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On the Asymptotics and Zeros of a Class of Fourier Integrals

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Abstract. We obtain the asymptotic expansion of the Fourier integrals

$$\int_0^\infty t^{\nu-1} \begin{matrix}\cos\\ \sin\end{matrix}(xt)\exp(-t^n/n)\,dt$$

for large complex values of x and integer $n > 2$ by means of the asymptotic theory of the Wright function. Asymptotic approximations for both the real and complex zeros of these integrals are considered. These results are extended to p-dimensional Fourier integrals of a similar structure.

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Key Words and Phrases: Fourier integrals, asymptotic expansion, zeros, Wright function

1. Introduction

The asymptotic expansion of the Fourier integrals

$$\int_0^\infty \begin{matrix}\cos\\ \sin\end{matrix}(xt)\exp(-t^n/n)\,dt \tag{1}$$

for large complex values of x and integer $n \geq 2$ has been considered in [1, 3, 5] and, in the case of the cosine integral, more recently in [15]. All these authors employed the method of steepest descents to derive the asymptotics which resulted in long and detailed calculations. Considerable interest in the real zeros of Fourier integrals originated with the seminal study of Pólya [14], who showed that the cosine integral in (1) when $n = 4, 6, \ldots$ has infinitely many real zeros; generalisations of these results have been obtained in [4] and more recently in [6, 7, 8]. The first-order asymptotics for the location of the zeros of the cosine integral in (1) were obtained in [5], with higher-order approximations for these zeros being given in [15].

Our aim in this paper is to show that the large-x asymptotics of the above integrals can be more readily obtained by making use of the well-established asymptotic theory of the Wright

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function defined below. We make a generalisation in (1) to include an algebraic power of t in the integrand and consider the integrals

$$\begin{matrix} C_{n,1} \\ S_{n,1} \end{matrix}(x;\nu)=\int_0^\infty t^{\nu-1}\begin{matrix}\cos\\ \sin\end{matrix}(xt)\,\exp(-t^n/n)\,dt, \qquad \mathrm{Re}\,(\nu)>\left\{\begin{matrix} 0 \\ -1 \end{matrix}\right., \tag{2}$$

where the subscript 1 denotes the dimension of the integral. We then use the asymptotics of the integrals in (2) to examine their real and complex zeros. An advantage of this approach is that the same procedure can be applied with little additional effort to the p-dimensional integrals possessing a similar structure given by

$$\begin{matrix} C_{n,p} \\ S_{n,p} \end{matrix}(x;\vec{\nu})=\int_0^\infty\cdots\int_0^\infty t_1^{\nu_1-1}\ldots t_p^{\nu_p-1}\begin{matrix}\cos\\ \sin\end{matrix}(xt_1\ldots t_p)\,\exp\left[-(t_1^n+\cdots+t_p^n)/n\right]dt_1\ldots dt_p, \tag{3}$$

where $\vec{\nu}=(\nu_1,\nu_2,\ldots,\nu_p)$, $n\geq 2$ is an integer and it is supposed for $C_{n,p}(x;\vec{\nu})$ that $\mathrm{Re}(\nu_r)>0$ and for $S_{n,p}(x;\vec{\nu})$ that $\mathrm{Re}(\nu_r)>-1$ $(1\leq r\leq p)$. An integral of this type with $p=2$ was given as the solution of a certain nth-order differential equation by Spitzer [17] well over a century ago.

The Wright function ${}_p\Psi_q(z)$ (a generalised hypergeometric function) is defined by

$${}_p\Psi_q(z)=\sum_{k=0}^\infty\frac{\prod_{r=1}^p\Gamma(\alpha_rk+a_r)}{\prod_{r=1}^q\Gamma(\beta_rk+b_r)}\frac{z^k}{k!}, \qquad \alpha_rk+a_r\neq 0,-1,-2,\ldots \tag{4}$$

where p and q are nonnegative integers, the parameters α_r and β_r are real and positive and a_r and b_r are arbitrary complex numbers. In the special case $\alpha_r=\beta_r=1$, the function ${}_p\Psi_q(z)$ reduces to a multiple of the ordinary hypergeometric function ${}_pF_q((a)_p;(b)_q;z)$ [16, p. 40]. The particular function of this class that we shall use has $q=0$ and the parameters $\alpha_r=1/n$, $a_r=\nu_r/n$. Following the notation used in [12, Chapter 3] for the solution of a certain nth-order differential equation, we denote this function by $U_{n,p}(z;\vec{\nu})$, where

$$U_{n,p}(z;\vec{\nu})=\sum_{k=0}^\infty\frac{(n^{p/n}z)^k}{k!}\prod_{r=1}^p\Gamma\left(\frac{k+\nu_r}{n}\right) \qquad (n>p\geq 1;\ |z|<\infty).$$

The integrals $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ in (3) will be shown to be expressed respectively in terms of the even and odd combinations

$$U_{n,p}(ix;\vec{\nu})\pm U_{n,p}(-ix;\vec{\nu}),$$

whence the asymptotics for large complex x can be easily constructed from knowledge of that of $U_{n,p}(z;\vec{\nu})$.

It will be established that, for general values of the parameters ν_r and $p\geq 1$, the integrals $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ possess a dominant algebraic expansion in the sectors $|\arg(\pm x)|<\pi p/(2n)$ and an exponentially large expansion in the complementary sectors $|\arg(\pm ix)|<\frac{1}{2}\pi(1-p/n)$. An infinite sequence of complex zeros of these integrals is found in the neighbourhood of the anti-Stokes lines $\arg x = \pm\pi p/(2n)$ (together with a symmetrical distribution in $\mathrm{Re}(x) < 0$), where the exponential and algebraic expansions are of comparable magnitude. For certain values of ν_r when n is even, however, the algebraic expansion vanishes to leave an exponentially small behaviour in the sectors $|\arg(\pm x)| < \pi p/(2n)$. In these cases, the integrals $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ are found to have an infinite sequence of real zeros. Asymptotic approximations to the zeros are obtained both in the general case and in the exponentially small case.

The structure of the paper is as follows. In Section 2 we present the asymptotic expansion of the function $U_{n,p}(z;\vec{\nu})$ for large complex values of z. From this we construct the asymptotics of the integrals (3) for large complex x in Section 3. The zeros (both real and complex) in the one-dimensional case $p = 1$ are examined in Section 4. Finally, in Section 5 we investigate the zeros when $p = 2$ in particular cases and make a conjecture on the real zeros for general p.

2. The Asymptotic Properties of $U_{n,p}(z;\vec{\nu})$

We present in this section the asymptotic expansion of the function $U_{n,p}(z;\vec{\nu})$ which is fundamental in our discussion of the Fourier integrals $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$. It was introduced in [11, 12] as the solution of the nth-order differential equations*

$$u^{(n)} \mp \sum_{r=0}^{p} a_r z^r u^{(r)} = 0 \qquad (n > p \geq 1), \tag{5}$$

where z is the independent variable and the coefficients a_r $(1 \leq r \leq p-1)$ are arbitrary constants with $a_0 \neq 0$ and (without loss of generality) $a_p = 1$. Four classes of solution of (5), exhibiting different types of asymptotic behaviour for large $|z|$, have been discussed in detail in [12, Chapter 3]. The polynomial $G(s)$ of degree p associated with (5) is defined by

$$G(s) = \sum_{r=0}^{p} (-)^r a_r (-s)_r = \prod_{r=1}^{p} (s + \nu_r), \tag{6}$$

where $(\alpha)_r = \Gamma(\alpha + r)/\Gamma(\alpha)$ is the Pochhammer symbol and $-\nu_r$ $(1 \leq r \leq p)$ are the zeros of $G(s)$. With $\Theta \equiv zd/dz$, so that the differential operator $z^r(d/dz)^r = \Theta(\Theta-1)\ldots(\Theta-r+1)$, the equation (5) can be written in the alternative form

$$u^{(n)} \mp \prod_{r=1}^{p} (\Theta + \nu_r) u = 0, \tag{7}$$

which is a transformation of the generalised hypergeometric differential equation [16, p. 42].

The solution of (5) and (7) with the upper sign that we consider here has the series expansion

$$U_{n,p}(z;\vec{\nu}) = \sum_{k=0}^{\infty} \frac{(n^{p/n} z)^k}{k!} \prod_{r=1}^{p} \Gamma\left(\frac{k+\nu_r}{n}\right) \qquad (n > p \geq 1). \tag{8}$$

*We exclude the trivial case $p = 0$.

Provided we impose the restriction that none of the ν_r equals a negative integer ($\nu_r \neq 0$ by hypothesis, since $a_0 \neq 0$), then (8) defines $U_{n,p}(z;\vec{\nu})$ as a uniformly and absolutely convergent series throughout the finite z-plane. Comparison with (4) shows that $U_{n,p}(z;\vec{\nu})$ is a particular case of the Wright function with $q=0$, $\alpha_r = 1/n$, $a_r = \nu_r/n$ and argument $n^{p/n}z$. It is easily verified by differentiation of the right-hand side of (8) that $U_{n,p}(z;\vec{\nu})$ satisfies the differential equation (7) with the upper sign. Since (5) is unaltered if z is replaced by Ωz, where Ω denotes an nth root of unity, a fundamental system of solutions of (5) is given by

$$\left.\begin{array}{c} U_{n,p}(\Omega_j z) \\ U_{n,p}(e^{\pi i/n}\Omega_j z) \end{array}\right\}, \qquad \Omega_j = \exp(2\pi i j/n), \qquad j = 0, 1, \ldots, n-1 \tag{9}$$

where the upper and lower sets of solutions correspond to the upper and lower signs, respectively. An integral representation of the solution is given by the Mellin-Barnes integral [12, p. 61]

$$U_{n,p}(z;\vec{\nu}) = \frac{1}{2\pi i}\int_{-\infty i}^{\infty i} \Gamma(-s)\prod_{r=1}^{p}\Gamma\left(\frac{s+\nu_r}{n}\right)(-n^{p/n}z)^{-s}ds \tag{10}$$

valid in the sector $|\arg(-z)| < \frac{1}{2}\pi(1+p/n)$, where, with the above-mentioned restrictions on ν_r, the path of integration can always be chosen to separate the poles of $\Gamma(-s)$ from those of $\Gamma((s+\nu_r)/n)$ $(1 \le r \le p)$.

