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Koszul Duality for Multigraded Algebras

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Abstract. Classical Koszul duality sets up an adjoint pair of functors, establishing an equivalence $F : D^b(A) \leftrightarrows D^b(A^!) : G$, where A is a quadratic algebra, $A^!$ is the quadratic dual, and D^b refers to the bounded derived category of complexes of graded modules over the graded algebra (i.e., A or $A^!$). This duality can be extended in many ways. We consider here two extensions: first we wish to allow a Λ-graded algebra, where Λ is any abelian group (not just $\mathbb{Z}$). Second, we will allow filtered algebras. In fact we are considering filtered quadratic algebras with an (internal) Λ-grading.

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1. Introduction

Koszul duality originated from Bernstein-Gelfand-Gelfand [2] in the mid 1970s. It led to the observation that for certain pairs of associative algebras A and $A^!$, there is a relationship between the categories of A-modules and $A^!$-modules. An example of such a pair is $S = k[x_1, x_2, \ldots, x_n]$, the polynomial algebra over a field k, and $E = \bigwedge_k(e_1, e_2, \ldots, e_n)$, the exterior algebra over k.

Bernstein-Gelfand-Gelfand constructed an adjoint pair of functors between the categories of bounded chain complexes of S-modules and bounded chain complexes of E-modules which induce an equivalence of the corresponding derived categories. This means that problems of homological algebra for S-modules can be translated into problems of homological algebra for E-modules and vice-versa. A key fact underlying this is that the Koszul complex of S and E which is given by

$$\cdots \to S_i \otimes E_n \to S_{i+1} \otimes E_{n-1} \to \cdots$$

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with differential

$$d(f \otimes e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) = \sum_{j=1}^{m} (-1)^m x_{i_j} f(e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_m})$$

is acyclic in all degrees greater than zero. Given this fact we say that S and E are Koszul algebras. This theory has been generalized by a number of people to algebras defined by homogeneous relations. A general reference is [5] for background on quadratic algebras and Koszul duality. Consider the free k-algebra $k < x_1, x_2, \ldots, x_n >$ and define $A = k < x_1, x_2, \ldots, x_n > /R$, where R is the ideal generated by homogeneous quadratic relations of the form $\Sigma c_{ij} x_i x_j = 0$. In our example above, the algebra S is defined by the quadratic relation $x_i x_j - x_j x_i = 0$ and the algebra E is defined by $x_i x_j + x_j x_i = 0$ and $x_i^2 = 0$.

More generally S and E can be replaced by a pair of dual quadratic algebras A and $A^!$. Note that this relation is symmetric, $(A^!)^! = A$. Here A and $A^!$ are *quadratic* algebras meaning that each is the quotient of a free k-algebra by homogeneous quadratic relations. Given a pair of quadratic algebras we may form the Koszul complex given by

$$\ldots \to A \otimes (A_2^!)^* \to A \otimes (A_1^!)^* \to A$$

and we say that A is a Koszul algebra if the Koszul complex is exact in nonzero degrees. By symmetry, if A is Koszul then $A^!$ is also Koszul.

Koszul duality is a relation between the complexes of A-modules and the complexes of $A^!$-modules and this establishes an equivalence of categories

$$F : D^b(A) \leftrightarrows D^b(A^!) : G$$

where D^b refers to the bounded derived category of complexes of graded modules over the graded algebra A (or $A^!$). For $M \in C^b(A)$ where $C^b(A)$ is the category of bounded chain complexes of graded left A-modules, the functor F is given by

$$(FM)_q^p = \bigoplus_{\substack{p=i+j \\ q=l-j}} A_l^! \otimes M_j^i.$$

For $N \in C^b(A^!)$, the functor G is explicitly described as

$$(GN)_q^p = \bigoplus_{\substack{p=i+j \\ q=l-j}} \mathrm{Hom}_k(A_{-l}, N_j^i).$$

Another example concerns a nondegenerate quadratic form, Q, in variables $x_0, x_1, \ldots, x_n$. Let $A = k[x_0, x_1, \ldots, x_n]/Q = \mathrm{Sym}(V)/Q$ be the homogeneous coordinate ring of the quadric $Q = 0$ in projective space $\mathbf{P}_k^n$. The dual $A^!$ is the graded Clifford algebra attached to Q. This is generated by elements $\xi \in V^*$ of tensor degree 1 and an element h of tensor degree 2 with the relations

$$\xi\eta + \eta\xi = 2Q(\xi, \eta)h, \quad \xi, \eta \in V^*.$$

The ring A is proven to be a Koszul ring in [4].

Not every quadratic algebra is Koszul. A counterexample is

$$A = \bigwedge_k (x, y, z, w)/(xy + zw).$$

Using the software package Singular to calculate the minimal free resolution of A, we observed that quadratic entries appear within the matrices, violating the property that an algebra is Koszul if and only if k has a linear free A-module resolution.

If we now allow the relations, R, of an algebra $A = k < x_1, x_2, \dots, x_n > /R$ to be nonhomogeneous, A will no longer be a graded algebra, but rather a filtered algebra. The dual of A will no longer be the quadratic algebra $A^!$, but rather the curved differential graded algebra (CDGA) $(A^!, d, c)$. That is, we have a differential $d : (A^!)^n \to (A^!)^{n+1}$ with the usual property that $d(xy) = dx(y) \pm (x)dy$ but with $d^2(x) = [c, x] = cx \pm xc$, where $c \in (A^!_2)^*$ is the curvature. In the case where $c = 0$ we refer to a CDGA as just a DGA (differential graded algebra). Two canonical examples of DGAs are the Koszul complex and the deRham Complex. Let us consider an example that illustrates dualizing a nonhomogeneous quadratic algebra. Consider the algebra,

$$U = k < x, y > /P, \quad P = (x^2 - y, xy - yx).$$

The associated graded algebra to U will be $A = k < x, y > /(x^2, xy - yx)$ which is dual to $A^! = k < \xi, \eta > /(\eta^2, \xi\eta + \eta\xi)$ which can be seen to be a curved differential graded algebra $(A^!, d, c)$ with $c = 0$, $d\xi = 0$ and $d\eta = -\xi^2$.

A canonical example of a nonhomogeneous quadratic algebra which is filtered by tensor degree is $U = U\mathfrak{g}$, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. The dual of $U\mathfrak{g}$ is the Chevellay-Eilenberg complex which is a CDGA (more specifically it is a DGA since $c = 0$).

Given a filtered algebra, we can add a Λ-grading to it. A Λ-graded filtered algebra, U, is an algebra with a filtration, F_i, and also a grading

$$U = \bigoplus_{\lambda \in \Lambda} U_\lambda \quad \text{for some abelian group } \Lambda.$$

If we allow V to be Λ-graded, and P is Λ-homogeneous, then we know that

$$U = T(V)/P$$

will be filtered by tensor degree and Λ-graded. Two examples of Λ-graded algebras are the universal enveloping algebra of a semisimple Lie algebra, and the coordinate ring of a projective toric variety.

There is no obvious way to extend Koszul duality to a Λ-graded situation when $\Lambda \neq \mathbb{Z}$. Consider the functor F given by

$$(FM)^p_q = \bigoplus_{\substack{p=i+j \\ q=l-j}} A^!_l \otimes M^i_j.$$

This functor is well defined for the case $\Lambda = \mathbb{Z}$ but if A is Λ-graded then we would have $i \in \mathbb{Z}$ but $j \in \Lambda$. Now the addition $i + j$ no longer makes sense since i is a scalar and j is a vector. The key step to our extension of Koszul duality is considering $A^!_l$ not as a quadratic algebra but as a CDGA, meaning

$$A^!_l = (A^!)^l_\lambda \quad l \in \mathbb{Z}, \lambda \in \Lambda,$$

with the multiplication

$$(A^!)^l_\lambda \times (A^!)^m_\mu \longrightarrow (A^!)^{l+m}_{\lambda+\mu}.$$

The benefit of switching over from a quadratic algebra point of view to a CDGA point of view is that now we are allowed two new levels of freedom. First we may insert a Λ-grading and second we can now let A be a nonhomogeneous filtered algebra.

We define $Com_\Lambda(A^!, d, c)$ to be the category of curved differential graded modules over the CDGA $(A^!, d, c)$. We define $Com_\Lambda(U)$ to be the category of complexes of Λ-graded left U-modules. The version of Koszul duality that we are most interested in is that introduced by Floystad [3] concerning the pair of adjoint functors

$$F : Com_\Lambda(A^!, d, c) \leftrightarrows Com_\Lambda(U) : G.$$

Here U is a Λ-graded filtered quadratic algebra and $(A^!, d, c)$ is a curved differential graded algebra (cdga) which is dual to U. We have generalized the functors F and G for this Λ-graded situation, i.e.,

$$F(N)^p_\lambda = \bigoplus_{\mu+\nu=\lambda} U_\mu \otimes_k N^p_\nu$$

$$G(M)^p_\lambda = \prod_{r\geq 0}\prod_{\mu} \mathrm{Hom}_k((A^!)^r_\mu, M^{p+r}_{\lambda+\mu}).$$

Notice that now integers are added to integers and elements of Λ are added to other elements of Λ.

