(bv_1+cv_2)r$, $\tilde{\Delta}=bK_2M_1v_2+cK_1M_2v_1$ on ∂D, where $v=(v_1,v_2)=(dy/ds,-dx/ds)$ is the outer unit normal vector on the boundary ∂D of the domain D such that $ds^2=dx^2+dy^2>0$, $|v|=1$;
$\iint\limits_D ()_x dxdy=\int\limits_{\partial D}()v_1ds$, $\iint\limits_D ()_y dxdy=\int\limits_{\partial D}()v_2ds$ are the Green's integral formulas. From the above conditions, we obtain

$$0\leq A \;=\; \begin{cases} xK_1\dot{M}_2+(y-1)K_1{}'M_2 & \text{in } G_1\\ xK_1\dot{M}_2+yK_1{}'M_2 & \text{in } G_1{}'\\ (x+1)K_1\dot{M}_2+(y-1)K_1{}'M_2 & \text{in } G_1{}''\\ (x+1)K_1\dot{M}_2+yK_1{}'M_2 & \text{in } G_1{}''' \end{cases};$$

$$0\leq A \;=\; \begin{cases} -K_1(M_2-x\dot{M}_2) & \text{in } G_2\\ M_2\big(K_1+(y-1)K_1{}'\big) & \text{in } G_2{}'\\ -K_1\big(M_2-(x+1)\dot{M}_2\big) & \text{in } G_2{}''\\ M_2\big(K_1+yK_1{}'\big) & \text{in } G_2{}''' \end{cases}.$$

Similarly we get

$$0\leq B \;=\; \begin{cases} xK_2\dot{M}_1+(y-1)K_2{}'M_1 & \text{in } G_1\\ xK_2\dot{M}_1+yK_2{}'M_1 & \text{in } G_1{}'\\ (x+1)K_2\dot{M}_1+(y-1)K_2{}'M_1 & \text{in } G_1{}''\\ (x+1)K_2\dot{M}_1+yK_2{}'M_1 & \text{in } G_1{}''' \end{cases};$$

$$0\leq B \;=\; \begin{cases} K_2(M_1+x\dot{M}_1) & \text{in } G_2\\ -M_1\big(K_2-(y-1)K_2{}'\big) & \text{in } G_2{}'\\ K_2\big(M_1+(x+1)\dot{M}_1\big) & \text{in } G_2{}''\\ -M_1\big(K_2-yK_2{}'\big) & \text{in } G_2{}''' \end{cases}.$$

Also

$$0\leq \Gamma \;=\; -\begin{cases} 2r+xr_x+(y-1)r_y & \text{in } G_1\\ 2r+xr_x+yr_y & \text{in } G_1{}'\\ 2r+(x+1)r_x+(y-1)r_y & \text{in } G_1{}''\\ 2r+(x+1)r_x+yr_y & \text{in } G_1{}''' \end{cases};$$

$$0\leq \Gamma \;=\; -\begin{cases} r+xr_x & \text{in } G_2\\ r+(y-1)r_y & \text{in } G_2{}'\\ r+(x+1)r_x & \text{in } G_2{}''\\ r+yr_y & \text{in } G_2{}''' \end{cases}.$$

Therefore,

$$\begin{aligned} I_D &= \int\int_D = \int\int_{G_1\cup G_1{}'\cup G_1{}''\cup G_1{}'''} + \int\int_{G_2\cup G_2{}'\cup G_2{}''\cup G_2{}'''} \\ &= (\iint\limits_{G_1} + \iint\limits_{G_1{}'} + \iint\limits_{G_1{}''} + \iint\limits_{G_1{}'''}) + (\iint\limits_{G_2} + \iint\limits_{G_2{}'} + \iint\limits_{G_2{}''} + \iint\limits_{G_2{}'''}) \geq 0. \end{aligned}$$

We claim that $I_{\partial D} = I_{Ext(D)\bigcup Int(D)} = I_{Ext(D)} + I_{Int(D)} \geq 0$, where

$$I_{Ext(D)} = (\int\limits_{\Gamma_0} + \int\limits_{\Gamma_0{}'} + \int\limits_{\Gamma_0{}''} + \int\limits_{\Gamma_0{}'''}) + (\int\limits_{\Gamma_2} + \int\limits_{\Gamma_2{}'}) + (\int\limits_{\gamma_2} + \int\limits_{\gamma_2{}'}) + (\int\limits_{\Delta_1} + \int\limits_{\Delta_1{}'}) + (\int\limits_{\delta_1} + \int\limits_{\delta_1{}'}) \geq 0,$$

and

$$I_{Int(D)} = (\int\limits_{\Gamma_1} + \int\limits_{\Gamma_1{}'}) + (\int\limits_{\gamma_1} + \int\limits_{\gamma_1{}'}) + (\int\limits_{\Delta_2} + \int\limits_{\Delta_2{}'}) + (\int\limits_{\delta_2} + \int\limits_{\delta_2{}'}) \geq 0.$$