The asymptotic expansion of $U_{n,p}(z;\vec{\nu})$ for large $|z|$ follows from that of the Wright function ${}_p\Psi_q(z)$ in (4) [18, 2]; see also [10]. We define the parameters

$$\kappa = 1 - \frac{p}{n}, \qquad \vartheta = \frac{1}{n}\sum_{r=1}^{p}\nu_r - \frac{1}{2}p \tag{11}$$

and introduce the formal exponential and algebraic asymptotic expansions defined respectively by

$$E(z) \;:=\; (2\pi)^{p/2}\kappa^{-\frac{1}{2}}(z^{1/\kappa}/n)^{\vartheta}\exp(\kappa z^{1/\kappa})\sum_{j=0}^{\infty}c_j(\kappa z^{1/\kappa})^{-j}, \tag{12}$$

$$H(z) \;:=\; n\sum_{r=1}^{p}(n^{p/n}z)^{-\nu_r}\, T_{n,p}(z;\vec{\nu}), \tag{13}$$

where, provided no two of the ν_r either coincide or differ by an integer multiple of n,

$$T_{n,p}(z;\vec{\nu}) := \sum_{k=0}^{\infty}\frac{(-)^k}{k!}\Gamma(nk+\nu_r)\sideset{}{'}\prod_{j=1}^{p}\Gamma\left(\frac{\nu_j-\nu_r}{n}-k\right)(n^{p/n}z)^{-nk}$$

with the prime denoting the omission of the term corresponding to $j = r$ in the product. The algebraic expansion $H(z)$ results from displacement of the integration path in (10) over the poles of the product of gamma functions and evaluation of the residues. When these restrictions on ν_r are not satisfied, the algebraic expansion is modified by the presence of logarithmic terms arising from the formation higher-order poles in the integrand of (10).

The coefficients c_j appearing in the exponential expansion $E(z)$ are independent of z with $c_0 = 1$ and are generated by the n-term recurrence relation [12, §3.4]

$$c_j = \frac{1}{n\kappa j}\left\{\sum_{s=1}^{n-1} c_{j-s}P_{s+1}^{(n)}(s-j) - \sum_{s=1}^{p-1} c_{j-s}Q_{s+1}(s-j)\right\} \qquad (j \geq 1) \tag{14}$$

with $c_{-1} = c_{-2} = \ldots = c_{2-n} = 0$, where

$$P_s^{(n)}(\chi) = \sum_{r=0}^{s}\sum_{k=0}^{r}(\vartheta+\chi)^{r-k}\kappa^k \binom{n-k}{r-k} S_n^{(n-k)} \mathfrak{S}_{n-r}^{(n-s)},$$

$$Q_s(\chi) = \sum_{r=0}^{s} a_{p-r}\kappa^r P_{s-r}^{(p-r)}(\chi),$$

the a_r are the coefficients in the differential equation (5) and $S_n^{(m)}$, $\mathfrak{S}_n^{(m)}$ are respectively the Stirling numbers of the first and second kind. Alternatively, these coefficients may be obtained by means of the algorithm described in [10]; see also [13, §2.2.4]. From [10, Appendix A], we have the explicit representation of the coefficient c_1 in the form

$$c_1 = \tfrac{1}{2}\kappa\left\{\sum_{r=1}^{p}\nu_r\left(\frac{\nu_r}{n}-1\right) - \frac{\vartheta(1-\vartheta)}{\kappa}\right\} + \frac{p}{12n}(n^2-np+1). \tag{15}$$

The first few values of the coefficients c_j obtained from (14) for different n, p and $\vec{\nu}$ are given in Table 1.

Table 1: The coefficients c_j $(1 \leq j \leq 5)$ for different n, p and $\vec{\nu}$.

j	$n=4,\ p=1$ $\nu=1$	$n=6,\ p=2$ $\vec{\nu}=(\frac{1}{2},2)$	$n=6,\ p=3$ $\vec{\nu}=(\frac{1}{2},\frac{3}{2},4)$
1	$\frac{7}{48}$	$\frac{161}{288}$	$\frac{7}{16}$
2	$\frac{385}{4608}$	$\frac{114625}{165888}$	$\frac{289}{512}$
3	$\frac{39655}{663552}$	$\frac{189038465}{143327232}$	$\frac{10061}{8192}$
4	$\frac{665665}{127401984}$	$\frac{608738148865}{165112971264}$	$\frac{2011691}{524288}$
5	$-\frac{1375739365}{6115295232}$	$\frac{704282046029485}{47552535724032}$	$\frac{132834185}{8388608}$

Then, we have the asymptotic expansion given by

Theorem 1. *For $n > p \geq 1$ and $|z| \to \infty$, the function $U_{n,p}(z;\vec{\nu})$ possesses the asymptotic expansion*†

$$U_{n,p}(z;\vec{\nu}) \sim \begin{cases} E(z) + H(ze^{\mp\pi i}) & |\arg z| < \pi\left(1 - \frac{p}{n}\right) \\ H(ze^{\mp\pi i}) & |\arg(-z)| < \frac{1}{2}\pi\left(1 + \frac{p}{n}\right) \end{cases}, \tag{16}$$

where the upper or lower signs in (16) *are chosen according as* $\arg z > 0$ *or* $\arg z < 0$, *respectively. The expansion of the fundamental systems in* (9) *follows immediately by rotation of the argument z by $2\pi j/n$ and $(2j+1)\pi/n$.*

The function $U_{n,p}(z;\vec{\nu})$ is exponentially large as $|z| \to \infty$ in the sector $|\arg z| < \frac{1}{2}\pi\kappa$, whereas in the complementary sector $|\arg(-z)| < \frac{1}{2}\pi(2-\kappa)$ the dominant asymptotic behaviour consists (in general) of p algebraic expansions, each with the controlling behaviour $z^{-\nu_r}$, $r = 1, 2, \ldots, p$. In the common sectors of validity, $\frac{1}{2}\pi\kappa < |\arg z| < \pi\kappa$, the expansions in (16) differ only through the presence of the series $E(z)$, which is exponentially small in these sectors. The rays $\arg z = \pm\pi\kappa$ are Stokes lines on which the expansion $E(z)$ is maximally subdominant. It was established in [9] that (in the sense of increasing $|\arg z|$) the expansion $E(z)$ switches off smoothly as these Stokes lines are crossed. The positive real axis is also a Stokes line where the algebraic expansion is maximally subdominant. The sectorial behaviour of $U_{n,p}(z;\vec{\nu})$ is illustrated in Fig. 1.

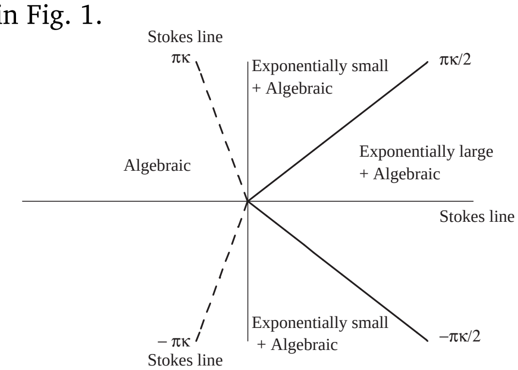


Figure 1: The sectorial behaviour of $U_{n,p}(z;\vec{\nu})$ for large $|z|$.

Finally, it is worth remarking that the expansion in (16) remains valid for noninteger values of $n > p$; in this case, of course, the function $U_{n,p}(z;\vec{\nu})$ is not a solution of (5) and the recurrence relation (14) can no longer be employed. The coefficients c_j in this case can be obtained by the algorithm described in [10].

†We remark that the first expansion in (16) was given in [12, 11] only in the narrower sector $|\arg z| \leq \frac{1}{2}\pi\kappa$ where the exponential expansion $E(z)$ is dominant.