Let $K_\Lambda(A^!, d, c)$ be the category $Com_\Lambda(A^!, d, c)$ with morphisms being chain homotopy equivalence classes of maps. The *null system*, N, of $K_\Lambda(A^!, d, c)$ is defined to be all of the complexes, X, such that $F(X)$ is acyclic. A similar definition is given for the null system of $K_\Lambda(U)$. We define $D_\Lambda(A^!, d, c)$ to be the category $K_\Lambda(A^!, d, c)/N$ where N is the null system of $Com_\Lambda(A^!, d, c)$. We also define $D_\Lambda(U)$ to be the category $K_\Lambda(U)/N$ where N is the null system of $Com_\Lambda(U)$.

The main result of this paper is that the functors F and G given in Proposition 4 induce an equivalence of categories between the quotient categories $D_\Lambda(A^!, d, c)$ and $D_\Lambda(U)$.

2. Graded and Filtered Algebras

Floystad's Koszul duality concerns a pair of adjoint functors

$$F : Com_\Lambda(A^!, d, c) \leftrightarrows Com_\Lambda(U) : G$$

which induce an equivalence of homotopy categories. Here U is a Λ-graded filtered quadratic algebra, and $(A^!, d, c)$ is a curved differential graded algebra (CDGA) dual to U. Here $\mathrm{Com}_\Lambda(U)$ is the category of complexes of Λ-graded left U-modules (of finite type). We let $\mathrm{Com}_\Lambda(A^!, d, c)$ be the category of curved differential graded left modules over the CDGA $(A^!, d, c)$. When $c = 0$ these are just the usual differential graded algebras. It is important to note that this form of duality is not symmetrical.

We will explain how classical Koszul duality relates to Floystad's Koszul duality in the special case $U = A$ is a $\mathbb{Z}$-graded Koszul quadratic algebra. So first let us recall the definitions of some of the key terms we will be using.

Definition 1. *Let k be a field. A Λ-graded associative k-algebra A with unit is an algebra together with a decomposition into k-subspaces,*

$$A = \bigoplus_{\lambda \in \Lambda} A_\lambda$$

which obeys the multiplication law

$$A_\lambda \cdot A_\mu \subset A_{\lambda+\mu}.$$

Definition 2. *A Λ-graded filtered algebra U over a field k is a Λ-graded algebra which has an increasing sequence $0 \subset F_0 \subset F_1 \subset \cdots F_i \subset \cdots \subset U$ of k-subspaces of U such that*

$$U = \bigcup_{i \in \mathbb{N}} F_i$$

and the following property of the algebra multiplication holds:

$$\forall m, n \in \mathbb{N},\ F_m \cdot F_n \subset F_{n+m}.$$

In addition, the grading must be compatible with the filtration, meaning

$$F_i = \bigoplus_{\lambda \in \Lambda} (F_i \cap U_\lambda) \text{ given the grading } U = \bigoplus_{\lambda \in \Lambda} U_\lambda.$$

Definition 3. *If U is a Λ-graded ring, then a Λ-graded module M is a left U-module with a decomposition*

$$M = \bigoplus_{\lambda \in \Lambda} M_\lambda$$

into k-subspaces such that $U_\lambda \cdot M_\mu \subset M_{\lambda+\mu}$.

Remark 1. *We allow the case where there is no Λ-grading in which case the algebra (resp. module) U is just considered a filtered algebra (resp. module). We also will allow the case where there is neither a filtration nor a grading in which case U is just an algebra.*

Definition 4. *Let $Com_\Lambda(U)$ be the category of chain complexes of Λ-graded left U-modules M^p (of finite type). The differentials will be U-linear maps which preserve the Λ degree, i.e.,*

$$M_\lambda^p \xrightarrow{d} M_\lambda^{p+1}.$$

If there is no Λ-grading then we use Com(U) to denote the category of chain complexes of left U-modules.

A general construction that we will be utilizing is that of a *quadratic* filtered algebra. Let k be a field, Λ an abelian group, and V a Λ-graded, finite dimensional vector space over k, i.e.,

$$V = \bigoplus_{\lambda\in\Lambda} V_\lambda.$$

We can form the tensor algebra

$$T(V) = V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

which will also be Λ-graded. For each $v_1 \otimes \ldots \otimes v_n \in T(V)$, each v_i has a Λ grading, λ_i, and the total degree will be the sum of all the λ_i. Let P be a Λ-graded sub-vector space of $k \oplus V \oplus (V \otimes V)$ such that $P \cap (k \oplus V) = 0$. Let $p_0(P)$, $p_1(P)$, and $p_2(P)$ be the projections of P onto k, V, and $(V \otimes V)$ respectively. Let $R = p_2(P)$, now we may then define

$$U = T(V)/ < P >$$

to be filtered by tensor powers and Λ-graded. In the case where $P = R$, $U = A$ is said to be a *quadratic* algebra defined by R and this quotient induces an epimorphism $\Phi : A \to \mathrm{gr}U$. Here $\mathrm{gr}U$ is the associated graded algebra of U defined by

$$U = \bigoplus_{i\in\mathbb{Z}} F_{i+1}/F_i.$$

If Φ is an isomorphism then we say that U is of *Poincaré-Birkho ff-Witt* (PBW) type.

Definition 5. *Given a quadratic algebra A defined by $R \subset V \otimes V$, we may dualize this inclusion and get an exact sequence*

$$0 \to R^\perp \to V^* \otimes V^* \to R^* \to 0.$$

The algebra $A^! = T(V^*)/R^\perp$ is called the quadratic dual algebra of A.

Now, assuming that $(k \oplus V) \cap P = 0$, the map $P \to p_2(P) = R$ is a bijection, so we can define maps $\alpha : R \to V$ and $\beta : R \to k$ as

$$\alpha : R \xrightarrow{p_2^{-1}} P \xrightarrow{p_1} V, \qquad \beta : R \xrightarrow{p_2^{-1}} P \xrightarrow{p_0} k.$$

Then

$$P = \{x + \alpha(x) + \beta(x) | x \in R\}.$$

Now let $A^!$ be the dual algebra of A. Dualizing the maps α and β we have

$$A^!_1 = V^* \xrightarrow{\alpha^*} A^!_2 = R^*, \qquad k \xrightarrow{\beta^*} A^!_2 = R^*.$$

Example 1. *First let us examine the classical case. Consider the tensor algebra, $T(V)$, of the Λ-graded vector space V, where $\Lambda = \mathbb{Z}$. Now assign $deg(v) = 1$ for $v \in V$. It is easy to see that the Λ-degree will equal the tensor degree meaning that the filtration and grading of $T(V)$ will be one in the same. Given $R \subset V \otimes V$,*

$$U = A = T(V)/ < R > .$$

Example 2. *Now let us examine a special case of a Λ-graded filtered algebra. Let $\mathfrak{g}$ be a semisimple Lie algebra, where Λ is the lattice of weights of $\mathfrak{g}$. It is known that we have a decomposition*

$$\mathfrak{g} = \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda, \ [\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}, \quad \mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [h,x] = \lambda(h)x \ \forall h \in \mathfrak{h}\}$$

where $\mathfrak{h}$ is a Cartan subalgebra. If we consider the relation $x \otimes y - [x,y]$ for $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$, we can see that $deg(x \otimes y) = deg(([x,y]) = \lambda + \mu$. By definition, the universal enveloping algebra of $\mathfrak{g}$ is

$$U = U_{\mathfrak{g}} = T\mathfrak{g}/J$$

where $T\mathfrak{g}$ is the tensor algebra on $\mathfrak{g}$ and $J =< x \otimes y - [x,y] >$. We can see that since J is generated by homogeneous relations, U is graded by Λ and is filtered by the tensor degree.

3. Curved Differential Graded Algebras

A Λ-graded curved differential graded algebra (CDGA) (B,d,c) over k is a cohomologically graded k-algebra such that

$$B = \bigoplus_{p \in \mathbb{Z}} B^p, \ B^p = \bigoplus_{\lambda \in \Lambda} B^p_\lambda, \quad B^p \cdot B^q \subset B^{p+q}, \ B^p_\lambda \cdot B^q_\mu = B^{p+q}_{\lambda+\mu},$$

where the differential d is a k-linear map such that $d : B^p \to B^{p+1}$ with

$$d^2(b) = [c,b] = cb + (-1)^{deg(b)deg(c)} bc, \quad d(b_1 b_2) = d(b_1)b_2 + (-1)^{|b_1|} b_1 d(b_2).$$

Note that when $\Lambda = 0$ we have the notion of a curved differential graded algebra, if $c = 0$ we have the notion of a Λ-graded differential graded algebra, and if both are zero then we simply have a differential graded algebra.

Definition 6. *A Λ-graded left curved differential graded module (CDGM) (N,d,c) over a CDGA (B,d,c) is a graded left B-module N with a k-linear map d_N such that*

$$N = \bigoplus_{i \in \mathbb{Z}} N^i, \ N^i = \bigoplus_{\lambda \in \Lambda} N^i_\lambda, \quad B^i_\lambda \cdot N^j_\mu \subset N^{i+j}_{\lambda+\mu},$$
$$d_N : N^i_\lambda \to N^{i+1}_\lambda, \ d^2_N(n) = cn, \quad d_N(bn) = d_B(b)n + (-1)^{|b|} b d_N(n), \ b \in B, \ n \in N.$$

Note that for a Λ-graded right curved differential graded module we have $N^j_\mu \cdot B^i_\lambda \subset N^{i+j}_{\lambda+\mu}$ and $d^2_N(n) = -cn$.