In fact, on Γ_0 with $b = x, c = y + 1$:

$$\begin{aligned} \tilde{A} = \big(xv_1 - (y+1)v_2\big)K_1M_2 \geq 0 \quad &, \quad \tilde{B} = \big(-xv_1 + (y+1)v_2\big)K_2M_1 \geq 0; \\ \tilde{\Gamma} = \big(xv_1 + (y+1)v_2\big)r \geq 0 \quad &; \quad \tilde{\Delta} = xK_2M_1v_2 + (y+1)K_1M_2v_1. \end{aligned}$$

$$\begin{aligned} I_{\Gamma_0} &= \int_{\Gamma_0} \tilde{Q}(u_x, u_y)ds + \int_{\Gamma_0} \tilde{\Gamma}u^2 ds \\ &= \int_{\Gamma_0} N^2\big(xv_1 + (y+1)v_2\big)H ds + \int_{\Gamma_0} \big(xv_1 + (y+1)v_2\big)ru^2 ds, \\ &= \int_{\Gamma_0} N^2\big(xdy - (y+1)dx\big)H \geq 0 \end{aligned}$$

where $u|_{\Gamma_0} = 0$ and

$$0 = du = u_x dx + u_y dy = N\big(v_1 dx + v_2 dy\big)\ ;\ u_x = Nv_1,\ u_y = Nv_2,$$

with a normalizing factor N, and $H|_{\Gamma_0} = K_1M_2{v_1}^2 + K_2M_1{v_2}^2 > 0$, as well as Γ_0 is a “star-liked” arc, such that $xdy - (y+1)dx|_{\Gamma_0} \geq 0$ and

$$\tilde{Q} = \tilde{Q}(u_x, u_y) = [\tilde{A}{u_x}^2 + \tilde{B}{u_y}^2 + 2\tilde{\Delta}u_x u_y]|_{\Gamma_0} = N^2\big(xv_1 + (y+1)v_2\big)H \geq 0.$$

Similarly, we obtain

$$0 \quad \leq \quad I_{\Gamma_0\cup\Gamma_0{}'\cup\Gamma_0{}''\cup\Gamma_0{}'''} = \int_{\Gamma_0} N^2\big(xdy - (y+1)dx\big)H + \int_{\Gamma_0{}'} N^2\big(xdy - ydx\big)H$$

$$+ \int_{\Gamma_0''} N^2\big((x+1)dy-(y-1)dx\big)H + \int_{\Gamma_0'''} N^2\big((x+1)dy-ydx\big)H.$$

Also on $\Gamma_2 \cup \Gamma_2{}'$ with $b = x, c = 0$:

$$\begin{aligned} I_{\Gamma_2\cup\Gamma_2{}'} &= \int_{\Gamma_2\cup\Gamma_2{}'} \tilde{Q}(u_x,u_y)ds + \int_{\Gamma_2\cup\Gamma_2{}'} \tilde{\Gamma}u^2 ds \\ &= \int_{\Gamma_2\cup\Gamma_2{}'} N^2(xv_1)Hds + \int_{\Gamma_2\cup\Gamma_2{}'} (xv_1)ru^2 ds = 0, \end{aligned}$$

where $u|_{\Gamma_2\cup\Gamma_2{}'} = 0$ and $H|_{\Gamma_2\cup\Gamma_2{}'} = K_1M_2{v_1}^2 + K_2M_1{v_2}^2 = 0$, because both Γ_2, $\Gamma_2{}'$ are characteristics.
Similarly we get

$$I_{\gamma_2\cup\gamma_2{}'} = \int_{\gamma_2\cup\gamma_2{}'} N^2\big((y-1)v_2\big)Hds + \int_{\gamma_2\cup\gamma_2{}'} \big((y-1)v_2\big)ru^2 ds = 0,$$

$$I_{\Delta_1\cup\Delta_1{}'} = \int_{\Delta_1\cup\Delta_1{}'} N^2\big((x+1)v_1\big)Hds + \int_{\Delta_1\cup\Delta_1{}'} \big((x+1)v_1\big)ru^2 ds = 0,$$

$$I_{\delta_1\cup\delta_1{}'} = \int_{\delta_1\cup\delta_1{}'} N^2\big(yv_2\big)Hds + \int_{\delta_1\cup\delta_1{}'} \big(yv_2\big)ru^2 ds = 0.$$

Also on $\Gamma_1 \cup \Gamma_1{}'$ with $b = x, c = 0$:

$$\tilde{A} = (xv_1)K_1M_2 \geq 0,\ \tilde{B} = (-xv_1)K_2M_1 \geq 0;$$

$$\tilde{\Gamma} = (xv_1)r \geq 0,\ \tilde{\Delta} = xK_2M_1v_2.$$

$$\begin{aligned} I_{\Gamma_1\cup\Gamma_1{}'} &= \int_{\Gamma_1\cup\Gamma_1{}'} \tilde{Q}(u_x,u_y)ds + \int_{\Gamma_1\cup\Gamma_1{}'} \tilde{\Gamma}u^2 ds \\ &= \int_{\Gamma_1\cup\Gamma_1{}'} \{x\big[(K_1M_2v_1){u_x}^2 + (-K_2M_1v_1){u_y}^2 + 2(K_2M_1v_2)u_xu_y\big]\}, \\ &\quad + \int_{\Gamma_1\cup\Gamma_1{}'} (xv_1)ru^2 ds > 0 \end{aligned}$$

because $r|_{\Gamma_1\cup\Gamma_1{}'} \leq 0$, $v_1|_{\Gamma_1\cup\Gamma_1{}'} < 0$, and $H = 0$ since both Γ_1, $\Gamma_1{}'$ are characteristics as well as $\tilde{A}\tilde{B} - (\tilde{\Delta})^2 = -x^2K_2M_1H = 0$.
Similarly, we get