3. The Asymptotic Expansion of $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ for $|x|\to\infty$

A Laplace integral representation for $U_{n,p}(z;\vec{\nu})$ has been given in [12, p. 124]. When $p=1$, this takes the form

$$U_{n,1}(z;\nu)=n^{\frac{1}{2}-\vartheta}\int_0^\infty t^{\nu-1}e^{zt-t^n/n}\,dt \qquad (\mathrm{Re}\,(\nu)>0), \tag{17}$$

which may be easily verified by expanding e^{zt} as a Maclaurin series followed by term-by-term integration. In a similar manner, we may establish that

$$U_{n,p}(z;\vec{\nu})=n^{p/2-\vartheta}\int_0^\infty\ldots\int_0^\infty t_1^{\nu_1-1}\ldots t_p^{\nu_p-1}\,e^{zt_1\ldots t_p}\,\exp\left[-(t_1^n+\ldots+t_p^n)/n\right]dt_1\ldots dt_p \tag{18}$$

for $\mathrm{Re}\,(\nu_r)>0$, $1\le r\le p$; see [12, p. 133]. This integral in the case $p=2$ was first considered in [17] as the solution of a certain nth-order differential equation. From (17) and (18), it then follows from the definition of the integrals $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ in (3) that

$$\begin{matrix}C_{n,p}\\S_{n,p}\end{matrix}(x;\vec{\nu})=\xi n^{\vartheta-\frac{1}{2}p}\{U_{n,p}(ix;\vec{\nu})\pm U_{n,p}(-ix;\vec{\nu})\}, \tag{19}$$

where $\xi=2^{-1}$ for $C_{n,p}(x;\vec{\nu})$ and $\xi=(2i)^{-1}$ for $S_{n,p}(x;\vec{\nu})$. From (8), we obtain the series representations

$$\begin{matrix}C_{n,p}\\S_{n,p}\end{matrix}(x;\vec{\nu})=n^{\vartheta-p/2}\sum_{k=0}^\infty\frac{(n^{p/n}x)^k}{k!}\prod_{r=1}^p\Gamma\left(\frac{k+\nu_r}{n}\right)\begin{matrix}\cos\\\sin\end{matrix}(\tfrac{1}{2}\pi k). \tag{20}$$

As these integrals are respectively even and odd functions of x, it is sufficient to restrict our attention to the sector $|\arg x|\le\frac{1}{2}\pi$.

We now introduce the formal asymptotic expansions

$$E_\pm:=\kappa^{-1/2-\vartheta}\left(\frac{2\pi}{n}\right)^{p/2}X^\vartheta\exp\left(Xe^{\pm\pi i/(2\kappa)}\right)\sum_{j=0}^\infty c_jX^{-j}\,e^{\pm\pi i(\vartheta-j)/(2\kappa)}, \tag{21}$$

$$H_{c,s}:=n^{1+\vartheta-\frac{1}{2}p}\sum_{r=1}^p(n^{p/n}x)^{-\nu_r}T_{n,p}^{(c,s)}(x;\vec{\nu}), \tag{22}$$

where the variable X is defined by

$$X:=\kappa x^{1/\kappa},$$

$$T_{n,p}^{(c,s)}(x;\vec{\nu}):=\sum_{k=0}^\infty\frac{(-)^k}{k!}\Gamma(nk+\nu_r)\prod_{j=1}^{p}{}'\,\Gamma\left(\frac{\nu_j-\nu_r}{n}-k\right)(n^{p/n}x)^{-nk}\begin{matrix}\cos\\\sin\end{matrix}\tfrac{1}{2}\pi(nk+\nu_r) \tag{23}$$

and the sub- and superscripts c, s refer to the expansion with cosine and sine, respectively. In addition, we introduce the expansions

$$E_c:=\frac{1}{2}(E_++E_-),\qquad E_s:=\frac{1}{2i}(E_+-E_-),$$

so that

$$E_{c,s} = \kappa^{-1/2-\vartheta} \left(\frac{2\pi}{n}\right)^{p/2} X^{\vartheta} \exp\left(X \cos\frac{\pi}{2\kappa}\right) \sum_{j=0}^{\infty} c_j X^{-j} \begin{matrix}\cos\\ \sin\end{matrix} \left(X \sin\frac{\pi}{2\kappa} + \frac{\pi}{2\kappa}(\vartheta - j)\right). \quad (24)$$

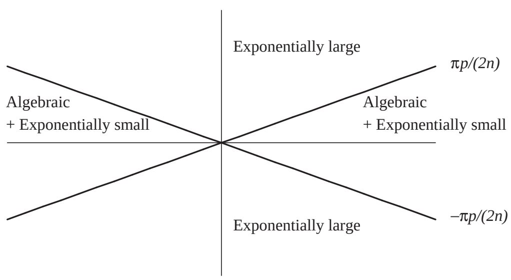
Then from Theorem 1, the asymptotic expansion of $C_{n,p}(x;\vec{v})$ and $S_{n,p}(x;\vec{v})$ when $p/n \le \frac{1}{2}$ ($\kappa \ge \frac{1}{2}$) is

$$\begin{matrix}C_{n,p}\\ S_{n,p}\end{matrix}(x;\vec{v}) \sim \begin{cases} H_{c,s} + E_{c,s} & |\arg x| \le \pi(\frac{1}{2} - \frac{p}{n}) \\ H_{c,s} \pm \xi E_- & \pi(\frac{1}{2} - \frac{p}{n}) < \arg x \le \frac{1}{2}\pi \\ H_{c,s} + \xi E_+ & -\frac{1}{2}\pi \le \arg x < \pi(\frac{1}{2} - \frac{p}{n}) \end{cases} \quad (25)$$

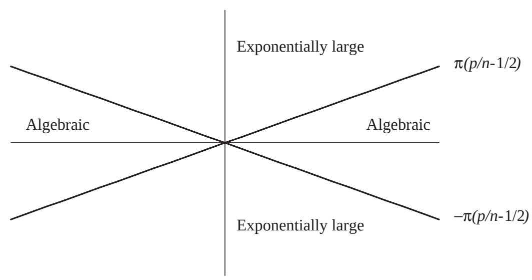
and when $p/n > \frac{1}{2}$ ($\kappa < \frac{1}{2}$)

$$\begin{matrix}C_{n,p}\\ S_{n,p}\end{matrix}(x;\vec{v}) \sim \begin{cases} H_{c,s} & |\arg x| \le \pi(\frac{p}{n} - \frac{1}{2}) \\ H_{c,s} \pm \xi E_- & \pi(\frac{p}{n} - \frac{1}{2}) < \arg x \le \frac{1}{2}\pi \\ H_{c,s} + \xi E_+ & -\frac{1}{2}\pi \le \arg x < \pi(\frac{p}{n} - \frac{1}{2}) \end{cases} \quad (26)$$

as $x \to \infty$; compare [12, §3.8.2]. The asymptotic structure of $C_{n,p}(x;\vec{v})$ and $S_{n,p}(x;\vec{v})$ as $|x| \to \infty$ is summarised in Fig. 2. When $\kappa \ge \frac{1}{2}$, this is seen to consist of an algebraic and a subdominant exponentially small expansion in the sectors $|\arg(\pm x)| < \pi p/(2n)$ and an exponentially large expansion (with a subdominant algebraic expansion) in the sectors $|\arg(\pm ix)| < \frac{1}{2}\pi\kappa$. When $\kappa < \frac{1}{2}$, there is a purely algebraic expansion in the sectors $|\arg(\pm x)| < \pi(p/n - \frac{1}{2})$ and an exponentially large (with a subdominant algebraic expansion) in the sectors $|\arg(\pm ix)| < \pi\kappa$.



(a) $\kappa \ge \frac{1}{2}$



(b) $\kappa < \frac{1}{2}$

Figure 2: The sectorial behaviour of $C_{n,p}(z;\vec{v})$ and $S_{n,p}(z;\vec{v})$ for large $|x|$.

We remark that the above expansions take into account the switching on or off of the exponential expansions due to the Stokes phenomenon as one crosses the rays $\arg x = \pm\pi(\frac{1}{2} - p/n)$. However, the details of the transition across these rays, together with those associated with the algebraic expansions on $\arg x = \pm\frac{1}{2}\pi$, would require further investigation of the Stokes phenomenon on the lines of that given in [9]. Finally, if some of the v_r are equal, or differ by integer multiples of n, then higher order poles will arise in the integrand of (10) and the algebraic expansions $H_{c,s}$ will be modified by the presence of logarithmic terms; see Section 5 for an example.

4. The Zeros in the Case $p = 1$

We first examine the case with $p = 1$, where from (19)

$$\begin{matrix} C_{n,1} \\ S_{n,1} \end{matrix}(x;\nu) = \int_0^\infty t^{\nu-1} \begin{matrix} \cos \\ \sin \end{matrix}(xt)\exp(-t^n/n)\,dt, \qquad \mathrm{Re}(\nu) > \left\{ \begin{matrix} 0 \\ -1 \end{matrix} \right.$$
$$= \xi n^{\vartheta-\frac{1}{2}}\{U_{n,1}(ix;\nu) \pm U_{n,1}(-ix;\nu)\}. \tag{27}$$

In this case the algebraic expansions in (22) simplify to

$$H_{c,s} = x^{-\nu} \sum_{k=0}^{\infty} \frac{(-)^k}{k!}\Gamma(nk+\nu)(n^{1/n}x)^{-nk} \begin{matrix} \cos \\ \sin \end{matrix} \tfrac{1}{2}\pi(nk+\nu). \tag{28}$$

Then, from (25), we have the asymptotic expansion[‡] for $n > 2$ ($\kappa > \frac{1}{2}$) given by

$$\begin{matrix} C_{n,1} \\ S_{n,1} \end{matrix}(x;\nu) \sim \begin{cases} H_{c,s} + E_{c,s} & |\arg x| \le \pi(\frac{1}{2} - \frac{1}{n}) \\ H_{c,s} \pm \xi E_- & \pi(\frac{1}{2} - \frac{1}{n}) < \arg x \le \frac{1}{2}\pi \\ H_{c,s} + \xi E_+ & -\frac{1}{2}\pi \le \arg x < \pi(\frac{1}{2} - \frac{1}{n}) \end{cases} \tag{29}$$

as $|x| \to \infty$. The coefficients c_j in the exponential expansions $E_\pm$ and $E_{c,s}$ in (21) and (24) can be computed by the recurrence relation (14). The first few values of these coefficients when $n = 4$ and $\nu = 1$ are given in the first column of Table 1.