We let $\mathrm{Com}_\Lambda(B,d,c)$ be the category of these curved differential graded modules, with the evident morphisms. When $c = 0$, we have simply a differential graded module.

Example 3. *The following example is due to Floystad [3]. Let k be a field and let U be the following filtered quadratic algebra,*

$$U = k[x]/(x^2 - (a-b)x + ab) = k[x]/(x-a) \oplus k[x]/(x-b).$$

The dual of U will be $A^! = k[\xi]$ which is the CDGA with differential

$$d(\xi^n) = \begin{cases} -(a+b)\xi^{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and curvature $c = ab\xi^2$. More details of this CDGA will be provided in Sections 5 and 6.

4. Koszul Algebras

Let A be a quadratic algebra over a field k, so $A = T(V)/(R)$. Its dual $A^!$ is the quadratic algebra over k given by $A^! = T(V^*)/(R^\perp)$ with $R^\perp \subset V^* \otimes V^* = (V \otimes V)^*$.

We define the *Koszul Complex* of A to be the complex (isomorphic to)

$$\ldots \to A \otimes (A_2^!)^* \to A \otimes (A_1^!)^* \to A$$

where $(A^!)_i = A_i^!$. We can note that $A_1^! = V^*$ and $A_2^! = (V^* \otimes V^*)/(R^\perp) = R^*$ so $(A_2^!)^* = R$. Before defining the differentials, recall the canonical isomorphism $W^* \otimes_k V = \mathrm{Hom}_k(W, V)$ defined by $\phi(\lambda \otimes_k v)(w) = \lambda(w)v$. Now if $A = \oplus_{j \in \mathbb{Z}} A_j$, then

$$A \otimes_k (A_i^!)^* = \bigoplus_{j \in \mathbb{Z}} A_j \otimes_k (A_i^!)^* = \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_k(A_i^!, A_j).$$

Since $A_i^!$ is finite dimensional, we know that

$$\bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}_k(A_i^!, A_j) = \mathrm{Hom}_k(A_i^!, \bigoplus_{j \in \mathbb{Z}} A_j) = \mathrm{Hom}_k(A_i^!, A)$$

via the isomorphism $A \otimes_k (A_i^!)^* = \mathrm{Hom}_k(A_i^!, A)$. The differential $\delta : A \otimes_k (A_{i+1}^!)^* \to A \otimes_k (A_i^!)^*$ carries over to a differential $d : \mathrm{Hom}_k(A_{i+1}^!, A) \to \mathrm{Hom}_k(A_i^!, A)$. We define d as follows. Let $f \in \mathrm{Hom}_k(A_{i+1}^!, A)$ then $df \in \mathrm{Hom}_k(A_i^!, A)$ is given by

$$df(\check{a}) = \sum_\alpha f(\check{v}_\alpha \check{a}) v_\alpha \text{ for } \check{a} \in A_i^!,$$

where $\{v_\alpha\}$ is any basis of $V = A_1$, and $\{\check{v}_\alpha\}$ is the corresponding dual basis of $V^* = A_1^!$. In fact, this formula for df does not depend on the choice of the basis $\{v_\alpha\}$. If $\{w_\alpha\}$ is another basis for V, i.e. $w_\alpha = \Sigma g_{\beta\alpha} v_\beta$, where $g_{\beta\alpha}$ is an invertible matrix with entries in k, then we may also define the dual basis of $\{w_\alpha\}$ as $\{\check{w}_\alpha\} = \Sigma h_{\epsilon\alpha} \check{v}_\epsilon$. Now,

$$\delta_{\alpha\beta} =< \check{v}_\alpha, v_\beta >=< \check{w}_\alpha, w_\beta >=< \check{w}_\alpha, \sum_\gamma g_{\gamma\beta} v_\gamma >= \sum_\gamma g_{\gamma\beta} < \check{w}_\alpha, v_\alpha > .$$

If we substitute in for $\check{w}_\alpha$, we have

$$\delta_{\alpha\beta} = \sum_\gamma g_{\gamma\beta} < \sum_\epsilon h_{\epsilon\alpha}\check{v}_\epsilon, v_\gamma >= \sum_{\gamma,\epsilon} g_{\gamma\beta} h_{\epsilon\alpha} < \check{v}_\epsilon, v_\gamma >= \sum_{\gamma,\epsilon} g_{\gamma\beta} h_{\epsilon\alpha}\delta_{\epsilon\gamma}.$$

Since $\delta_{\epsilon\gamma}$ vanishes unless $\epsilon = \gamma$ we may rewrite the last term above as

$$\delta_{\alpha\beta} = \sum_\gamma g_{\gamma\beta} h_{\gamma\alpha}$$

which can be rewritten as

$$\delta_{\alpha\beta} = \sum_\gamma ({}^t g)_{\beta\gamma} h_{\gamma\alpha} = ({}^t g h)_{\beta\alpha}.$$

This shows that ${}^t gh = 1$, i.e., $h = ({}^t g)^{-1}$. So, if we have another basis for V,

$$w_\alpha = \sum_\beta g_{\beta\alpha} v_\beta,$$

we know that

$$\check{w}_\alpha = \sum_\gamma ({}^t g)^{-1}_{\gamma\alpha}\check{v}_\gamma$$

and we would like to show that

$$df(\check{a}) = \sum_\alpha f(\check{v}_\alpha \check{a}) v_\alpha = \sum_\alpha f(\check{w}_\alpha \check{a}) w_\alpha.$$

Observe that

$$\sum_\alpha f(\check{w}_\alpha \check{a}) w_\alpha = \sum_\alpha f(\sum_\gamma ({}^t g)^{-1}_{\gamma\alpha}\check{v}_\gamma \check{a}) \sum_\beta g_{\beta\alpha} v_\beta$$

allowing us to simplify to

$$\sum_\alpha f(\check{w}_\alpha \check{a}) w_\alpha = \sum_{\alpha,\beta,\gamma} ({}^t g)^{-1}_{\gamma\alpha} g_{\beta\alpha} f(\check{v}_\gamma \check{a})\check{v}_\beta = \sum_{\alpha,\beta,\gamma} g_{\beta\alpha} g^{-1}_{\alpha\gamma} f(\check{v}_\gamma \check{a})\check{v}_\beta.$$

Since $\Sigma g_{\beta\alpha}(g^{-1})_{\alpha\gamma} = \delta^\gamma_\beta$ we may conclude that

$$\sum_\alpha f(\check{w}_\alpha \check{a}) w_\alpha = \sum_{\beta,\gamma} (\sum_\alpha g_{\beta\alpha}(g^{-1})_{\alpha\gamma}) f(\check{v}_\gamma \check{a})\check{v}_\beta = \sum_\beta f(\check{v}_\gamma \check{a})\check{v}_\beta.$$

Now we will define δ so that the following commutes:

$$\begin{array}{ccc} A \otimes (A^!_{i+1})^* & \xrightarrow{\delta} & A \otimes (A^!_i)^* \\ \phi \big\downarrow & & \phi \big\downarrow \\ \mathrm{Hom}(A^!_{i+1}, A) & \xrightarrow{d} & \mathrm{Hom}(A^!_i, A). \end{array}$$

Since an element of $A \otimes (A^!_{i+1})^*$ is a sum of tensors $a \otimes \lambda$, $a \in A$, $\lambda \in (A^!_{i+1})^* = \mathrm{Hom}_k(A^!_{i+1}, k)$ it is enough to define $\delta(a \otimes \lambda)$. Set

$$\delta(a \otimes \lambda) = \sum_\alpha a v_\alpha \otimes \lambda_\alpha,$$

where $\lambda_\alpha = \lambda(\check{v}_\alpha \cdot -) \in (A^!_i)^* = \mathrm{Hom}_k(A^!_i, k)$ is the map $\check{a} \mapsto \lambda(\check{v}_\alpha \check{a})$. Let $f = \phi(a \otimes \lambda)$ and recall that $\phi(a \otimes \lambda)(\check{b}) = \lambda(\check{b})(a)$ giving us,

$$d\phi(a \otimes \lambda)(\check{a}) = df(\check{a}) = \sum_\alpha f(\check{v}_\alpha \check{a}) v_\alpha = \sum_\alpha \phi(a \otimes \lambda)(\check{v}_\alpha \check{a}) v_\alpha,$$

so we have

$$d\phi(a \otimes \lambda)(\check{a}) = \sum_\alpha \phi(a \otimes \lambda)(\check{a}\check{v}_\alpha) v_\alpha = \sum_\alpha \lambda(\check{a}\check{v}_\alpha) a v_\alpha.$$