$$I_{\gamma_1\cup\gamma_1{}'} = \int_{\gamma_1\cup\gamma_1{}'} \tilde{Q}(u_x,u_y)ds + \int_{\gamma_1\cup\gamma_1{}'} \tilde{\Gamma}u^2 ds$$

$$= \int_{\gamma_1 \cup \gamma_1{}'} \{(y-1)\left[(-K_1 M_2 \nu_2) u_x{}^2 + (K_2 M_1 \nu_2) u_y{}^2 + 2(K_1 M_2 \nu_1) u_x u_y\right]\}, + \int_{\gamma_1 \cup \gamma_1{}'} \big((y-1)\nu_2\big) r u^2 ds > 0$$

$$\begin{aligned} I_{\Delta_2 \cup \Delta_2{}'} &= \int_{\Delta_2 \cup \Delta_2{}'} \tilde{Q}(u_x, u_y) ds + \int_{\Delta_2 \cup \Delta_2{}'} \tilde{\Gamma} u^2 ds \\ &= \int_{\Delta_2 \cup \Delta_2{}'} \{(x+1)\left[(K_1 M_2 \nu_1) u_x{}^2 + (-K_2 M_1 \nu_1) u_y{}^2 + 2(K_2 M_1 \nu_2) u_x u_y\right]\}, \\ &\quad + \int_{\Delta_2 \cup \Delta_2{}'} \big((x+1)\nu_1\big) r u^2 ds > 0 \end{aligned}$$

$$\begin{aligned} I_{\delta_2 \cup \delta_2{}'} &= \int_{\delta_2 \cup \delta_2{}'} \tilde{Q}(u_x, u_y) ds + \int_{\delta_2 \cup \delta_2{}'} \tilde{\Gamma} u^2 ds \\ &= \int_{\delta_2 \cup \delta_2{}'} \{y\left[(-K_1 M_2 \nu_2) u_x{}^2 + (K_2 M_1 \nu_2) u_y{}^2 + 2(K_1 M_2 \nu_1) u_x u_y\right]\} \\ &\quad + \int_{\delta_2 \cup \delta_2{}'} \big(y \nu_2\big) r u^2 ds > 0. \end{aligned}$$

From (7) we get $0 = I_D + I_{\partial D} \geq 0$ with $I_D \geq 0$ and $I_{\partial D} \geq 0$. These relations yield

$$I_D = I_{\partial D} = 0. \tag{8}$$

It is clear that $\iint_{\bar{G}_1} \Gamma u^2 dxdy = -\iint_{\bar{G}_1} \big(2r + x r_x + (y-1) r_y\big) u^2 dxdy = 0.$

Therefore, we get $u(x,y) = 0$ everywhere in $\bar{G}_1$. Alternatively

$$\iint_{\bar{G}_1} \left[\big(x K_1 \dot{M}_2 + (y-1) K_1{}' M_2\big) u_x{}^2 + \big(x K_2 \dot{M}_1 + (y-1) K_2{}' M_1\big) u_y{}^2\right] dxdy = 0,$$

yielding $u_x \equiv 0$; $u_y \equiv 0$ in $\bar{G}_1$. Thus, in $\bar{G}_1 : u(x,y) \equiv 0$.
Similarly

$$u(x,y) \equiv 0 \text{ in } \bar{G}_1 \cup \bar{G}_1' \cup \bar{G}_1'' \cup \bar{G}_1'''. \tag{9}$$

It is clear that $\iint_{\bar{G}_2} \Gamma u^2 dxdy = -\iint_{\bar{G}_2} (r + x r_x) u^2 dxdy = 0.$

Therefore, we get $u(x,y) \equiv 0$ everywhere in $\bar{G}_2$. Alternatively

$$\iint_{\bar{G}_2} \left[\big(x K_1 \dot{M}_2\big) u_x{}^2 + \big(x K_2 \dot{M}_1\big) u_y{}^2\right] dxdy = 0,$$

yielding $u_x \equiv 0$; $u_y \equiv 0$ in $\bar{G}_2$. Thus in $\bar{G}_2 : u(x,y) \equiv 0$.
Similarly

$$u(x,y) \equiv 0 \text{ in } \bar{G}_2 \cup \bar{G}_2' \cup \bar{G}_2'' \cup \bar{G}_2'''.$$

From (8) we get

$$I_{\partial D} = I_{Ext(D)} + I_{Int(D)} = I_{\Gamma_0 \cup \Gamma_0{}' \cup \Gamma_0{}'' \cup \Gamma_0{}'''}$$
$$+ \left(I_{\Gamma_1 \cup \Gamma_1{}'} + I_{\gamma_1 \cup \gamma_1{}'} + I_{\Delta_2 \cup \Delta_2{}'} + I_{\delta_2 \cup \delta_2{}'} \right) = 0.$$

Thus,

$$I_{\Gamma_1 \cup \Gamma_1{}'} = 0. \tag{10}$$

Also alternatively, by a well-known theorem on hyperbolic equations if $u|_{\Gamma_2 \cup \Gamma_2{}'} = 0$ (from the boundary condition) and $I_{\Gamma_1 \cup \Gamma_1{}'} = 0$ (from (10)), then $u(x,y) \equiv 0$ everywhere in $\bar{G}_2$.