4.1. Real zeros

When n is even and the parameter ν is an odd (resp. even) integer for $C_{n,1}(x;\nu)$ (resp. $S_{n,1}(x;\nu)$), the algebraic expansion H_c (resp. H_s) in (28) vanishes to leave an exponentially small expansion in the sector $|\arg x| < \pi(\frac{1}{2} - 1/n)$. From (29) and (24), we then have when $\kappa > \frac{1}{2}$

$$\begin{matrix} C_{n,1} \\ S_{n,1} \end{matrix}(x;\nu) \sim \kappa^{-\vartheta}\left(\frac{2\pi}{n\kappa}\right)^{1/2} X^{\vartheta} \exp\left(X\cos\frac{\pi}{2\kappa}\right) \sum_{j=0}^{\infty} c_j X^{-j} \begin{matrix} \cos \\ \sin \end{matrix}\left(X\sin\frac{\pi}{2\kappa} + \frac{\pi}{2\kappa}(\vartheta - j)\right) \tag{30}$$
$$n \text{ even}, \nu = \left\{ \begin{matrix} 2m+1 \\ 2m+2 \end{matrix} \right. \qquad (m = 0, 1, 2, \ldots)$$

as $|x| \to \infty$ in $|\arg x| < \pi(\frac{1}{2} - 1/n)$. In this case, $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ possess an infinite sequence of real zeros; see the appendix.

The leading-order approximation for the real zeros is then given by

$$\begin{matrix} \cos \\ \sin \end{matrix}\Psi = 0, \qquad \Psi = X\sin\frac{\pi}{2\kappa} + \frac{\pi\vartheta}{2\kappa}$$

[‡]We exclude the case $n = 2$ ($\kappa = \frac{1}{2}$) since the integrals in (27) can be evaluated in terms of parabolic cylinder functions.

to produce the zeroth-order approximation

$$\Psi^{(0)} = (k+\epsilon)\pi, \qquad \epsilon = \left\{ \begin{array}{l} \frac{1}{2} \\ 1 \end{array} \right. \qquad (k = 0, 1, 2, \ldots).$$

To next order we have

$$\begin{array}{c}\cos\\ \sin\end{array}\Psi + \frac{c_1}{X}\begin{array}{c}\cos\\ \sin\end{array}\left(\Psi - \frac{\pi}{2\kappa}\right) = 0,$$

which leads to

$$\Psi^{(1)} = (k+1)\pi \mp \arctan\left(\begin{array}{c}\Lambda\\ \Lambda^{-1}\end{array}\right), \qquad \Lambda = \frac{X + c_1 \cos\frac{\pi}{2\kappa}}{c_1 \sin\frac{\pi}{2\kappa}}.$$

Thus we obtain for $X \to \infty$

$$\Psi^{(1)} = \Psi^{(0)} + \frac{c_1}{X}\sin\frac{\pi}{2\kappa}.$$

This yields the zeroth and first-order approximations $X^{(0)}$, $X^{(1)}$ to the (positive) real zeros of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ given by

$$X^{(0)} = \left(k + \epsilon - \frac{\vartheta}{2\kappa}\right)\frac{\pi}{\sin\frac{\pi}{2\kappa}}, \qquad X^{(1)} = X^{(0)} + \frac{c_1}{X^{(0)}} \qquad (X = \kappa x^{1/\kappa}). \tag{31}$$

This approximation procedure in the case of the real zeros of $C_{n,1}(x;1)$ for even n has been carried out to fourth order in [15].

We remark that when $n = 2$ ($\kappa = \frac{1}{2}$) the functions $C_{n,1}(x;\nu)$ (with ν odd) and $S_{n,1}(x;\nu)$ (with ν even) can be expressed in terms of Hermite polynomials $H_n(z)$ as

$$\begin{array}{c}C_{2,1}\\ S_{2,1}\end{array}(x;\nu) = (-)^m 2^{-\frac{1}{2}\nu}\,\pi^{\frac{1}{2}}\,e^{-x^2/2}\,H_{\nu-1}(x/\sqrt{2}), \qquad \nu = \left\{\begin{array}{l} 2m+1\\ 2m+2\end{array}\right. \tag{32}$$

for nonnegative integer m. By a well-known property of the Hermite polynomials, it follows that the functions on the left-hand side of (32) possess a *finite* number of real zeros.

4.2. Complex zeros

For simplicity, we shall restrict attention to real positive values of ν. Provided the algebraic expansions $H_{c,s}$ do not vanish identically (which can only arise when n is an even integer and ν is either odd (resp. even)), the complex zeros of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ will be situated near the anti-Stokes lines arg $x = \pm\pi/(2n)$, where the expansions $H_{c,s}$ and $E_\mp$ are comparable in magnitude. We consider only the neighbourhood of the ray arg $x = \pi/(2n)$ where

$$\begin{array}{c}C_{n,1}\\ S_{n,1}\end{array}(x;\nu) \sim H_{c,s} \pm \xi E_-$$

for large $|x|$, since E_+ is a subdominant expansion in the sector $0 < \arg x < \pi(\frac{1}{2} - 1/n)$. To leading order the complex zeros of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ are then described by[§]

$$x^{-\nu} \begin{matrix}\cos\\ \sin\end{matrix} (\tfrac{1}{2}\pi\nu)\,\Gamma(\nu) \pm \xi \left(\frac{2\pi}{n\kappa}\right)^{1/2} (-ix)^{\vartheta/\kappa} \exp(Xe^{-\pi i/(2\kappa)}) = 0.$$

If we put

$$x = re^{i\phi+\pi i/(2n)}, \tag{33}$$

with $r = |x|$, then we find

$$\exp\{i\kappa r^{1/\kappa} \cos\phi/\kappa\} - \Upsilon e^{i\Phi} = 0, \tag{34}$$

where

$$\Upsilon = \lambda r^{(\nu-\frac{1}{2})/\kappa} \exp(\kappa r^{1/\kappa} \sin(\phi/\kappa)), \qquad \lambda = \frac{(\pi/2n\kappa)^{1/2}}{\Gamma(\nu) \left| \begin{matrix}\cos\\ \sin\end{matrix} (\frac{1}{2}\pi\nu) \right|},$$

$$\Phi = \tfrac{1}{2}\pi(1 \mp \tfrac{1}{2}) + (\nu - \tfrac{1}{2})\phi/\kappa + \pi\delta,$$

with $\delta = 1$ if $\cos\frac{1}{2}\pi\nu$ or $\sin\frac{1}{2}\pi\nu > 0$, and $\delta = 0$ if $\cos\frac{1}{2}\pi\nu$ or $\sin\frac{1}{2}\pi\nu < 0$.

The solution of (34) requires $\kappa r^{1/\kappa}\cos(\phi/\kappa) = \Phi + 2k\pi$, $\Upsilon = 1$ to yield

$$\kappa r^{1/\kappa}\cos(\phi/\kappa) = (2k+\tfrac{1}{2})\pi + (\delta \mp \tfrac{1}{4})\pi + (\nu - \tfrac{1}{2})\frac{\phi}{\kappa},$$

$$\kappa r^{1/\kappa}\sin(\phi/\kappa) = -\log(\lambda r^{(\nu-\frac{1}{2})/\kappa}),$$

where $k = 0, 1, 2, \ldots$. If the parameter ν is such that $|\phi| \ll 1$, then we find approximately

$$\kappa r^{1/\kappa} \simeq (2k+\tfrac{1}{2})\pi + (\delta \mp \tfrac{1}{4})\pi, \qquad k = 0, 1, 2, \ldots, \tag{35}$$

$$\phi \simeq -\kappa \arcsin\left\{\frac{\log(\lambda r^{(\nu-\frac{1}{2})/\kappa})}{\kappa r^{1/\kappa}}\right\}, \tag{36}$$

where the upper or lower sign corresponds to $C_{n,1}(x;\nu)$ or $S_{n,1}(x;\nu)$, respectively. The asymptotic distribution of the complex zeros is then obtained from (33).

[§]When ν is an odd (resp. even) integer and n is odd the leading term in the algebraic expansion in (28) corresponds to $k = 1$. The modification required in this case is easily carried out.

4.3. Numerical results

The zeros of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ have been calculated by means of the secant method in *Mathematica* applied to the combinations $U_{n,1}(ix;\nu) \pm U_{n,1}(-ix;\nu)$ in (27). The function $U_{n,1}(z;\nu)$ was computed by suitable truncation of its series representation in (8) and asymptotic estimates obtained from (33), (35) and (36) for the complex zeros and (31) for the real zeros were employed to initiate the process. The complex zeros, together with their asymptotic approximations, are presented in Tables 2 and 3 for $n = 4$ and $n = 5$ and different values of ν. It will be observed that these zeros arise in conjugate pairs (when ν is real) situated near the anti-Stokes lines $\arg x = \pm\pi/(2n)$. It should also be noted that as ν increases some real zeros are present; this is discussed more fully at the end of this section.

Table 2: The complex zeros x_k of $C_{n,1}(x;\nu)$ in the right-half plane for different n and ν.