Now to verify that the diagram commutes, we must check that $d\phi(a \otimes \lambda)(\check{a}) = \phi\delta(a \otimes \lambda)(\check{a})$. Given that $\delta(a \otimes \lambda) = \sum_\alpha a v_\alpha \otimes \lambda_\alpha$ we know the following,

$$\phi\delta(a \otimes \lambda)(\check{a}) = \sum_\alpha \phi(a v_\alpha \otimes \lambda_\alpha)(\check{a}) = \sum_\alpha \lambda_\alpha(\check{a}) a v_\alpha = \sum_\alpha \lambda(\check{v}_\alpha \check{a}) a v_\alpha$$

showing that the diagram commutes. Now we will need to show that $d^2 = \delta^2 = 0$. If we can show that $\delta^2 = 0$ then by duality we will know that $d^2 = 0$. The map δ is defined as follows

$$\delta(a \otimes \lambda) = \sum_\alpha a v_\alpha \otimes \lambda(\check{v}_\alpha \cdot -),$$

where $\delta : A \otimes_k (A^!_{i+1})^* \to A \otimes_k (A^!_i)^*$. Now define $\delta' : A \otimes_k (A^!_i) \to A \otimes_k (A^!_{i+1})$ by $\delta' = xe$ where $e = \sum \check{v}_\alpha \otimes v_\alpha \in A^! \otimes A$. Now we want to show that $A \otimes A^!_{i+1} = [\mathrm{Hom}_k(A^!_{i+1}, A)]^*$. Since any k-linear map can be extended canonically to an A-linear map, and any A-linear map comes from a k-linear map, we know that

$$(A^!_i)^* \otimes A = \mathrm{Hom}_A(A^!_i \otimes A, A) = \mathrm{Hom}_k(A^!_i, A).$$

So $A \otimes A^!_{i+1}$ and $A \otimes (A^!_{i+1})^*$ are dual in the A-linear sense. Now since $(\delta')^2 x = xe^2$ and assuming that $e^2 = 0$ that implies that $(\delta')^2 = 0$ implying that $\delta^2 = 0$ which, by duality, tells us that $d^2 = 0$. Now to show that $e^2 = 0$ first note that

$$\begin{aligned} A^!_2 \otimes A_2 &= ((V^{\otimes 2})^*/R^\perp) \otimes (V^{\otimes 2}/R) \\ &= R^* \otimes (V^{\otimes 2}/R) \\ &= \mathrm{Hom}(R, V^{\otimes 2}/R). \end{aligned}$$

Now consider the following diagram:

$$\begin{array}{ccc} (A^!_1 \otimes A_1) \otimes (A^!_1 \otimes A_1) & \xrightarrow{m} & A^!_2 \otimes A_2 \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle q}\big\downarrow \\ \mathrm{Hom}(V^{\otimes 2}, V^{\otimes 2}) & \xrightarrow{\phi} & \mathrm{Hom}(R, V^{\otimes 2}/R), \end{array}$$

where m is ring multiplication defined by

$$m[(\check{a}_1 \otimes a_1) \otimes (\check{a}_2 \otimes a_2)] = \check{a}_1\check{a}_2 \otimes a_1 a_2$$

with canonical isomorphisms p and q. Now given $e = \sum \check{v}_\alpha \otimes v_\alpha \in A^! \otimes A$, we know that $e^2 = (\sum \check{v}_\alpha \otimes v_\alpha)(\sum \check{v}_\beta \otimes v_\beta) = \sum_{\alpha,\beta} \check{v}_\alpha \check{v}_\beta \otimes v_\alpha v_\beta$ by definition of the ring multiplication m. We will show that $\sum_{\alpha,\beta} \check{v}_\alpha \check{v}_\beta \otimes v_\alpha v_\beta = 0$.

Given $f \in \mathrm{Hom}(V^{\otimes 2}, V^{\otimes 2})$, then $g = \phi(f)$ is the composite

$$R \hookrightarrow V^{\otimes 2} \xrightarrow{f} V^{\otimes 2} \to V^{\otimes 2}/R.$$

Now we will check that the diagram commutes. To define p explicitly first recall,

$$(A_1^! \otimes A_1) \otimes (A_1^! \otimes A_1) = (V^* \otimes V) \otimes (V^* \otimes V)$$

so we know that

$$p : (V^* \otimes V) \otimes (V^* \otimes V) \to \mathrm{Hom}(V^{\otimes 2}, V^{\otimes 2}).$$

Now for $(\check{v}_\alpha \otimes v_\beta) \otimes (\check{v}_\gamma \otimes v_\delta) \in (V^* \otimes V) \otimes (V^* \otimes V)$ and $(v_\epsilon \otimes v_\theta) \in V^{\otimes 2}$ define p as follows

$$p((\check{v}_\alpha \otimes v_\beta) \otimes (\check{v}_\gamma \otimes v_\delta))(v_\epsilon \otimes v_\theta) = \delta_\epsilon^\alpha \delta_\theta^\gamma v_\beta \otimes v_\delta.$$

Let us check that $p(e \otimes e) = id \in \mathrm{Hom}(V^{\otimes 2}, V^{\otimes 2})$, where

$$e \otimes e = \sum (\check{v}_\alpha \otimes v_\alpha) \otimes \sum (\check{v}_\beta \otimes v_\beta).$$

We evaluate $e \otimes e$ on $v_\epsilon \otimes v_\theta$ and we have

$$(e \otimes e)(v_\epsilon \otimes v_\theta) = \sum_{\alpha,\beta} \delta_\epsilon^\alpha \delta_\theta^\beta v_\alpha \otimes v_\beta$$

which will equal zero when either $\alpha = \epsilon$ or $\beta = \theta$ meaning that

$$(e \otimes e)(v_\epsilon \otimes v_\theta) = (v_\epsilon \otimes v_\theta),$$

so $e \otimes e$ is the identity map. The map ϕ clearly takes $id \in \mathrm{Hom}(V^{\otimes 2}, V^{\otimes 2})$ to the zero map in $\mathrm{Hom}(R, V^{\otimes 2}/R)$. To define the isomorphism q recall that

$$\begin{aligned} A_2^! \otimes A_2 &= ((V^*)^{\otimes 2}/R^\perp) \otimes (V^{\otimes 2}/R) \\ &= R^* \otimes (V^{\otimes 2}/R). \end{aligned}$$

If we can show that the diagram commutes, that will prove that $q(e^2) = 0$. For an equivalence class of elements $\overline{\check{v}_\alpha \otimes \check{v}_\gamma} \otimes \overline{v_\beta \otimes v_\delta} \in (V^*)^{\otimes 2}/R^\perp \otimes (V^{\otimes 2}/R)$ and for an element $x \in R$ define q as follows

$$q(\overline{\check{v}_\alpha \otimes \check{v}_\gamma} \otimes \overline{v_\beta \otimes v_\delta})(x) = (\check{v}_\alpha \otimes \check{v}_\gamma(x)) \cdot v_\beta v_\delta = (\check{v}_\alpha \otimes \check{v}_\gamma(x)) \cdot \overline{v_\beta \otimes v_\delta},$$

where $\check{v}_\alpha \otimes \check{v}_\gamma \in V^* \otimes V^* = ((V^{\otimes 2})^*$ is a linear functional. To see that q is well defined, note that $\overline{\check{v}_\alpha \otimes \check{v}_\gamma}$ is unique up to $(\check{v}_\alpha \otimes \check{v}_\gamma(x) + \rho)$ where $\rho \in R^\perp$. Since $\rho \in R^\perp$, $\rho(x) = 0$ and in the quotient $(V^{\otimes 2}/R)$ so the map q is well defined. Note that $x \in R$ is also contained in $(V^{\otimes 2})$ meaning that any linear functional on $V^{\otimes 2}$ restricts to R. So to see that the diagram commutes, observe that

$$\begin{aligned}\phi p[(\check{v}_\alpha \otimes v_\beta) \otimes (\check{v}_\gamma \otimes v_\delta)](x) &= [(\check{v}_\alpha \otimes \check{v}_\gamma)(x)] \cdot \overline{v_\beta \otimes v_\delta} \\ &= [(\overline{\check{v}_\alpha \otimes \check{v}_\gamma})(x) \cdot (\check{v}_\beta \otimes v_\delta)] \\ &= qm[(\check{v}_\alpha \otimes v_\beta) \otimes (\check{v}_\gamma \otimes v_\delta)](x).\end{aligned}$$

We have shown that the diagram commutes proving that $q(e^2) = 0$, so $e^2 = 0$ since q is an isomorphism.

Thus we may define the Koszul complex $K^i = A \otimes (A^!_i)^*$. The notation $K(A)$ will also be used to indicate the Koszul complex of a quadratic algebra A.

Proposition 1. *[Proposition 2.9.1, [1]]. Let A be a Koszul ring. Then $A^!$ is Koszul as well.*

Proposition 2 (Koszul Algebra). *Let A be a quadratic k-algebra, then following conditions are equivalent:*

(a) $H^i(K(A))_n = 0$ *if* $i > 0$ *and if* $i = 0$, $n \neq 0$, *we have* $H^0(K(A))_0 = k$.

(b) $H^i_j(A, M)$, *(Hochschild Homology) where A is considered as an A-bimodule, vanishes for any* $\mathbb{Z}^+$*-graded A-bimodule M and* $i < -j$.

(c) $Ext^i_j(k, k) = 0$ *for all* $i \neq j$, *where* $Ext^i(k, k)$ *is taken in the category of left A-modules.*

(d) $K^\bullet$ *is a resolution of A in the category of A-bimodules.*

(e) $K^\bullet \otimes_A k$ *is a resolution of k in the category of left A-modules.*

(f) There exists a free resolution of k such that the i'th syzygies are all generated in degree i.