Pertinent to the above, there is the following general uniqueness approach:
First, from the maximum principle, if $u|_{\overline{G}_2 \bigcup \overline{G}_2' \bigcup \overline{G}_2'' \bigcup \overline{G}_2'''} = 0$, it follows that $u|_{\overline{G}_1 \bigcup \overline{G}_1' \bigcup \overline{G}_1'' \bigcup \overline{G}_1'''} = 0$. Second, from the uniqueness of the solution of the Cauchy problem, if $u|_{\overline{G}_1 \bigcup \overline{G}_1' \bigcup \overline{G}_1'' \bigcup \overline{G}_1'''} = 0$, it follows $u|_{\overline{G}_2 \bigcup \overline{G}_2' \bigcup \overline{G}_2'' \bigcup \overline{G}_2'''} = 0$. Thus, $u(x,y) \equiv 0$ everywhere in D, completing the proof of the uniqueness theorem.
Note that the case: $r = r(x,y) = 0$ in D and $K_1{}'(0) = \dot{M}_i(0) = 0$ $(i = 1,2)$ yield also uniqueness results for the problem (ET).

3. The Exterior Frankl Problem

Consider the quaterelliptic-quaterhyperbolic equation (1) with eight parabolic lines of degeneracy in a bounded doubly connected mixed domain $\tilde{D}$ with a piecewise smooth boundary

$$\partial \tilde{D} = Ext(\tilde{D}) \cup Int(\tilde{D}) = \left[ExtEl(\tilde{D}) \cup ExtHn(\tilde{D}) \right] \cup Int(\tilde{D}),$$

where $\tilde{D}$ is a part of D, and $Int(\tilde{D}) = Int(D)$, as well as

$$ExtEl(\tilde{D}) = \Gamma_0 \cup \Gamma_0{}' \cup \Gamma_0{}'' \cup \Gamma_0{}'''$$

is the elliptic exterior boundary of $\tilde{D}$ and

$$ExtHn(\tilde{D}) = (\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2') \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}_2') \cup (\tilde{\Delta}_1 \cup \tilde{\Delta}_1') \cup (\tilde{\delta}_1 \cup \tilde{\delta}_1')$$

is the non-characteristic hyperbolic exterior boundary of $\tilde{D}$, such that:

$$Ext(\tilde{D}) = ExtEl(\tilde{D}) \cup ExtHn(\tilde{D}) = \left(\Gamma_0 \cup \Gamma_0{}' \cup \Gamma_0{}'' \cup \Gamma_0{}''' \right)$$
$$\cup \left[(\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2') \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}_2') \cup (\tilde{\Delta}_1 \cup \tilde{\Delta}_1') \cup (\tilde{\delta}_1 \cup \tilde{\delta}_1') \right]$$

is the exterior boundary of $\tilde{D}$, with the following non-characteristics:

$$\tilde{\Gamma}_2,\ \tilde{\Gamma}_2',\ \tilde{\gamma}_2,\ \tilde{\gamma}_2',\ \tilde{\Delta}_1,\ \tilde{\Delta}_1',\ \tilde{\delta}_1,\ \tilde{\delta}_1' :$$

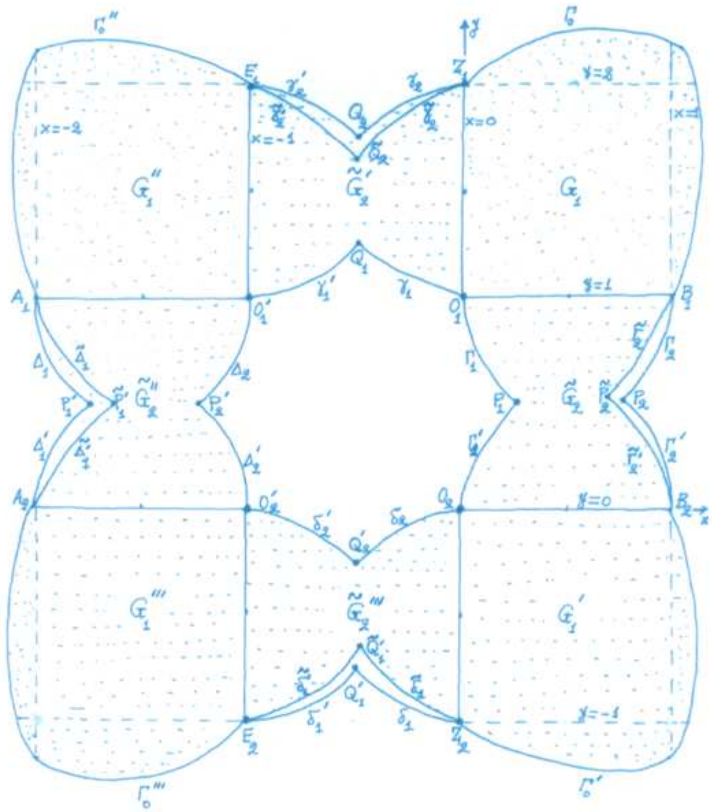

Figure 2

$$
\begin{aligned}
\tilde{\Gamma}_2 &: \quad \sqrt{M(x)}dx \geq \sqrt{-K(y)}dy\,; \\
\tilde{\Delta}_1' &: \quad 0 \leq \sqrt{M(x)}dx \leq \sqrt{-K(y)}dy\,; \\
\tilde{\Gamma}_2' &: \quad \sqrt{M(x)}dx \leq -\sqrt{-K(y)}dy \leq 0\,; \\
\tilde{\Delta}_1 &: \quad \sqrt{M(x)}dx \geq -\sqrt{-K(-y)}dy\,; \\
\tilde{\gamma}_2 &: \quad 0 \geq \sqrt{-M(x)}dx \geq \sqrt{K(y)}dy\,; \\
\tilde{\delta}_1' &: \quad 0 \leq \sqrt{-M(x)}dx \leq \sqrt{K(y)}dy\,; \\
\tilde{\gamma}_2' &: \quad 0 \geq \sqrt{-M(x)}dx \geq -\sqrt{K(y)}dy\,; \\
\tilde{\delta}_1 &: \quad 0 \leq \sqrt{-M(x)}dx \leq -\sqrt{K(y)}dy\,,
\end{aligned}
$$