	$n=4,\ \nu=\frac{1}{2}$		$n=4,\ \nu=\frac{2}{3}$	
k	x_k	Asymptotic x_k	x_k	Asymptotic x_k
0	$3.1041 \pm 1.6890i$	$3.0410 \pm 1.6533i$	$3.2753 \pm 1.1203i$	$3.2777 \pm 1.1125i$
1	$6.4566 \pm 2.9845i$	$6.4330 \pm 2.9742i$	$6.6574 \pm 2.4767i$	$6.6433 \pm 2.4690i$
2	$9.2953 \pm 4.1250i$	$9.2821 \pm 4.1194i$	$9.4904 \pm 3.6411i$	$9.4816 \pm 3.6367i$
3	$11.8692 \pm 5.1697i$	$11.8604 \pm 5.1660i$	$12.0592 \pm 4.7025i$	$12.0528 \pm 4.6995i$
4	$14.2687 \pm 6.1487i$	$14.2622 \pm 6.1459i$	$14.4544 \pm 5.6941i$	$14.4494 \pm 5.6919i$
5	$16.5406 \pm 7.0783i$	$16.5354 \pm 7.0761i$	$16.7225 \pm 6.6341i$	$16.7184 \pm 6.6322i$
	$n=4,\ \nu=\frac{3}{2}$		$n=5,\ \nu=\frac{1}{2}$	
k	x_k	Asymptotic x_k	x_k	Asymptotic x_k
0	1.8582	$1.0126 \pm 0.2153i$	$3.3130 \pm 1.6108i$	$3.2058 \pm 1.5728i$
1	$5.3221 \pm 0.6644i$	$5.3306 \pm 0.7204i$	$7.1928 \pm 2.7755i$	$7.1544 \pm 2.7627i$
2	$8.5087 \pm 1.9237i$	$8.4553 \pm 1.9032i$	$10.6035 \pm 3.8433i$	$10.5819 \pm 3.8362i$
3	$11.2300 \pm 3.0161i$	$11.1794 \pm 2.9958i$	$13.7597 \pm 4.8439i$	$13.7450 \pm 4.8391i$
4	$13.7215 \pm 4.0352i$	$13.6754 \pm 4.0164i$	$16.7445 \pm 5.7959i$	$16.7335 \pm 5.7923i$
5	$16.0595 \pm 4.9977i$	$16.0161 \pm 4.9817i$	$19.6017 \pm 6.7107i$	$19.5931 \pm 6.7078i$

When n is even and ν is odd (resp. even), the zeros of $C_{n,1}(x;\nu)$ (resp. $S_{n,1}(x;\nu)$) are all real; see the appendix. The zeroth-order approximation $x_k^{(0)}$ for these zeros is given by the first equation in (31) with $\epsilon = \frac{1}{2}$ (resp. 1). For example, when $n = 4$, we find $\kappa = \frac{3}{4}$ and $\vartheta = \frac{1}{4}\nu - \frac{1}{2}$, so that

$$x_k^{(0)} = \left(\frac{8\pi}{3\sqrt{3}}(k + \epsilon + \tfrac{1}{3} - \tfrac{1}{6}\nu)\right)^{3/4} \qquad (k = 0, 1, 2, \dots).$$

The first-order approximation $x_k^{(1)}$ is described by the second equation in (31), with the coefficient c_1 obtained from (15). The calculation of the real zeros of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ is presented in Table 4.

Table 3: The complex zeros x_k of $S_{n,1}(x;\nu)$ in the right-half plane for different n and ν.

k	$n=4,\ \nu=\frac{1}{2}$ x_k	Asymptotic x_k	$n=4,\ \nu=1$ x_k	Asymptotic x_k
0	$4.0244\pm 2.0323i$	$3.9764\pm 2.0087i$	$4.2375\pm 1.3161i$	$4.2550\pm 1.3196i$
1	$7.1994\pm 3.2811i$	$7.1796\pm 3.2726i$	$7.4819\pm 2.5264i$	$7.4758\pm 2.5234i$
2	$9.9589\pm 4.3936i$	$9.9471\pm 4.3886i$	$10.2567\pm 3.6293i$	$10.2497\pm 3.6261i$
3	$12.4831\pm 5.4199i$	$12.4750\pm 5.4164i$	$12.7869\pm 4.6542i$	$12.7801\pm 4.6512i$
4	$14.8473\pm 6.3852i$	$14.8412\pm 6.3826i$	$15.1536\pm 5.6211i$	$15.1471\pm 5.6183i$
5	$17.0930\pm 7.3047i$	$17.0872\pm 7.3022i$	$17.3992\pm 6.5430i$	$17.3930\pm 6.5404i$
k	$n=4,\ \nu=\frac{3}{2}$ x_k	Asymptotic x_k	$n=5,\ \nu=\frac{1}{2}$ x_k	Asymptotic x_k
0	$4.0787,\ 4.6474$	$4.4355\pm 0.4157i$	$4.3501\pm 1.9128i$	$4.2766\pm 1.8857i$
1	$7.7747\pm 1.6364i$	$7.7229\pm 1.6162i$	$8.0767\pm 3.0502i$	$8.0444\pm 3.0395i$
2	$10.5755\pm 2.7507i$	$10.5238\pm 2.7302i$	$11.4121\pm 4.0989i$	$11.3927\pm 4.0925i$
3	$13.1152\pm 3.7862i$	$13.0680\pm 3.7670i$	$14.5199\pm 5.0859i$	$14.5063\pm 5.0815i$
4	$15.4858\pm 4.7625i$	$15.4430\pm 4.7448i$	$17.4695\pm 6.0278i$	$17.4592\pm 6.0244i$
5	$17.7038\pm 5.6866i$	$17.6938\pm 5.6766i$	$20.2996\pm 6.9344i$	$20.2913\pm 6.9317i$

The manner in which the zeros change as ν increases is shown in Fig. 3(a) for the case of $C_{n,1}(x;\nu)$ when $n=4$; a similar behaviour applies to $S_{n,1}(x;\nu)$. This figure shows the first complex zeros x_0 and x_1 (and their conjugates) for values of ν increasing from 0.1 to 1 in steps of 0.1. As ν increases, the zeros approach the real axis and eventually coalesce to form real zeros. This is found to occur for $\nu \doteq 0.8216$ in the case of x_0 and $\nu \doteq 0.9875$ in the case of x_1. The zeros labelled A, B, C, D indicate the zeros when $\nu=1$; the next real zero in the sequence when $\nu=1$ (which results from the coalesence of x_2 and its conjugate) is labelled E. The remaining complex zeros exhibit a cascade effect since they all progressively coalesce to become real as ν increases in the interval $(0.9875, 1]$. An alternative depiction of the zeros as ν increases in the interval $[\frac{1}{2}, 1]$ is shown in Fig. 4.

As ν increases beyond the value $\nu=1$, the zeros labelled B, C and D, E in Fig. 4(a) approach one another, coalesce and then move off into the complex plane as new conjugate pairs. The loci of these complex zeros (in the upper half-plane) are indicated in Fig. 3(b). As ν continues to increase these loci form loops that return to the real axis, resulting in coalescence and the formation of real zeros again. The loop formed by B and C exists for ν in the interval $(1.0853, 1.8733)$ and that formed by D and E exists in the interval $(1.0025, 2.6560)$. A similar behaviour is exhibited by the other zeros with the result that when $\nu=3$, all the zeros are again real. This pattern then repeats itself for ν in the intervals $[3,5]$, $[5,7]$ and so on. It then becomes clear from this discussion that the finite number of real zeros of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ for a given value of ν (apart from odd or even integer values) is difficult to predict.

In the case of odd integer n we find a similar behaviour of the zeros. Fig. 5 shows the distribution of the zeros of $C_{n,1}(x;\nu)$ when $n=3$ for ν in the interval $[\frac{1}{2}, 1]$. When $\nu=\frac{1}{2}$, the zeros lie close to the anti-Stokes lines $\arg x = \pm\pi/6$. As ν increases, the first complex zero

Table 4: The real zeros x_k of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ on the positive axis, together with their zeroth and first-order approximations, for different even n and integer ν. The corresponding value of the coefficient c_1 is given.

	$n=4,\ \nu=1,\ c_1=\frac{7}{48}$ $C_{n,1}(x;\nu)$			$n=6,\ \nu=1,\ c_1=\frac{11}{36}$ $C_{n,1}(x;\nu)$		
k	x_k	$x_k^{(0)}$	$x_k^{(1)}$	x_k	$x_k^{(0)}$	$x_k^{(1)}$
0	2.441968	2.4063	2.4512	2.500814	2.3407	2.4517
1	4.797244	4.7842	4.7985	4.932583	4.9032	4.9427
2	6.813581	6.8060	6.8140	7.232399	7.2095	7.2326
3	8.647288	8.6422	8.6475	9.389764	9.3743	9.3903
4	10.359390	10.3556	10.3595	11.454280	11.4425	11.4545
5	11.981848	11.9788	11.9819	13.447433	13.4380	13.4476
	$n=4,\ \nu=2,\ c_1=-\frac{5}{48}$ $S_{n,1}(x;\nu)$			$n=6,\ \nu=2,\ c_1=-\frac{1}{36}$ $S_{n,1}(x;\nu)$		
k	x_k	$x_k^{(0)}$	$x_k^{(1)}$	x_k	$x_k^{(0)}$	$x_k^{(1)}$
0	3.246903	3.2615	3.2421	3.446131	3.4114	3.4054
1	5.478116	5.4852	5.4770	5.843645	5.8473	5.8445
2	7.430167	7.4346	7.4297	8.088659	8.0892	8.0874
3	9.221748	9.2249	9.2215	10.210708	10.2115	10.2102
4	10.903062	10.9055	10.9029	12.247705	12.2484	12.2474
5	12.501541	12.5035	12.5015	14.218715	14.2193	14.2185

and its conjugate coalesce to form a pair of real zeros, followed by the second complex zero and its conjugate, with the other zeros remaining complex when $\nu = 1$. The real zero with the greatest real part moves off to infinity as $\nu \to 1$, with the result that when $\nu = 1$ there are just 3 real zeros[¶] together with an infinite string of complex zeros and their conjugates. As ν increases further, more real zeros can form but their number always remains finite.