Example 4. *Let us consider the special case where A is the symmetric algebra on* V, $S(V)$. *The quadratic dual* $A^!$ *will be the exterior algebra on* V^*, $E(V^*)$. *Therefore we may represent* $K(A)$ *in the following way*

$$\ldots \xrightarrow{d_2} A \otimes \wedge^2(V) \xrightarrow{d_1} A \otimes V \xrightarrow{d_0} A,$$

where $d_0 : a \otimes v \to av$ *or more specifically for this example,* $d_0 : p(x) \otimes x_i \to x_i p(x)$ *with* $p(x) \in S(V)$ *and* $x_i \in \wedge^i(V)$. *More generally, we may define* $d_i : A \otimes \wedge^p(V) \to A \otimes \wedge^{p-1}(V)$ *as follows,*

$$d_i : a \otimes v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_p} \to \Sigma a v_{i_j} (-1)^{j-1} v_{i_1} \wedge \cdots \wedge \widehat{v}_{i_j} \wedge \cdots \wedge v_{i_p}.$$

Note that the Koszul complex is graded by tensor degree since A and $A^!$ are both graded. For $a \otimes v \in (A \otimes V)_n$, $deg(a) = n - 1$ and $deg(v) = 1$ so $deg(a \otimes v) = n$. Also note that $H^i(K^*)_n = 0$ for all $i > 0$ and for $i = 0$ with $n \neq 0$. The only nontrivial homology group is $H^0(K^*)_0 = k$.

5. Duality

Let $U = T(V)/ < P >$ be a Λ-graded filtered quadratic algebra. Let $A = T(V)/ < R >$, where $R = \beta(P)$ as in Section 2. We define a dual Λ-graded curved differential graded algebra $(A^!, d, c)$ as follows:

$$A^! = T(V^*)/R^{\perp},$$

$$\alpha : R \xrightarrow{p_2^{-1}} P \xrightarrow{p_1} V, \quad \beta : R \xrightarrow{p_2^{-1}} P \xrightarrow{p_0} k,$$

with $A^!_1 = V^*$, $d = \alpha^*$, and $c = \beta^*(1)$.

Theorem 1. *Assume A is Koszul. Then U is of PBW-type if and only if the map α^* extends to an antiderivation d on $A^!$ such that, letting $c = \beta^*(1), (A^!, d, c)$ is a curved differential graded algebra. In particular, when $c = 0$, giving $A^!$ the structure of a differential graded algebra is equivalent to giving a subspace $P \subset V \oplus (V \otimes V)$ with $p_2(P) = R$ such that $grU = A$.*

Theorem 2. *Assume A is Koszul. Then U is of PBW-type if and only if*

1. *$im(\alpha \otimes id - id \otimes \alpha) \subseteq R \subseteq V \otimes V$ (this map is defined on $(R \otimes V) \cap (V \otimes R)$).*
2. *$\alpha \circ (\alpha \otimes id - id \otimes \alpha) = \beta \otimes id - id \otimes \beta$.*
3. *$\beta \circ (\alpha \otimes id - id \otimes \alpha) = 0$.*

These two theorems amount to giving $A^!$ the structure of a CDGA. The following lemma will be made use of in Section 6.

Lemma 1. *The element in $U \otimes_k A^!$*

$$\sum x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha + \sum x_\alpha \otimes d(\check{x}_\alpha) + 1 \otimes c \tag{1}$$

and the element in $A^! \otimes_k U$

$$\sum \check{x}_\beta \check{x}_\alpha \otimes x_\alpha x_\beta + d(\check{x}_\alpha) \otimes x_\alpha + c \otimes 1$$

are both zero.

Proof. Consider the pairing

$$(U \otimes A^!_2) \otimes (A^!_2)^* \to U.$$

Denoting the element in (1) as m, we show that $\langle m, - \rangle : (A^!_2)^* \to U$ is zero. Note that

$$\langle m, r \rangle = \sum \left\langle x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha, r \right\rangle + \sum \left\langle x_\alpha \otimes d(\check{x}_\alpha), r \right\rangle + \langle 1 \otimes c, r \rangle .$$

Also note that for an element r in $R = (A^!_2)^*$ we have

$$\sum x_\alpha x_\beta \left\langle \check{x}_\beta \check{x}_\alpha, r \right\rangle = r, \quad \sum x_\alpha \left\langle d(\check{x}_\alpha), r \right\rangle = \alpha(r), \quad \langle c, r \rangle = \beta(r),$$

and since $r + \alpha(r) + \beta(r) = 0$ in U, the lemma is proven.

Example 5. *We may first consider the case where $U = A$ is a Λ-graded filtered algebra. The dual will be the cdga $(A^!, d = 0, c = 0)$.*

Example 6. *Let $U = U\mathfrak{g}$ be the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then the dual $(A^!, d, c = 0)$ is the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$.*

Example 7. *The symmetric algebra on V, $S(V)$, is defined by $R = (x \otimes y - y \otimes x)_{x,y \in V}$. The quadratic dual algebra of $S(V)$ is the exterior algebra $E(V^*)$ defined by the relations $(x \otimes x)_{x \in V^*}$ in $V^* \otimes V^*$.*

Example 8. *Continuing Example 3, we can show explicitly that Theorems 1 and 2 amount to giving $A^!$ the structure of a CDGA. We have the following algebra*

$$U = k[x]/(x^2 - (a-b)x + ab) = k[x]/(x-a) \oplus k[x]/(x-b).$$

Given U, we know that $A^! = k[\xi]$ assuming $< x, \xi >= 1$. By definition we know

$$\alpha(x^2) = -(a+b)x \quad \text{and} \quad \beta(x^2) = ab,$$

so now to calculate the differential we have

$$< \alpha^*(\xi), x^2 >=< \xi, \alpha(x^2) >=< \xi, -(a+b)x >= -(a+b).$$

This tells us that $\alpha^*(\xi)$ is the element of $A^!_2$ for which $< \alpha^*(\xi), x^2 >= -(a+b)$, so $\alpha^*(\xi) = \xi^2$. This tells us that $d\xi = -(a+b)\xi^2$. Now let us calculate $d(\xi^2)$. We have

$$d(\xi^2) = d\xi \cdot \xi - \xi \cdot d\xi = -(a+b)\xi^2 \cdot \xi + \xi \cdot (a+b)\xi^2 = 0.$$

Therefore we have the differential

$$d(\xi^n) = \begin{cases} -(a+b)\xi^{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

It is easy to see that $d^2 = 0$.

6. The Duality Functors F and G

We let U be a Λ-graded filtered quadratic algebra such that $A = \mathrm{gr}(U)$ is Koszul, and we let $(A^!, d, c)$ be the dual cdga. Let $T = U \otimes A^!$. This is a $U - A^!$ bimodule and we give it the grading of $A^!$. Let d be the endomorphism defined by

$$u \otimes a \longmapsto \sum u x_\alpha \otimes \check{x}_\alpha a + u \otimes d(a)$$

where x_α is a basis for V and $\check{x}_\alpha$ is the dual basis for V^*. This definition is independent of the choice of this basis and one can check that this gives $U \otimes A^!$ the structure of a right cgd module over $(A^!, d, c)$.

Lemma 2. *For $T = U \otimes A^!$ a Λ-graded $(U, A^!)$-bimodule, T is a right $A^!$-CDGM.*

Proof. It is clear that T is a Λ-graded $(U, A^!)$-bimodule. We only need to check that $d^2(u \otimes a) = -u \otimes ac$. We define d as follows

$$d(u \otimes a) = \sum u x_\alpha \otimes \check{x}_\alpha a + u \otimes d(a).$$

We have

$$\begin{aligned} d^2(u \otimes a) &= \sum u x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha a + \sum u x_\alpha \otimes \check{x}_\alpha d(a) + \sum u x_\alpha \otimes d(\check{x}_\alpha a) + u \otimes d^2(a) \\ &= \sum u x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha a + \sum u x_\alpha \otimes d(\check{x}_\alpha) a + u \otimes ca - u \otimes ac \\ &= -u \otimes ac. \end{aligned}$$

The minus sign is as we would expect since T is a right $A^!$-module.