or

$$
\tilde{\Gamma}_2 \cup \tilde{\Delta}_1' : 0 \leq \frac{dy}{dx} \leq \frac{\sqrt{M(x)}}{\sqrt{-K(y)}} \quad ; \quad \tilde{\Gamma}_2' \cup \tilde{\Delta}_1 : 0 \geq \frac{dy}{dx} \geq -\frac{\sqrt{M(x)}}{\sqrt{-K(y)}}\,;
$$

$$\tilde{\gamma}_2 \cup \tilde{\delta}_1' : \frac{dy}{dx} \geq \frac{\sqrt{-M(x)}}{\sqrt{K(y)}} \quad ; \quad \tilde{\gamma}_2' \cup \tilde{\delta}_1 : \frac{dy}{dx} \leq -\frac{\sqrt{-M(x)}}{\sqrt{K(y)}},$$

satisfying the non-characteristic relation

$$H = K_1 M_2 {v_1}^2 + K_2 M_1 {v_2}^2 \geq 0, \text{ or } K(y)(dy)^2 + M(x)(dx)^2 \geq 0,$$

and intersecting characteristics Γ_1, Γ_1', γ_1, γ_1', Δ_2, Δ_2', δ_2, δ_2', only once.
Note that

$$(\tilde{\Gamma}_2 \cup \tilde{\Delta}_1) \cup (\tilde{\Gamma}_2' \cup \tilde{\Delta}_1') : (\sqrt{M(x)}dx - \sqrt{-K(y)}dy)(\sqrt{M(x)}dx + \sqrt{-K(y)}dy) = H;$$
$$(\tilde{\gamma}_2 \cup \tilde{\delta}_1) \cup (\tilde{\gamma}_2' \cup \tilde{\delta}_1') : (\sqrt{-M(x)}dx - \sqrt{K(y)}dy)(\sqrt{-M(x)}dx + \sqrt{K(y)}dy) = -H.$$

Let us consider the intersection points of the hyperbolic characteristics:
$\tilde{\Gamma}_2 \cap \tilde{\Gamma}_2' = \{\tilde{P}_2\}$, where $\tilde{P}_2 = (\tilde{x}_2, \frac{1}{2})$, $\quad 0 < x_1 < \frac{1}{2} < \tilde{x}_2 < x_2 < 1$; $\tilde{\Delta}_1 \cap \tilde{\Delta}_1' = \{\tilde{P}_1'\}$, where $\tilde{P}_1' = (\tilde{x}_1', \frac{1}{2})$, $-2 < x_1 < \tilde{x}_1' < -1$; $\tilde{\gamma}_2 \cap \tilde{\gamma}_2' = \{\tilde{Q}_2\}$, where $\tilde{Q}_2 = (-\frac{1}{2}, \tilde{y}_2)$, $1 < y_1 < \frac{3}{2} < \tilde{y}_2 < y_2 < 2$; $\tilde{\delta}_1 \cap \tilde{\delta}_1' = \{\tilde{Q}_1'\}$, where $\tilde{Q}_1' = (-\frac{1}{2}, \tilde{y}_1')$, $-1 < {y_1}' < \tilde{y}_1' < 0$.
Let the right hyperbolic domain $\tilde{G}_2 \subset G_2 = \{(x,y) \in D : 0 < x < 1, 0 < y < 1\}$ with boundary $\partial \tilde{G}_2 = (O_1B_1) \cup (O_2B_2) \cup (\Gamma_1 \cup {\Gamma_1}') \cup (\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2')$.
Let the upper hyperbolic domain $\tilde{G}_2' \subset {G_2}' = \{(x,y) \in D : -1 < x < 0, 1 < y < 2\}$ with boundary $\partial \tilde{G}_2' = (O_1Z_1) \cup ({O_1}'E_1) \cup (\gamma_1 \cup {\gamma_1}') \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}_2')$.
Let the left hyperbolic domain $\tilde{G}_2'' \subset {G_2}'' = \{(x,y) \in D : -2 < x < -1, 0 < y < 1\}$ with boundary $\partial \tilde{G}_2'' = ({O_1}'A_1) \cup ({O_2}'A_2) \cup (\tilde{\Delta}_1 \cup \tilde{\Delta}_1') \cup (\Delta_2 \cup {\Delta_2}')$.
Let the lower hyperbolic domain $\tilde{G}_2''' \subset {G_2}''' = \{(x,y) \in D : -1 < x < 0, -1 < y < 0\}$ with boundary $\partial \tilde{G}_2''' = (O_2Z_2) \cup ({O_2}'E_2) \cup (\tilde{\delta}_1 \cup \tilde{\delta}_1') \cup (\delta_2 \cup {\delta_2}')$.
Assume boundary conditions on the above exterior boundary $Ext(\tilde{D})$:

$$u = \begin{cases} \varphi_1(s) \text{ on } \quad \Gamma_0; & \varphi_2(s) \text{ on } {\Gamma_0}' \\ \varphi_3(s) \text{ on } \quad {\Gamma_0}''; & \varphi_4(s) \text{ on } {\Gamma_0}''' \\ \tilde{\psi}_1(x) \text{ on } \quad \tilde{\Gamma}_2; & \tilde{\psi}_2(x) \text{ on } \tilde{\Gamma}_2' \\ \tilde{\psi}_3(x) \text{ on } \quad \tilde{\gamma}_2; & \tilde{\psi}_4(x) \text{ on } \tilde{\gamma}_2' \\ \tilde{\psi}_5(x) \text{ on } \quad \tilde{\Delta}_1; & \tilde{\psi}_6(x) \text{ on } \tilde{\Delta}_1' \\ \tilde{\psi}_7(x) \text{ on } \quad \tilde{\delta}_1; & \tilde{\psi}_8(x) \text{ on } \tilde{\delta}_1' \end{cases} \tag{11}$$

with continuous prescribed values.

The Exterior Frankl Problem or Problem (EF): consists of finding a solution u of the quaterelliptic -quaterhyperbolic equation (1) with eight parabolic lines in $\tilde{D}(\subset D)$ and which assumes continuous prescribed values (11).

Uniqueness Theorem 2. *Consider the quaterelliptic - quaterhyperbolic equation (1) with eight parabolic lines and the boundary condition (11). Assume the above mixed doubly connected domain $\tilde{D}(\subset D)$ and the following conditions:*

(R_1) $r \le 0$ *on the interior boundary* $Int(\tilde{D})(= Int(D))$,

$$(R_2)\quad \begin{cases} x\,dy - (y-1)dx \ge 0 \text{ on } \Gamma_0 \\ x\,dy - y\,dx \ge 0 \text{ on } {\Gamma_0}' \\ (x+1)dy - (y-1)dx \ge 0 \text{ on } {\Gamma_0}'' \\ (x+1)dy - y\,dx \ge 0 \text{ on } {\Gamma_0}''' \end{cases},$$

$$(R_3)\quad \begin{cases} 2r + x r_x + (y-1) r_y \le 0 \text{ in } G_1 \\ 2r + x r_x + y r_y \le 0 \text{ in } {G_1}' \\ 2r + (x+1) r_x + (y-1) r_y \le 0 \text{ in } {G_1}'' \\ 2r + (x+1) r_x + y r_y \le 0 \text{ in } {G_1}''' \\ r + x r_x \le 0 \text{ in } \tilde{G}_2 \\ r + (y-1) r_y \le 0 \text{ in } \tilde{G}_2' \\ r + (x+1) r_x \le 0 \text{ in } \tilde{G}_2'' \\ r + y r_y \le 0 \text{ in } \tilde{G}_2''' \end{cases},$$

(R_4) $K_i > 0,\ M_i > 0 \quad (i = 1,2)$, *in* $G_1 \cup {G_1}' \cup {G_1}'' \cup {G_1}'''$,

$$(R_5)\quad \begin{cases} K_1 < 0, & M_1 > 0 \text{ in } \tilde{G}_2 \cup \tilde{G}_2'' \\ K_1 > 0, & M_1 < 0 \text{ in } \tilde{G}_2' \cup \tilde{G}_2''' \end{cases},$$

$$(R_6)\quad \begin{cases} \dot{M}_1 \ge 0,\ \dot{M}_2 \ge 0 \quad ; \quad {K_1}' \ge 0,\ {K_2}' \ge 0 \text{ in } G_1 \\ \dot{M}_1 \ge 0,\ \dot{M}_2 \ge 0 \quad ; \quad {K_1}' \le 0,\ {K_2}' \le 0 \text{ in } {G_1}' \\ \dot{M}_1 \le 0,\ \dot{M}_2 \le 0 \quad ; \quad {K_1}' \ge 0,\ {K_2}' \ge 0 \text{ in } {G_1}'' \\ \dot{M}_1 \le 0,\ \dot{M}_2 \le 0 \quad ; \quad {K_1}' \le 0,\ {K_2}' \le 0 \text{ in } {G_1}''' \end{cases},$$

(R_7) $K_2 > 0,\ M_2 > 0$ *in* $\tilde{D}(\subset D)$,

$$(R_8)\quad \begin{cases} \dot{M}_1 \ge 0, & \dot{M}_2 \le 0 \text{ in } \tilde{G}_2 \\ {K_1}' \ge 0, & {K_2}' \le 0 \text{ in } \tilde{G}_2' \\ \dot{M}_1 \le 0, & \dot{M}_2 \ge 0 \text{ in } \tilde{G}_2'' \\ {K_1}' \le 0, & {K_2}' \ge 0 \text{ in } \tilde{G}_2''' \end{cases}.$$

Let $()_x = \partial()/\partial x$, $\quad ()^{\cdot} = d()/dx$, $\quad ()_y = \partial()/\partial y$, $\quad ()' = d()/dy$, *where* $f = f(x,y)$ *is continuous in* $\tilde{D}(\subset D)$, $\quad r = r(x,y)$ *is once-continuously differentiable in* $\tilde{D}(\subset D)$, $K_i = K_i(y)\,(i = 1,2)$ *are once-continuously differentiable for* $y \in [-k_1, k_2]$ *with* $-k_1 = inf\{y : (x,y) \in \tilde{D}(\subset D)\}$ *and* $k_2 = sup\{y : (x,y) \in \tilde{D}(\subset D)\}$, *and* $M_i = M_i(x)$ $(i = 1,2)$ *are once-continuously differentiable for* $x \in [-m_1, m_2]$ *with* $-m_1 = inf\{x : (x,y) \in \tilde{D}(\subset D)\}$ *and* $m_2 = sup\{x : (x,y) \in \tilde{D}(\subset D)\}$. *Then the Problem (EF) has at most one quasi-regular solution in* $\tilde{D}(\subset D)$.