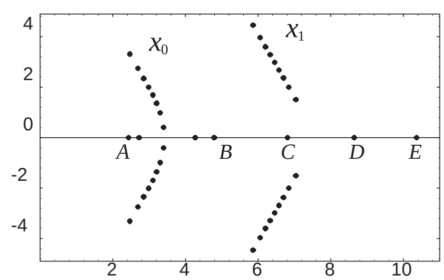
5. The Case $p \geq 2$

For general real ν, an infinite string of complex zeros of $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ will be situated in the right-half plane near the anti-Stokes lines arg $x = \pm\pi p/(2n)$. In the neighbourhood of arg $x = \pi p/(2n)$ we have from (25) and (26)

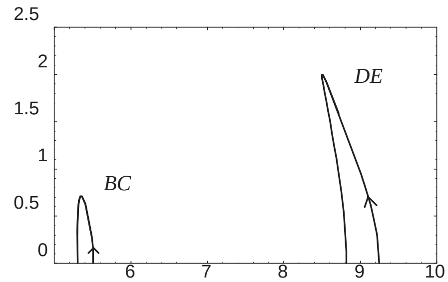
$$\begin{matrix} C_{n,p} \\ S_{n,p} \end{matrix}(x;\vec{\nu}) \sim H_{c,s} \pm \xi E_- \tag{37}$$

for large $|x|$, where the exponential expansion E_- is defined in (21) and, in the simplest situation where the parameters ν_r do not coincide or differ by integer multiples of n, the

[¶]The statement made in [5, p. 69] that $C_{n,1}(x;1)$ has no real zeros when n is odd is seen to be incorrect.



(a) The zeros x_0 and x_1 and their conjugates of $C_{n,1}(x;\nu)$ when $n=4$ and $\nu = 0.1(0.1)1$. The zeros labelled A, B, C, D and E indicate the real zeros when $\nu = 1$.



(b) The loci in the upper-half plane of the zeros B, C and D, E after coalescence. The arrows indicate the sense of increasing ν.

Figure 3

algebraic expansions $H_{c,s}$ are defined by (22) and (23). We recall that $\xi = 2^{-1}$ for $C_{n,p}(x;\vec{\nu})$ and $\xi = (2i)^{-1}$ for $S_{n,p}(x;\vec{\nu})$. If one of these parameters, say ν_1, is much smaller than the others, then the term containing $(n^{p/n}x)^{-\nu_1}$ will be the dominant term in $H_{c,s}$ as $|x| \to \infty$ and a similar procedure to that described in Section 4.2 for the case $p = 1$ can be followed. If, on the other hand, the ν_r are comparable then the complex zeros can be estimated by direct solution of the leading-order form of (37). To illustrate, we consider the case $p = 2$ with $\nu_1 = \frac{1}{2}$ and $\nu_2 = \frac{3}{2}$. Then (37) to leading order yields

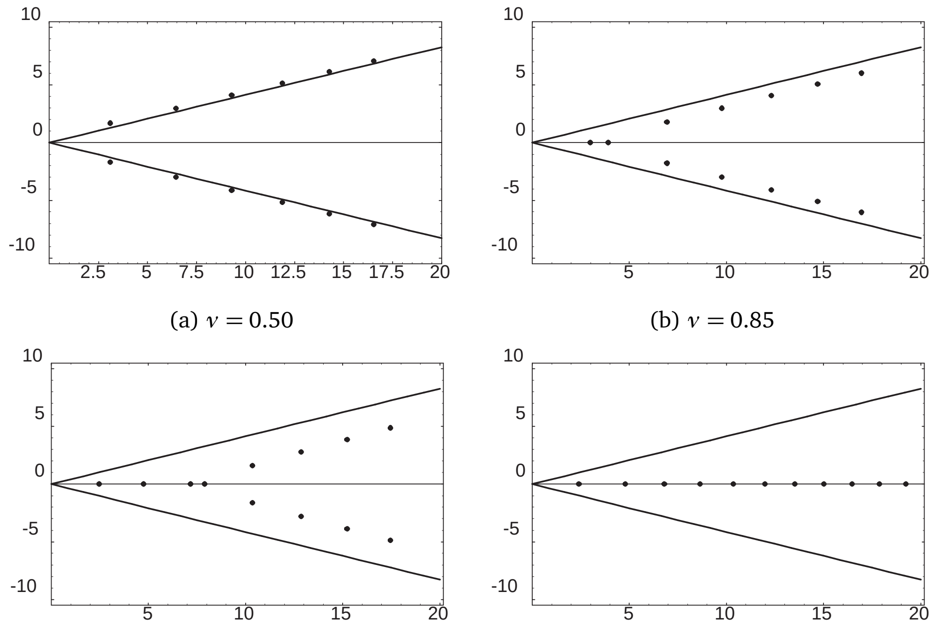
$$n^{\vartheta}\left\{(n^{2/n}x)^{-\frac{1}{2}}\Gamma(\tfrac{1}{2})\Gamma(1/n)\frac{\cos}{\sin}(\tfrac{1}{4}\pi) + (n^{2/n}x)^{-\frac{3}{2}}\Gamma(\tfrac{3}{2})\Gamma(-1/n)\frac{\cos}{\sin}(\tfrac{3}{4}\pi)\right\}$$
$$\pm \frac{2\pi\xi}{n\kappa^{\frac{1}{2}}}(-ix)^{\vartheta/\kappa}\exp(Xe^{-\pi i/(2\kappa)}) = 0. \tag{38}$$

Solution of this equation can be carried out using the secant method in *Mathematica*.

When $p \geq 2$, it becomes possible to encounter a more complicated structure for the algebraic expansion. When some of the ν_r either coincide or differ by integer multiples of n, some of the poles in the integrand of (10) are of higher order and logarithmic terms can appear. This is discussed fully in [12, §3.5]. As an example, we let $n = 6$, $p = 2$ and consider the two cases $(\nu_1, \nu_2) = (1, 1)$ and $(\nu_1, \nu_2) = (1, 7)$. For the first case all the poles in (10) in $\mathrm{Re}(s) < 0$ are double, whereas for the second case the pole at $s = -1$ is simple with those at $s = -7-6k$ $(k = 0, 1, 2, \ldots)$ being double. If we write $\nu_{1,2} \equiv a \pm 3m$, where $m = 0, 1, 2, \ldots$, we have $a = 1$, $m = 0$ for the first case and $a = 4$, $m = 1$ for the second case[‖]. From [12, p. 81], the algebraic expansion of $U_{6,2}(z;\vec{\nu})$ in (22) when $\nu_{1,2} = a \pm 3m$ becomes

$$H(z) = 6(6^{\frac{1}{3}}z)^{-a+3m}\sum_{k=0}^{m-1}\frac{(-)^k}{k!}\Gamma(a-3m+6k)\Gamma(m-k)(6^{\frac{1}{3}}z)^{-6k}$$

[‖]In terms of the differential equation (5) we have the coefficients $a_0 = 1$, $a_1 = 3$ for the first case and $a_0 = 7$, $a_1 = 9$ for the second case.



(a) $\nu = 0.50$ (b) $\nu = 0.85$

(c) $\nu = 0.99$ (d) $\nu = 1$

Figure 4: The distribution of the zeros of $C_{n,1}(x;\nu)$ in the right-half plane when $n = 4$ and $\nu = [0.50, 0.85, 0.99, 1]$. The rays $\arg x = \pm\pi/8$ are the anti-Stokes lines.

$$+(-)^m 6(6^{\frac{1}{3}}z)^{-a-3m}\sum_{k=0}^{\infty}\frac{\Gamma(a+3m+6k)}{k!(k+m)!}(6^{\frac{1}{3}}z)^{-6k}$$
$$\times\{6\log(6^{\frac{1}{3}}z)+\psi(k+1)+\psi(k+m+1)-6\psi(a+3m+6k)\},$$

where ψ denotes the psi function and the first sum is interpreted as zero when $m = 0$. Then, some routine algebra shows that the algebraic expansions associated with $C_{6,2}(x;\vec{\nu})$ and $S_{6,2}(x;\vec{\nu})$ are