The pair of adjoint functors

$$F : Com_\Lambda(A^!, d, c) \leftrightarrows Com_\Lambda(U) : G$$

is given by

$$F(N) = T \otimes_{A^!} N, \quad G(M) = \mathrm{Hom}_U(T, M).$$

Explicitly, we have

$$F(N)^p_\lambda = \bigoplus_{\mu+\nu=\lambda} U_\mu \otimes_k N^p_\nu$$

with

$$d(u \otimes n) = \sum_\alpha u x_\alpha \otimes \check{x}_\alpha n + u \otimes d_N(n). \tag{2}$$

Since $\deg_\Lambda(\check{x}_\alpha) = -\deg_\Lambda(x_\alpha)$ one can show that d preserves the Λ-grading. Since $a x_\alpha \in U_{\mu+deg(x_\alpha)}$ and $\check{x}_\alpha n \in N^{p+1}_{\nu-deg(x_\alpha)}$, the sum $\mu + \nu = \lambda$ remains unchanged in the tensor. The map d is also U-linear, this can easily be seen since

$$\begin{aligned} d(u_1 u \otimes n) &= \sum u_1 u x_\alpha \otimes x_\alpha n + u_1 u \otimes d_N(n) \\ &= u_1 \sum u x_\alpha \otimes x_\alpha n + u \otimes d_N(n) \\ &= u_1 d(u \otimes n). \end{aligned}$$

Lemma 3. *$d^2 = 0$ in Equation (2) and if N is in $Com_\Lambda(A^!, d, c)$ then $F(N)$ is in $Com_\Lambda(U)$.*

Proof. All of the axioms necessary to show that $F(N) \in Com_\Lambda(U)$ are obvious except for that $d^2 = 0$, so let us verify this. For $u \otimes n \in F(N)$ we have

$$d^2(a \otimes n) = \sum_\alpha (\sum_\beta a x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha n + a x_\alpha \otimes d_N(\check{x}_\alpha n)) + \sum_\gamma a x_\gamma \otimes \check{x}_\gamma d_N(n) + a \otimes d^2_N(n).$$

The term $\sum_\alpha ax_\alpha \otimes d_N(\check{x}_\alpha n)$ may be rewritten as

$$\sum_\alpha ax_\alpha \otimes d(\check{x}_\alpha)n - ax_\alpha \otimes \check{x}_\alpha d_N(n)$$

which allows us to simplify and apply Lemma 1 as follows,

$$\begin{aligned} d^2(a\otimes n) &= \sum_\alpha(\sum_\beta ax_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha n + ax_\alpha + d(\check{x}_\alpha)n) + a\otimes cn \\ &= a(\sum_\alpha(\sum_\beta x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha + x_\alpha + d(\check{x}_\alpha)) + 1\otimes c)n \\ &= 0. \end{aligned}$$

Next, $G(M) = \mathrm{Hom}_U(T,M) = \mathrm{Hom}_k(A^!, M)$ has the structure of a graded $A^!$-module, defined as

$$(a\cdot f)(b) = (-1)^{q(p+r)} f(ba),\ a\in (A^!)^q,\ b\in (A^!)^r,\ f\in \mathrm{Hom}^p_k(A^!,M).$$

One can verify that $a_1\cdot(a_2\cdot f) = (a_1a_2)\cdot f$. Also it carries an internal Λ-grading:

$$G(M)^p_\lambda = \prod_{r\geq 0}\prod_\mu \mathrm{Hom}_k((A^!)^r_\mu, M^{p+r}_{\lambda+\mu})$$

with d is given by

$$d(f)(a) = (-1)^{|f|+1}\sum x_\alpha f(\check{x}_\alpha a) + (-1)^{|f|+1} f(d_{A^!}(a)) + d_M(f(a))). \tag{3}$$

Note that our sign conventions differ from that in [3].

Example 9. *Continuing Example 3 the algebra U has two simple modules of dimension 1 over k namely*

$$k_a = k[x]/(x-a) \quad \textit{and} \quad k_b = k[x]/(x-b).$$

By definition, the CDG-module $G(k_a)^p = \prod_{r\geq 0} Hom(A^!_r, k_a^{p+r})$. Note that k_a^{p+r} is nonzero only if $p=-r$. Now since $Hom(A^!_r, k_a) = (x^r)$ we can check that for the CDG-module $G(k_a)$ we have

$$\cdots(x^3)\overset{a\xi}{\to}(x^2)\overset{b\xi}{\to}(x)\overset{a\xi}{\to}(1)\to(0)\to\cdots.$$

The module multiplication for $v\in k_a$ is defined as $x\cdot v = av$. Now let us consider $FG(k_a)$ which has the form

$$\cdots\to U\otimes(A^!_p)^*\otimes k_a \to U\otimes (A^!_{p-1})^*\otimes k_a \to \cdots.$$

The differential for $FG(k_a)$ is given by

$$d(u\otimes \dot{\xi}^p\otimes 1) = ux\otimes \dot{\xi}^{p-1}\otimes 1 + (-1)^{p+1}(u\otimes \dot{\xi}^{p-1}\otimes a + u\otimes d(\dot{\xi}^p)\otimes 1).$$

It is easy to check that $d^2=0$ and the complex $FG(k_a)$ will be quasi-isomorphic to k_a by Proposition 4.

Lemma 4. *$d^2 = c$ in Equation* (3) *and if M is in $Com_\Lambda(U)$ then $G(M)$ is in $Com_\Lambda(A^!, d, c)$.*

Proof. To verify that $d^2(f) = cf$ we have

$$d(d(f)(a)) = d[(-1)^{|f|+1}\sum x_\alpha f(\check{x}_\alpha a)] + d[(-1)^{|f|+1} f(d_{A^!}(a))] + d[d_M(f(a))].$$

Let us examine the first term of this differential. Let $g(a) = (-1)^{|f|+1}\sum x_\alpha f(\check{x}_\alpha a)$ and let $\left|f\right| = p$ so we have

$$d(g)(a) = (-1)^{p+2}\sum_\beta x_\beta g(\check{x}_\beta a) + (-1)^{p+2} g(d_{A^!}(a)) + d_M(g(a))$$

which can be rewritten as

$$d(g)(a) = -\sum_{\beta,\alpha} x_\beta x_\alpha f(\check{x}_\alpha \check{x}_\beta a) - \sum_\gamma x_\gamma f(\check{x}_\gamma d_{A^!}(a)) + (-1)^{p+1}\sum_\delta d_M(x_\delta f(\check{x}_\delta a)).$$

Now recall from Lemma 1 that

$$\sum_{\alpha,\beta} x_\alpha x_\beta \otimes \check{x}_\beta \check{x}_\alpha + \sum x_\alpha \otimes d(\check{x}_\alpha) + 1 \otimes c = 0 \tag{4}$$

and since we know

$$d_{A^!}(\check{x}_\gamma a) = d_{A^!}(\check{x}_\gamma)a - \check{x}_\gamma d_{A^!}(a)$$

we have

$$\sum x_\gamma f(\check{x}_\gamma d_{A^!}(a)) = \sum x_\gamma f(d_{A^!}(\check{x}_\gamma)a) - \sum x_\gamma f(d_{A^!}(\check{x}_\gamma a)).$$

So we may rewrite $d(g)(a)$ as

$$d(g)(a) = f(ca) + \sum_\gamma x_\gamma f(d_{A^!}(\check{x}_\gamma a)) + (-1)^{p+1} d_M(\sum_\delta x_\delta f(\check{x}_\delta a)).$$

Now let us examine the second term of $d(d(f)(a))$. Let $h(a) = (-1)^{p+1} f(d_{A^!}(a))$ so we have

$$\begin{aligned} dh(a) &= (-1)^{p+2}\sum x_\alpha h(\check{x}_\alpha a) + (-1)^{p+2} h(d_{A^!}(a)) + d_M(h(a)) \\ &= -\sum x_\alpha f(d_{A^!}(\check{x}_\alpha a)) - f(d_{A^!}(d_{A^!}(a))) + (-1)^{p+1} d_M(f(d_{A^!}(a))) \\ &= -\sum x_\alpha f(d_{A^!}(\check{x}_\alpha a)) + (-1)^{p+1} d_M(f(d_{A^!}(a))) - f(ca) + f(ac). \end{aligned}$$

For the third term of $d(d(f)(a))$, let $k(a) = d_M(f(a))$. So we have

$$\begin{aligned} dk(a) &= (-1)^{p+2}\sum x_\alpha k(\check{x}_\alpha a) + (-1)^{p+2} k(d_{A^!}(a)) + d_M(k(a)) \\ &= (-1)^{p+2}\sum x_\alpha d_M(f(\check{x}_\alpha a)) + (-1)^{p+2} d_M(f(d_{A^!}(a))) + d_M(d_M(f(a))) \\ &= (-1)^{p+2}\sum x_\alpha d_M(f(\check{x}_\alpha a)) + (-1)^{p+2} d_M(f(d_{A^!}(a))). \end{aligned}$$

Note that since d_M is U-linear we have

$$d(\sum x_\delta f(\check{x}_\delta a)) = \sum x_\delta d(f(\check{x}_\delta a)).$$

Now we can list all remaining terms of the differential $d^2(f)(a)$ as so

$$\begin{aligned} d^2(f)(a) =& f(ca) + \sum_\gamma x_\gamma f(d_{A^!}(\check{x}_\gamma a)) + (-1)^{p+1} d_M(\sum_\delta x_\delta f(\check{x}_\delta a)) \\ &- \sum x_\alpha f(d_{A^!}(\check{x}_\alpha a)) + (-1)^{p+1} d_M(f(d_{A^!}(a))) - f(ca) + f(ac) \\ &+ (-1)^{p+1} \sum x_\alpha d_M(f(\check{x}_\alpha a)) + (-1)^{p+2} d_M(f(d_{A^!}(a))) \\ =& f(ac) = (c \cdot f)(a). \end{aligned}$$

We will now check that $d(a \cdot f) = da \cdot f + (-1)^{|a|} a \cdot df$. Let $\left|f\right| = p$, $|a| = q$, and $|b| = r$. So we have

$$d(a \cdot f)(b) = (-1)^{p+q+1} \sum x_\alpha (a \cdot f)(\check{x}_\alpha b) + (-1)^{p+q+1} (a \cdot f)(d(b)) + d(a \cdot f)(b)$$

which may be rewritten as

$$d(a \cdot f)(b) = (-1)^{qp+qr+p+1} \sum x_\alpha f((\check{x}_\alpha ba)) + (-1)^{qp+qr+p+1} f((d(b)a)) + (-1)^{qp+qr} d(f(ba)).$$