Proof. We apply the well-known energy integral method, and use the above mixed type equation (1) as well as the boundary condition (11). First, we assume two quasi-regular solutions u_1, u_2 of the Problem (EF). Then we claim that $u = u_1 - u_2 = 0$ holds in the domain

$\tilde{D}(\subset D)$. In fact, we investigate

$$0 = \tilde{J} = 2 < \tilde{l}u, Lu >_0 = \iint_{\tilde{D}} 2\tilde{l}uLudxdy$$

where $\tilde{l}u = \tilde{b}(x)u_x + \tilde{c}(y)u_y$, and $Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0$ in $\tilde{D}(\subset D)$ with choices

$$\tilde{b} = \tilde{b}(x) = \begin{cases} x & \text{in } G_1 \cup {G_1}' \cup \tilde{G}_2 \\ x+1 & \text{in } {G_1}'' \cup {G_1}''' \cup \tilde{G}_2'' \\ 0 & \text{in } \tilde{G}_2' \cup \tilde{G}_2''' \end{cases},$$

and

$$\tilde{c} = \tilde{c}(y) = \begin{cases} y & \text{in } {G_1}' \cup {G_1}''' \cup \tilde{G}_2''' \\ y-1 & \text{in } G_1 \cup {G_1}'' \cup \tilde{G}_2' \\ 0 & \text{in } \tilde{G}_2 \cup \tilde{G}_2'' \end{cases}.$$

The rest of the proof is similar to the proof of the uniqueness theorem 1 (for the exterior Tricomi problem), except clearly proving in additional that the following condition holds on the non-characteristic hyperbolic exterior boundary
$ExtHn(\tilde{D}) = (\tilde{\Gamma}_2 \cup \tilde{\Gamma}_2') \cup (\tilde{\gamma}_2 \cup \tilde{\gamma}_2') \cup (\tilde{\Delta}_1 \cup \tilde{\Delta}_1') \cup (\tilde{\delta}_1 \cup \tilde{\delta}_1')$:

$$0 < \tilde{b}\nu_1 + \tilde{c}\nu_2 = \begin{cases} x\nu_1 & \text{on } \tilde{\Gamma}_2 \cup \tilde{\Gamma}_2' \\ (y-1)\nu_2 & \text{on } \tilde{\gamma}_2 \cup \tilde{\gamma}_2' \\ (x+1)\nu_1 & \text{on } \tilde{\Delta}_1 \cup \tilde{\Delta}_1' \\ y\nu_2 & \text{on } \tilde{\delta}_1 \cup \tilde{\delta}_1' \end{cases}.$$

Uniqueness Theorem 3. *Consider the quaterelliptic - quaterhyperbolic equation (1) with eight parabolic lines and the boundary condition (11). Assume the above mixed doubly connected domain $\tilde{D}(\subset D)$ and the following conditions:*

(R_1) $r \leq 0$ on the interior boundary $Int(\tilde{D})(= Int(D))$,

$$(R_2) \begin{cases} xdy - (y-1)dx \geq 0 \text{ on } \Gamma_0 \\ xdy - ydx \geq 0 \text{ on } {\Gamma_0}' \\ (x+1)dy - (y-1)dx \geq 0 \text{ on } {\Gamma_0}'' \\ (x+1)dy - ydx \geq 0 \text{ on } {\Gamma_0}''' \end{cases},$$

$$(R_3) \begin{cases} 2r + xr_x + (y-1)r_y \leq 0 \text{ in } G_1 \\ 2r + xr_x + yr_y \leq 0 \text{ in } {G_1}' \\ 2r + (x+1)r_x + (y-1)r_y \leq 0 \text{ in } {G_1}'' \\ 2r + (x+1)r_x + yr_y \leq 0 \text{ in } {G_1}''' \\ r + xr_x \leq 0 \text{ in } \tilde{G}_2 \\ r + (y-1)r_y \leq 0 \text{ in } \tilde{G}_2' \\ r + (x+1)r_x \leq 0 \text{ in } \tilde{G}_2'' \\ r + yr_y \leq 0 \text{ in } \tilde{G}_2''' \end{cases},$$

(R_4) $K_i > 0$, $M_i > 0 \quad (i = 1,2)$, *in* $G_1 \cup G_1{}' \cup G_1{}'' \cup G_1{}'''$,

(R_5) $\begin{cases} K_1 < 0\,, & M_1 > 0 \text{ in } \tilde{G}_2 \cup \tilde{G}_2'' \\ K_1 > 0\,, & M_1 < 0 \text{ in } \tilde{G}_2' \cup \tilde{G}_2''' \end{cases}$,

(R_6) $b(x)K_i(y)\dot{M}_j(x) + c(y)K_i{}'(y)M_j(x) > 0$, $(1 \le i \ne j \le 2)$ *in* $G_1 \cup G_1{}' \cup G_1{}'' \cup G_1{}'''$, *where*

$$b(x) = \begin{cases} x & \text{in } G_1 \cup G_1{}' \\ x+1 & \text{in } G_1{}'' \cup G_1{}''' \end{cases} ;$$

$$c(y) = \begin{cases} y-1 & \text{in } G_1 \cup G_1{}'' \\ y & \text{in } G_1{}' \cup G_1{}''' \end{cases} .$$