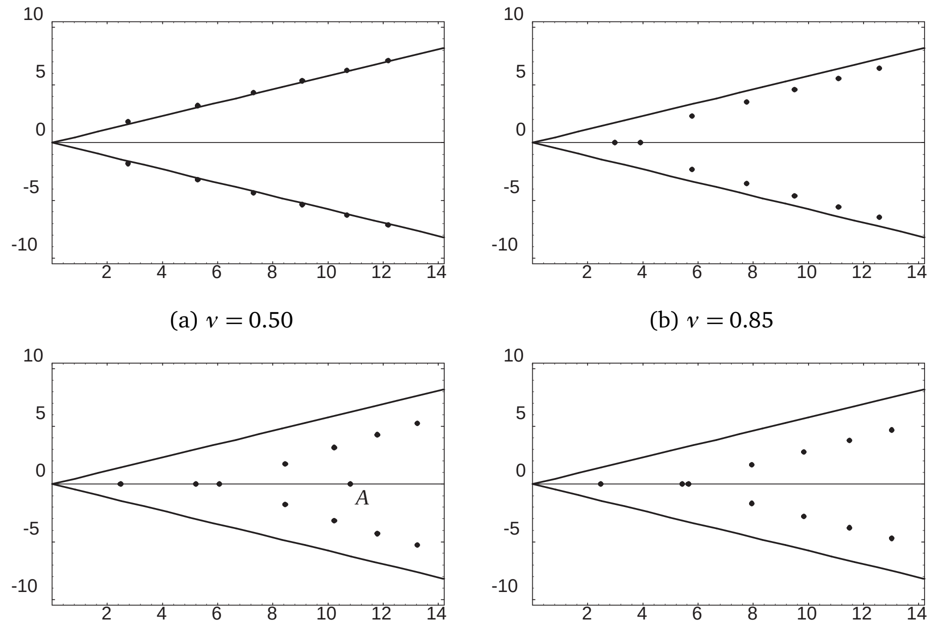
$$H_c = \frac{\pi}{2x}\sum_{k=0}^{\infty}\frac{(-)^k(6k)!}{(k!)^2}(6^{\frac{1}{3}}x)^{-6k}, \tag{39}$$

$$H_s = \frac{1}{x}\sum_{k=0}^{\infty}\frac{(-)^k(6k)!}{(k!)^2}(6^{\frac{1}{3}}x)^{-6k}\{\log(6^{\frac{1}{3}}x)+\tfrac{1}{3}\psi(k+1)-\psi(6k+1)\} \tag{40}$$

when $(\nu_1, \nu_2) = (1, 1)$, and

$$H_c = \frac{\pi}{12x^7}\sum_{k=0}^{\infty}\frac{(-)^k(6k+6)!}{k!\,(k+1)!}(6^{\frac{1}{3}}x)^{-6k}, \tag{41}$$

$$H_s = \frac{1}{x}+\frac{1}{6x^7}\sum_{k=0}^{\infty}\frac{(-)^k(6k+6)!}{k!\,(k+1)!}(6^{\frac{1}{3}}x)^{-6k}\{\log(6^{\frac{1}{3}}x)+\tfrac{1}{6}\psi(k+1)+\tfrac{1}{6}\psi(k+2)-\psi(6k+7)\} \tag{42}$$



(a) $\nu = 0.50$ (b) $\nu = 0.85$

(c) $\nu = 0.999$ (d) $\nu = 1$

Figure 5: The distribution of the zeros of $C_{n,1}(x;\nu)$ in the right-half plane when $n = 3$ and $\nu = [0.50, 0.85, 0.999, 1]$. In (c) the zero labelled A moves off to infinity as $\nu \to 1$. The rays $\arg x = \pm\pi/6$ are the anti-Stokes lines.

when $(\nu_1, \nu_2) = (1, 7)$. The leading terms in these expansions can then be used in (38) to estimate the corresponding zeros. We show some results for the complex zeros when $n = 4$ and $n = 6$ with $p = 2$ in Table 5.

We now consider the conditions for $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ to have all real zeros**. From (25), such zeros can only occur when $\frac{1}{2} < \kappa < 1$ (that is, when $p/n < \frac{1}{2}$), since when $\kappa < \frac{1}{2}$ the expansions in (26) on the real axis are purely algebraic with no exponentially small contribution. The special case $\kappa = \frac{1}{2}$, where there can be finitely many real zeros, is discussed below. When $\kappa > \frac{1}{2}$, an infinite sequence of real zeros will arise when the expansions $H_{c,s}$ vanish. From (22) and (23), this will only occur *when n is even and the ν_r are distinct odd (resp. even) integers which do not differ by integer multiples of n* (Condition A). From (24) and (25), we then find for n even and the parameters ν_r satisfying the above condition that

$$\begin{matrix} C_{n,p} \\ S_{n,p} \end{matrix}(x;\vec{\nu}) \sim \kappa^{-\frac{1}{2}-\vartheta}\left(\frac{2\pi}{n}\right)^{p/2} X^{\vartheta} \exp\left(X\cos\frac{\pi}{2\kappa}\right)\sum_{j=0}^{\infty} c_j X^{-j} \begin{matrix} \cos \\ \sin \end{matrix}\left(X\sin\frac{\pi}{2\kappa} + \frac{\pi}{2\kappa}(\vartheta - j)\right)$$

as $|x| \to \infty$ in the sector $|\arg x| < \pi(\frac{1}{2} - p/n)$. If some of the integer ν_r either coincide or differ by an integer multiple of n, then the algebraic expansion will not vanish — compare the expansions in (39) – (42) — and complex zeros will arise. In this latter case, it is still

**This is a conjecture as we have no proof that integrals with $p \geq 2$ can have all real zeros.

Table 5: The complex zeros x_k in the right-half plane when $p = 2$ for different n and $\vec{\nu}$.

	$S_{4,2}(x;\vec{\nu})$, $(\nu_1,\nu_2)=(\frac{1}{2},\frac{3}{2})$		$C_{6,2}(x;\vec{\nu})$, $(\nu_1,\nu_2)=(\frac{1}{2},\frac{3}{2})$	
k	x_k	Asymptotic x_k	x_k	Asymptotic x_k
0	$2.2338 \pm 2.6142i$	$2.2188 \pm 2.6008i$	$2.7381 \pm 2.5479i$	$2.6624 \pm 2.5189i$
1	$3.3260 \pm 3.6474i$	$3.3209 \pm 3.6428i$	$5.1859 \pm 3.8429i$	$5.1620 \pm 3.8322i$
2	$4.1507 \pm 4.4378i$	$4.1480 \pm 4.4354i$	$7.1864 \pm 4.9344i$	$7.1735 \pm 4.9282i$
3	$4.8405 \pm 5.1042i$	$4.8388 \pm 5.1026i$	$8.9491 \pm 5.9093i$	$8.9406 \pm 5.9051i$
4	$5.4454 \pm 5.6916i$	$5.4442 \pm 5.6905i$	$10.5570 \pm 6.8058i$	$10.5509 \pm 6.8027i$
5	$5.9905 \pm 6.2230i$	$5.9896 \pm 6.2221i$	$12.0532 \pm 7.6444i$	$12.0484 \pm 7.6419i$
	$C_{6,2}(x;\vec{\nu})$, $(\nu_1,\nu_2)=(1,1)$		$S_{6,2}(x;\vec{\nu})$, $(\nu_1,\nu_2)=(1,1)$	
k	x_k	Asymptotic x_k	x_k	Asymptotic x_k
0	$3.0327 \pm 2.2880i$	$2.9696 \pm 2.2521i$	$3.4932 \pm 2.7751i$	$3.4448 \pm 2.7529i$
1	$5.4610 \pm 3.5587i$	$5.4396 \pm 3.5468i$	$5.8058 \pm 4.0221i$	$5.7869 \pm 4.0128i$
2	$7.4473 \pm 4.6490i$	$7.4355 \pm 4.6424i$	$7.7366 \pm 5.0838i$	$7.7257 \pm 5.0782i$
3	$10.7994 \pm 6.5263i$	$10.7937 \pm 6.5231i$	$9.4550 \pm 6.0386i$	$9.4477 \pm 6.0348i$
4	$12.2888 \pm 7.3683i$	$12.2844 \pm 7.3658i$	$11.0314 \pm 6.9204i$	$11.0260 \pm 6.9175i$
5	$13.6931 \pm 8.1648i$	$13.6896 \pm 8.1628i$	$12.5033 \pm 7.7475i$	$12.4991 \pm 7.7452i$

possible[††] to have some real zeros in addition to the complex zeros situated near the anti-Stokes lines $\arg x = \pm\pi p/(2n)$.

The procedure for the calculation of the real zeros follows that described in Section 4.1 for the case $p = 1$. For example, when $n = 6$, $p = 2$ ($\kappa = \frac{2}{3}$), we have the leading-order approximation from (31)

$$X^{(0)} = \sqrt{2}\left(k + \epsilon + \tfrac{3}{4} - \tfrac{1}{8}(\nu_1 + \nu_2)\right)\pi, \qquad X = \tfrac{2}{3}x^{3/2},$$

where $\epsilon = \frac{1}{2}$ for $C_{n,p}(x;\vec{\nu})$ and $\epsilon = 1$ for $S_{n,p}(x;\vec{\nu})$. The first-order approximation $X^{(1)}$ can be similarly computed according to (31); typical results are shown in Table 6.

Finally, we briefly discuss the case of even n and odd (resp. even) integer values of ν_r when $\kappa = \frac{1}{2}$ (that is, when $p = \frac{1}{2}n$). Although the functions $C_{n,\frac{1}{2}n}(x;\vec{\nu})$ and $S_{n,\frac{1}{2}n}(x;\vec{\nu})$ are also exponentially small as $x \to \pm\infty$ when the ν_r satisfy Condition A, it transpires that they can be evaluated as polynomials multiplied by $\exp(-x^2/2)$ and so possess finitely many real zeros. This situation may be compared with the case $n = 2$, $p = 1$ in (32), where $C_{2,1}(x;\nu)$ (resp. $S_{2,1}(x;\nu)$) for odd (resp. even) integer ν is expressible in terms of Hermite polynomials.

To show this, we consider only the case of $C_{n,\frac{1}{2}n}(x;\vec{\nu})$; the treatment of $S_{n,\frac{1}{2}n}(x;\vec{\nu})$ is

[††]For example, the function $C_{6,2}(x;\vec{\nu})$ with $\vec{\nu} = (1,7)$ has 4 positive real zeros.

Table 6: The real zeros x_k and their approximations on the positive axis when $p = 2$ for different even n and integer $\vec{\nu}$ satisfying Condition A. The corresponding value of the coefficient c_1 is given.