Recalling that

$$f(d(b)a) = f(d(ba)) + (-1)^{r+1} f(b(da)),$$

we have

$$\begin{aligned} d(a \cdot f)(b) =& (-1)^{qp+qr+p+1} \sum x_\alpha f((\check{x}_\alpha ba)) + (-1)^{qp+qr+p+1} f(d(ba)) \\ &+ (-1)^{qp+qr+p+r} f(b(da)) + (-1)^{qp+qr} d(f(ba)). \end{aligned}$$

Let us now calculate $(da \cdot f)(b) + (-1)^q a \cdot d(f(b))$. For the first term, we have

$$(da \cdot f)(b) = (-1)^{qp+qr+p+r} f(b(da))$$

and for the second term, we have

$$(-1)^q (a \cdot df)(b) = (-1)^{qp+qr+p+1} \sum x_\alpha f((\check{x}_\alpha ba)) + (-1)^{qp+qr+p+1} f(d(ba)) + (-1)^{qp+qr} d(f(ba))$$

which proves that $d(a \cdot f) = da \cdot f + (-1)^{|a|} a \cdot df$. These define a structure of a CDGM over the CDGA $(A^!, d, c)$. More explicitly

$$(A^!)^p_\mu \times G(M)^q_\nu \to G(M)^{p+q}_{\mu+\nu}.$$

Note that when $A^! = E(V^*)$ with dim $V < \infty$, the above direct products are direct sums since dim $A^! < \infty$.

Proposition 3. *We have:*

1. *For N in $Com_\Lambda(A^!, d)$ and M in $Com_\Lambda(U)$ there is a canonical isomorphism of differential graded Λ-graded vector spaces*

$$Hom_U(F(N), M) = Hom_{A^!}(N, G(M)).$$

2. *The functors F and G are adjoint, i.e.,*

$$Hom_{Com_\Lambda(U)}(F(N), M) = Hom_{Com_\Lambda(A^!,d)}(N, G(M)).$$

3. *The functors F and G are exact.*

Proof. Since $F(N) = T \otimes_{A^!} N$ and $G(M) = \mathrm{Hom}_U(T, M)$, the first statement follows directly from the adjointness of Hom and tensor product. Explicitly both complexes have (p, λ)'th term equal to

$$\prod_{r\in\mathbb{Z}} \prod_{\mu} \mathrm{Hom}_k(N^r_\mu, M^{p+r}_{\lambda+\mu})$$

with differential d given as follows for $f \in \prod_{r\in\mathbb{Z}} \mathrm{Hom}_k(N^r, M^{p+r})$ and $n \in N^r$,

$$d(f)(n) = (-1)^r d_M f(n) + (-1)^{r+1} f d_N(n) + (-1)^{r+1} \sum_\alpha x_\alpha f(\check{x}_\alpha n).$$

To see that the functors F and G are adjoint we note that the two sides are the cycles of degree 0 in the complexes described in the first statement. The functor F is exact since as a module, $F(N) = T \otimes_{A^!} N = U \otimes_k N$ and k is a field. For the same reason, the functor G is exact since $G(M) = \mathrm{Hom}_U(T, M) = \mathrm{Hom}_k(A^!, M)$.

Lemma 5. *Let M be a U-module considered as a complex situated in degree 0. Then $FG(M) \to M$ is a quasi-isomorphism.*

Note that $FG(M)$ is the complex

$$\cdots \to U \otimes (A^!_p)^* \otimes M \to U \otimes (A^!_{p-1})^* \otimes M \to \cdots \to U \otimes M$$

and the differential is given by

$$d(u \otimes a^* \otimes m) = \sum u x_\alpha \otimes \check{x}_\alpha a^* \otimes m \pm \sum u \otimes a^* \check{x}_\alpha \otimes x_\alpha m \pm u \otimes d^*(a^*) \otimes m.$$

Proof. Proof is from [3]. The complex $FG(M)$ has a filtration $F_i G(M)$ defined as

$$\cdots \to F_{i-2} U \otimes (A^!_2)^* \otimes M \to F_{i-1} U \otimes (A^!_1)^* \otimes M \to F_i U \otimes M.$$

We claim that for $i \geq 0$, we have $H^p(F_i G(M)) = M$ for $p = 0$ and zero otherwise. This follows by induction from the exact sequence

$$0 \to F_{i-1} G(M) \to F_i G(M) \to F_i G(M)/F_{i-1} G(M) \to 0$$

by noting that the term $F_iGM/F_{i-1}G(M)$ is a homogeneous part of the Koszul complex for A and $A^!$ tensored with M (over k) i.e.

$$A_0 \otimes (A^!_i)^* \otimes M \to \cdots \to A_i \otimes (A^!_0)^* \otimes M$$

with differential

$$d(u \otimes a^* \otimes m) = \sum u x_\alpha \otimes a^* \check{x}_\alpha \otimes m.$$

We want to show that $F_iG(M) \to M$ is a quasi-isomorphism, that is, $H^p(F_iG(M)) = 0$ for all i and with $p < 0$, and $H^0(F_iG(M)) = M$ for all i. We know that $FG(M) = \varinjlim F_iG(M)$ so therefore

$$H^p(FG(M)) = H^p(\varinjlim F_iG(M)) = \varinjlim H^p(F_iG(M))$$

and thus $FG(M) \to M$ is a quasi-isomorphism provided that $F_iG(M) \to M$ is for all i. Now we will prove by induction that $H^p(F_iG(M)) = 0$ for all $p < 0$ and $H^0(F_iG(M)) = M$ for all i. The case when $i = 0$ is trivial since $F_0G(M) = M$. Now assume $i > 0$ and consider the following commutative diagram of exact sequences,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & F_iG(M) \arrow[r] \arrow[d, "\alpha"'] & F_{i+1}G(M) \arrow[r] \arrow[d, "\beta"'] & F_{i+1}G(M)/F_iG(M) \arrow[r] \arrow[d, "\gamma"'] & 0 \arrow[d] \\
0 \arrow[r] & M \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

The map α is a quasi-isomorphism by the induction hypothesis and the map γ is a quasi-isomorphism since A is a Koszul algebra. Now by applying the 5-Lemma we know that the map β is also a quasi-isomorphism. We know that $FG(M)$ is $\varinjlim F_iG(M)$ and since taking the filtered direct limit is an exact functor it commutes with cohomology so we get the lemma.

Corollary 1. *For $M^\bullet$ a bounded complex indexed as follows*

$$0 \to M^b \to M^{b+1} \to \cdots \to M^{t-1} \to M^t \to 0.$$

$H^p(F_iG(M^\bullet)) = 0$ *for all i and for $p < b$.*

Proposition 4. *Assuming that A and $A^!$ are Koszul, the natural morphisms coming from the adjunction*

$$FG(M) \to M, \quad N \to GF(N)$$

are quasi-isomorphisms.

Proof. Assume M is bounded and indexed as follows

$$0 \to M^b \to M^{b+1} \to \cdots \to M^{t-1} \to M^t \to 0.$$

Let $\sigma^{>b}M$ be the truncation $M^{b+1} \to M^{b+2} \to \cdots$ and so $M^b[-b]$ will just be a module considered as a one term complex. We may now form the following short exact sequence

$$0 \to \sigma^{>b}M \to M \to M^b[-b] \to 0.$$

Consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & FG(\sigma^{>b}M)) & \longrightarrow & FG(M) & \longrightarrow & FG(M^b[-b]) & \longrightarrow & 0 \\ \downarrow & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \downarrow \\ 0 & \longrightarrow & \sigma^{>b}M & \longrightarrow & M & \longrightarrow & M^b[-b] & \longrightarrow & 0. \end{array}$$

The map α is a quasi-isomorphism by induction on the length of the truncation, the map γ is a quasi-isomorphism by Lemma 5 so by the 5-Lemma we know that the map β is also a quasi-isomorphism.