(R_7) $K_2 > 0$, $M_2 > 0$ *in* $\tilde{D}(\subset D)$,

(R_8)

$$\begin{cases} M_i(x) - (-1)^i \tilde{b}(x)\dot{M}_i(x) > 0 & \text{in } \tilde{G}_2 \cup \tilde{G}_2'' \\ K_i(y) - (-1)^i \tilde{c}(x)K_i{}'(y) > 0 & \text{in } \tilde{G}_2' \cup \tilde{G}_2''' \end{cases} \quad (i \in \{1,2\}),$$

where

$$\tilde{b}(x) = \begin{cases} x & \text{in } \tilde{G}_2 \\ x+1 & \text{in } \tilde{G}_2'' \\ 0 & \text{in } \tilde{G}_2' \cup \tilde{G}_2''' \end{cases} ;$$

$$\tilde{c}(y) = \begin{cases} y-1 & \text{in } \tilde{G}_2' \\ y & \text{in } \tilde{G}_2''' \\ 0 & \text{in } \tilde{G}_2 \cup \tilde{G}_2'' \end{cases} .$$

Let us denote $()_x = \partial()/\partial x$, $\quad ()^{\cdot} = d()/dx$, $\quad ()_y = \partial()/\partial y$, $\quad ()' = d()/dy$, *where* $f = f(x,y)$ *is continuous in* $\tilde{D}(\subset D)$, $\quad r = r(x,y)$ *is once-continuously differentiable in* $\tilde{D}(\subset D)$, $K_i = K_i(y)\,(i = 1,2)$ *are once-continuously differentiable for* $y \in [-k_1, k_2]$ *with* $-k_1 = inf\{y : (x,y) \in \tilde{D}(\subset D)\}$ *and* $k_2 = sup\{y : (x,y) \in \tilde{D}(\subset D)\}$, *and* $M_i = M_i(x)$ $(i = 1,2)$ *are once-continuously differentiable for* $x \in [-m_1, m_2]$ *with* $-m_1 = inf\{x : (x,y) \in \tilde{D}(\subset D)\}$ *and* $m_2 = sup\{x : (x,y) \in \tilde{D}(\subset D)\}$. *Then the Problem (EF) has at most one quasi-regular solution in* $\tilde{D}(\subset D)$.

4. Open Problems

4.1. Extend "quasi-regularity" of solutions to "regularity" by fixing singularities at the following twelve points:

$$O_1 = (0,1),\, O_1{}' = (-1,1),\, O_2 = (0,0), O_2{}' = (-1,0);$$
$$A_1 = (-2,1),\, B_1 = (1,1),\, A_2 = (-2,0),\, B_2 = (1,0);$$
$$E_1 = (-1,2),\, Z_1 = (0,2),\, E_2 = (-1,-1),\, Z_2 = (0,-1).$$

4.2. Investigate the exterior Tricomi and Frankl problems in a multiply connected mixed domain.

4.3. Establish “well-posedness” of solutions for the exterior Tricomi and Frankl problems, in the sense that there is at most one quasi-regular solution and a weak solution exists.

4.4. Solve the n - dimensional Tricomi and Frankl problems in a multiply connected mixed domain.

4.5. Establish the extremum principle for the exterior Tricomi problem:
“A solution of the exterior Tricomi (or Frankl) problem, vanishing on the exterior boundary of the considered mixed domain, achieves neither a positive maximum nor a negative minimum on open arcs of the type-degeneracy curves.”

4.6. Solve the Tricomi problem for PDE of second order:

4.6.1 $K(y-x^m-x^n)u_{xx}+u_{yy}+r(x,y)u=f(x,y);$
4.6.2 $u_{xx}+M(x-y^m-y^n)u_{yy}+r(x,y)u=f(x,y);$
4.6.3 $K(x^m+y^n-1)u_{xx}+u_{yy}+r(x,y)u=f(x,y)$, for example $m=n=2$ or $=2/3$;
4.6.4 $K\big((y-x^m)(y-x^n)\big)u_{xx}+u_{yy}+r(x,y)u=f(x,y);$
4.6.5 $K(y-x^n)u_{xx}+M(x-y^m)u_{yy}+r(x,y)u=f(x,y);$
4.6.6 $K(y^k-x^m\pm x^n)u_{xx}+M(x^k-y^m\pm y^n)u_{yy}+r(x,y)u=f(x,y);$
4.6.7 $K\big(y^m(y-x^n)\big)u_{xx}+M\big(x^m(x-y^n)\big)u_{yy}+r(x,y)u=f(x,y);$
4.6.8 $K\big((y-x^m)(y-x^n)\big)u_{xx}+M\big((x-y^\alpha)(x-y^\beta)\big)u_{yy}+r(x,y)u=f(x,y).$

4.7. Solve the Tricomi problem for PDE of fourth order:

$$\big(sgn(y-x^l)|y-x^l|^k\frac{\partial^2}{\partial x^2}+sgn(x-y^n)|x-y^n|^m\frac{\partial^2}{\partial y^2}+r\big)^2u=f.$$

4.8. Solve the 3 - dimensional Tricomi problem for mixed type PDE of second order:

$$sgn(z)|z|^k(u_{xx}\pm u_{yy})+sgn(xy)|x|^m|y|^nu_{zz}+ru=f.$$

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