	$(\nu_1, \nu_2) = (1, 3)$, $c_1 = \frac{5}{36}$ $C_{6,2}(x; \vec{\nu})$			$(\nu_1, \nu_2) = (3, 5)$ $c_1 = -\frac{7}{36}$ $C_{6,2}(x; \vec{\nu})$		
k	x_k	$x_k^{(0)}$	$x_k^{(1)}$	x_k	$x_k^{(0)}$	$x_k^{(1)}$
0	2.945831	2.9233	2.9477	1.283599	1.4054	1.2535
1	5.150494	5.1428	5.1506	4.092473	4.1094	4.0921
2	6.955470	6.9512	6.9555	6.072944	6.0808	6.0729
3	8.550716	8.5479	8.5507	7.765281	7.7701	7.7653
4	10.008981	10.0070	10.0090	9.288354	9.2917	9.2884
5	11.367831	11.3662	11.3678	10.694775	10.6974	10.6948
	$(\nu_1, \nu_2) = (2, 4)$ $c_1 = -\frac{7}{36}$ $S_{6,2}(x; \vec{\nu})$			$(\nu_1, \nu_2) = (4, 6)$, $c_1 = \frac{5}{36}$ $S_{6,2}(x; \vec{\nu})$		
k	x_k	$x_k^{(0)}$	$x_k^{(1)}$	x_k	$x_k^{(0)}$	$x_k^{(1)}$
0	3.524152	3.5414	3.5181	2.254113	2.2309	2.2726
1	5.613666	5.6216	5.6123	4.648327	4.6405	4.6502
2	7.361496	7.3663	7.3610	6.527559	6.5233	6.5281
3	8.920297	8.9237	8.9200	8.166473	8.1636	8.1667
4	10.352462	10.3550	10.3523	9.654715	9.6526	9.6549
5	11.691308	11.6933	11.6912	11.035942	11.0343	11.0360

similar. From (20) when $p = \frac{1}{2}n$, we find

$$C_{n,\frac{1}{2}n}(x; \vec{\nu}) = \pi^{\frac{1}{2}} n^{\vartheta - p/2} \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}nx^2)^k}{k!\,\Gamma(k+\frac{1}{2})} \prod_{r=1}^{p} \Gamma\left(\frac{2k+\nu_r}{n}\right).$$

Application of the multiplication formula for the gamma function

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}(1-m)} m^{mz-\frac{1}{2}} \prod_{r=0}^{m-1} \Gamma(z + \frac{r}{m}), \qquad (m = 2, 3, \ldots)$$

with $m = \frac{1}{2}n$ to the factor $\Gamma(k + \frac{1}{2})$, then leads to the representation

$$\hat{C}_{n,\frac{1}{2}n}(x; \vec{\nu}) \equiv \frac{2^{\frac{1}{2}} C_{n,\frac{1}{2}n}(x; \vec{\nu})}{(2\pi)^{p/2} n^{\vartheta - p/2}} = \sum_{k=0}^{\infty} \frac{\Xi(k)}{k!} (-\tfrac{1}{2}x^2)^k, \tag{43}$$

where

$$\Xi(k) = \prod_{r=1}^{p} \frac{\Gamma\left(\frac{2k}{n} + \frac{\nu_r}{n}\right)}{\Gamma\left(\frac{2k}{n} + \frac{2r-1}{n}\right)}.$$

Whenever the ν_r are distinct odd integers such that $\Xi(k)$ reduces to a polynomial in k, the sum in (43) may be evaluated in closed form in terms of derivatives of $\exp(-x^2/2)$. For example, in the particular case $n = 4$, $p = 2$, where

$$\Xi(k) = \frac{\Gamma(\frac{1}{2}k + \frac{1}{4}\nu_1)\,\Gamma(\frac{1}{2}k + \frac{1}{4}\nu_2)}{\Gamma(\frac{1}{2}k + \frac{1}{4})\,\Gamma(\frac{1}{2}k + \frac{3}{4})},$$

we see that when ν_1, ν_2 are distinct odd integers whose difference is not a multiple of 4, $\Xi(k)$ reduces to a polynomial in k. The degree of this polynomial depends on $\vec{\nu}$: when $(\nu_1, \nu_2) = (1, 3)$ we have $\Xi(k) = 1$, when $(\nu_1, \nu_2) = (1, 7)$ we have $\Xi(k) = \frac{1}{2}k + \frac{3}{4}$, when $(\nu_1, \nu_2) = (1, 11)$ we have $\Xi(k) = (\frac{1}{2}k + \frac{3}{4})(\frac{1}{2}k + \frac{7}{4})$, and so on. Thus we find[‡‡]

$$\begin{aligned}
\hat{C}_{4,2}(x; (1,3)) &= e^{-x^2/2} \\
\hat{C}_{4,2}(x; (1,7)) &= \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}x^2)^k}{k!}(\tfrac{1}{2}k + \tfrac{3}{4}) = \tfrac{1}{4}(3 - x^2)\,e^{-x^2/2} \\
\hat{C}_{4,2}(x; (1,11)) &= \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}x^2)^k}{k!}(\tfrac{1}{2}k + \tfrac{3}{4})(\tfrac{1}{2}k + \tfrac{7}{4}) = \tfrac{1}{16}(x^4 - 12x^2 + 21)\,e^{-x^2/2}.
\end{aligned}$$

When ν_1, ν_2 are odd integers that either coincide or differ by a multiple of 4, $\Xi(k)$ contains a gamma function in the numerator. It follows from the asymptotic theory of the Wright function [18, 2, 10] that the large-x behaviour of $C_{4,2}(x; \vec{\nu})$ must then contain a non-vanishing algebraic component, with the result that there will be infinite strings of complex zeros in this case.

6. Concluding Remarks

The asymptotic expansion of the integrals $C_{n,p}(x; \vec{\nu})$ and $S_{n,p}(x; \vec{\nu})$ defined in (3) for large complex x has been obtained by application of the asymptotic theory of a particular case of the Wright function. The case corresponding to $p = 1$, where the integrals are one-dimensional Fourier integrals, extends the results of previous authors. The zeros of $C_{n,p}(x; \vec{\nu})$ and $S_{n,p}(x; \vec{\nu})$ have been considered which, in general, are found to lie in infinite strings in the complex plane situated near the anti-Stokes lines $\arg x = \pm\pi p/(2n)$ in the right-half plane, with a symmetrical distribution in the left-half plane.

An infinite sequence of real zeros of $C_{n,p}(x; \vec{\nu})$ (resp. $S_{n,p}(x; \vec{\nu})$) is found to occur only when n is even, $p/n < \frac{1}{2}$ and the parameters ν_r $(1 \le r \le p)$ are distinct odd (resp. even) integers which do not differ by integer multiples of n. In this case, the integrals in (3) may also be written over doubly infinite intervals in the form

$$C_{n,p}(x; \vec{\nu}) = 2^{-p} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} t_1^{2m_1} \ldots t_p^{2m_p}\, e^{ixt_1 \ldots t_p} \exp\left[-(t_1^n + \cdots + t_p^n)/n\right] dt_1 \ldots dt_p,$$

[‡‡]We employ the result $\sum_{k=0}^{\infty} k^r (-z)^k/k! = (-)^r (z\,d/dz)^r e^{-z}$ for $r = 0, 1, 2, \ldots$.

with $\nu_r = 2m_r + 1$, and

$$iS_{n,p}(x;\vec{\nu}) = 2^{-p}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} t_1^{2m_1+1}\ldots t_p^{2m_p+1}\, e^{ixt_1\ldots t_p}\,\exp\left[-(t_1^n+\cdots+t_p^n)/n\right]dt_1\ldots dt_p,$$

with $\nu_r = 2m_r + 2$, where m_r $(1 \le r \le p)$ are distinct nonnegative integers which do not differ by a multiple of $\frac{1}{2}n$.

Finally, from (9), we remark that the integrals $C_{n,p}(x;\vec{\nu})$ and $S_{n,p}(x;\vec{\nu})$ are the even and odd solutions of the nth-order differential equation (5), with the upper or lower sign chosen according as $\frac{1}{2}n$ is even or odd,respectively, and the coefficients a_r given in terms of the ν_r by (6).

Appendix: The Zeros of $C_{n,1}(x;\nu)$ and $S_{n,1}(x;\nu)$ for Even n and Integer ν

Let $n = 1,2,\ldots$, $m = 0,1,2,\ldots$ and define

$$\psi_n(z) := \int_{-\infty}^{\infty}\exp(-t^{2n}/2n)e^{izt}\,dt$$

for complex z. Then $\psi_1(z) = \sqrt{2\pi}\exp(-z^2/2)$ has no zeros. In [14], Pólya proved that for $n \ge 2$, $\psi_n(z)$ has infinitely many zeros all of which are real. These results were extended in [7], where the following theorem was established:

Theorem 2. *For $k = 0,1,2,\ldots$ and $n = 1,2,\ldots$ all the zeros of $\psi_n^{(k)}(z)$ are real and simple.*

Then, from (2), some straightforward rearrangement shows that

$$C_{2n,1}(x;2m+1) = \frac{1}{2}\int_{-\infty}^{\infty} t^{2m}\exp(-t^{2n}/2n)e^{ixt}\,dt = \frac{(-)^m}{2}\,\psi_n^{(2m)}(x)$$

and

$$S_{2n,1}(x;2m+2) = \frac{1}{2i}\int_{-\infty}^{\infty} t^{2m+1}\exp(-t^{2n}/2n)e^{ixt}\,dt = \frac{(-)^m}{2}\,\psi_n^{(2m+1)}(x).$$

It then follows from the above theorem that when $n \ge 2$ and $m = 0,1,2,\ldots$ the zeros of $C_{2n,1}(x;2m+1)$ and $S_{2n,1}(x;2m+2)$ are all real and, moreover, simple.

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