Now let us assume that the complex M is bounded above so $M = \varinjlim \sigma^{\geq p}M$ where the $\sigma^{\geq p}M$ are bounded. Since we know that

$$G(M)^p_\lambda = \prod_{r\geq 0}\prod_{\mu} \mathrm{Hom}_k((A^!)^r_\mu, M^{p+r}_{\lambda+\mu})$$

we can now show that G commutes with direct limit. We know that $M = \varinjlim \sigma^{\leq s}M$ and even $[M]^l = [\varinjlim \sigma^{\leq s}M]^l$. Now note that $[\varinjlim \sigma^{\leq s}M]^l = 0$ if $l < s$ and equals M^l if $l \geq s$. Since M is bounded above, we know that $M^l = 0$ for $l >> 0$ which tells us that there exists an s_0 such that for all $s \geq s_0$ we have $(\sigma^{\leq s}M)^{p+r} = M^{p+r}$ for all p and $r \geq 0$. Therefore

$$G(\varinjlim \sigma^{\geq p}M) = G(M).$$

Since F is a left adjoint it also commutes with the direct limit so we have that

$$FG(M) = FG(\varinjlim \sigma^{\geq p}M) = \varinjlim FG(\sigma^{\geq p}M) \to \varinjlim \sigma^{\geq p}M = M$$

is a quasi-isomorphism since $\varinjlim$ is exact in the category of vector spaces.
Next suppose that M is bounded below e.g. $M = \sigma^{>b}M$ and indexed as follows

$$0 \to M^b \to M^{b+1} \to \cdots .$$

We know by Lemma 5 that for a module M over U, $F_iG(M)$ is exact in cohomological degrees < 0. First we must define the filtration $F_iG(M)$ for when M is a bounded complex, not just a module. Let

$$[F_iG(M)]^a = F_{i+a}U \otimes \prod_{p\geq 0} \mathrm{Hom}(A^!_p, M^{p+a}).$$

If M is bounded above, in particular bounded, then

$$Gr^F_i FG(M) = \bigoplus_{p\geq 0} A_{i+a} \otimes (A^!_p)^* \otimes M^{p+a}.$$

Let M be a complex bounded below i.e. $M^\bullet = 0$ for $i < b$. We want to show that $FG(M) \to M$ is a quasi-isomorphism in degrees $< b$. Also we know that $H^i(M) = 0$ for $i < b$. Since we know that $FG(M) = \varinjlim F_i G(M)$ and since we know that taking cohomology commutes with taking direct limits, it suffices to show that $H^i(F_\nu G(M)) = 0$ for $i < b$ and for all $\nu >> 0$ which amounts to showing that $H^i(FG(M)) = 0$ for $i < b$. By our proof of Lemma 5 we know that $H^i(F_\nu G(M)) = 0$ for $i < b$ and for all $\nu >> 0$ is true for the case where M is a module considered as a one term complex situated in deg b. Now if we allow for M to be a bounded complex, again by our proof of Lemma 5 we can induct on the length of the complex and again show our intended result. Now let $M = \varprojlim \sigma^{\leq p} M$ and note that each $\sigma^{\leq p} M$ is bounded since M is bounded below. We know that

$$F_\nu G(M) = F_\nu G(\varprojlim \sigma^{\leq p} M) = F_\nu(\varprojlim G(\sigma^{\leq p} M)) = \varprojlim(F_\nu G(\sigma^{\leq p} M)).$$

The first equality is easy to see since $M = \varprojlim \sigma^{\leq p} M$, the second equality follows from the fact that G is a right adjoint and the third equality follows from the fact that each $F_\nu G(M)$ is finite dimensional, and therefore commutes with inverse limit. Now we would like to show that even though inverse limit does not usually commute with cohomology, in our case we do have

$$H^i(\varprojlim F_\nu G(\sigma^{\leq p} M)) = \varprojlim H^i(F_\nu G(\sigma^{\leq p} M)).$$

This equality follows from the fact that each $F_\nu G(\sigma^{\leq p} M)$ and $H^i(F_\nu G(\sigma^{\leq p} M))$ satisfy the Mittag-Leffler condition (See Proposition 5). We also know that for $i < b$ the following is true,

$$H^i(\varprojlim \sigma^{\leq p} M) = \varprojlim H^i(\sigma^{\leq p} M),$$

since $M = 0$ for $i < b$ and we know by our proof of Lemma 5 that since each $\sigma^{\leq p} M$ is bounded we have

$$\varprojlim H^i(F_\nu G(\sigma^{\leq p} M)) = \varprojlim H^i(\sigma^{\leq p} M),$$

proving that $H^i(F_\nu G(M)) = 0$ for $i < b$ and for all $\nu >> 0$.

Now let M be an arbitrary complex. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & FG(\sigma^{>p})M \arrow[r] \arrow[d, "\alpha"'] & FG(M) \arrow[r] \arrow[d, "\beta"'] & FG(\sigma^{\leq p} M) \arrow[r] \arrow[d, "\gamma"'] & 0 \arrow[d] \\
0 \arrow[r] & \sigma^{>p} M \arrow[r] & M \arrow[r] & \sigma^{\leq p} M \arrow[r] & 0
\end{tikzcd}$$

and the resulting cohomology diagram:

$$\begin{tikzcd}
H^i(FG(\sigma^{>p} M)) \arrow[r] \arrow[d, "\alpha"] & H^i(FG(M)) \arrow[r] \arrow[d, "\beta"] & H^i(FG(\sigma^{\leq p}(M)) \arrow[r] \arrow[d, "\gamma"] & H^{i+1}(FG(\sigma^{>p} M)) \arrow[d, "\alpha'"] \\
H^i(\sigma^{>p} M) \arrow[r] & H^i(M) \arrow[r] & H^i(\sigma^{\leq p} M) \arrow[r] & H^{i+1}(\sigma^{>p} M).
\end{tikzcd}$$

We know that the map γ is an isomorphism in all degrees i since $\sigma^{\leq p} M$ is bounded above. We have also shown that α is an isomorphism in degrees $i \leq p$ since $\sigma^{>p} M$ is bounded below. Similarly, α' is an isomorphism in degrees $i + 1 \leq p$. The 5-Lemma now shows that β is an isomorphism in degrees $i \le p-1$. Since p is arbitrary, β is an isomorphism in all degrees i.

We now would like to show that $N \to GF(N)$ is a quasi-isomorphism. The complex $GF(k)$ is the complex

$$\cdots \to (A^!_p)^* \otimes U \to (A^!_{p-1})^* \otimes U \to \cdots \to U.$$

By the same argument as in the proof of Lemma 5 the map $k \to GF(k)$ is a quasi-isomorphism. So if $N = N^0$ we have $GF(N) = GF(k) \otimes_k N$ and so $N \to GF(N)$ is also a quasi-isomorphism.

By induction on the length of the truncations, we know that $N \to GF(N)$ is a quasi-isomorphism for bounded N. Now let N be bounded above. We know that $N = \varinjlim \sigma^{>p} N$ for $p \to -\infty$ and we also know that for N bounded above,

$$\sigma^{>p} N \to GF(\sigma^{>p} N)$$

is a quasi-isomorphism for all p since each $\sigma^{>p} N$ is bounded. Since we know that direct limit commutes with taking cohomology, we would like to show that

$$GF(\varinjlim \sigma^{>p} N) = G \varinjlim F(\sigma^{>p} N) = \varinjlim GF(\sigma^{>p} N).$$

The first equality is clear since F is a left adjoint but it remains to show that G commutes with direct limit. We know

$$GF(\sigma^{>p} N)^i = \prod_{r \ge 0} \mathrm{Hom}_k((A^!)^r, F(\sigma^{>p} N)^{i+r})$$

and since $p - i + 1 \le r \le t - i$ we have

$$GF(\sigma^{>p} N)^i = \bigoplus_{r=p-i+1}^{t-i} \mathrm{Hom}_k((A^!)^r, F(\sigma^{>p} N)^{i+r}).$$

If we take the direct limit of both sides as $p \to -\infty$ we have

$$\varinjlim GF(\sigma^{>p} N)^i = \bigoplus_{0}^{t-i} \mathrm{Hom}_k((A^!)^r, F(\sigma^{>p} N)^{i+r})$$

which we would like to have equal to $G \varinjlim F(\sigma^{>p} N)$. We know

$$\begin{aligned} G \varinjlim F(\sigma^{>p} N) &= \prod_{r \ge 0} \mathrm{Hom}_k((A^!)^r, [\varinjlim F(\sigma^{>p} N)]^{i+r}) \\ &= \prod_{r \ge 0} \mathrm{Hom}_k((A^!)^r, \varinjlim U \otimes (\sigma^{>p} N)^{i+r}) \\ &= \prod_{r \ge 0} \varinjlim \mathrm{Hom}_k((A^!)^r, U \otimes (\sigma^{>p} N)^{i+r}). \end{aligned}$$

The third equality is due to the fact that each $(A^!)^r$ is finite dimensional and since we know that r is bounded, i.e., $p-i+1 \leq r \leq t-i$ we have for $p \to -\infty$

$$G \varinjlim F(\sigma^{>p}N) = \varinjlim \bigoplus_{r=0}^{t-i} \mathrm{Hom}_k((A^!)^r, F(\sigma^{>p}N)^{i+r})$$

which eventually stabilizes in this range of r giving us

$$G \varinjlim F(\sigma^{>p}N) = \varinjlim GF(\sigma^{>p}N)$$

showing that G can commute with direct limit and showing that for N bounded above, we have $N \to GF(N)$ is a quasi-isomorphism.

Now let N be arbitrary. We know for $p \to \infty$,

$$\varprojlim \sigma^{\leq p}N = N$$

and we know that each $\sigma^{\leq p}N$ is bounded above so for all p we know that

$$\sigma^{\leq p}N \to GF(\sigma^{\leq p}N)$$

is a quasi-isomorphism. By Lemma 7 we know

$$\varprojlim \sigma^{\leq p}N = \varprojlim GF(\sigma^{\leq p}N)$$

and we also know that

$$\varprojlim GF(\sigma^{\leq p}N) = G \varprojlim F(\sigma^{\leq p}N)$$

since G is a right adjoint. It remains to show that F can commute with inverse limit. We know that $F(N)^i = U \otimes N^i$ by definition and we know that for a fixed i and for $p \to \infty$ we have

$$\varprojlim F(\sigma^{\leq p}N)^i = \varprojlim U \otimes (\sigma^{\leq p}N)^i$$

and since p will become greater that i after finitely many steps, both sides will equal $U \otimes N^i$ showing our intended equality and thus proving that for arbitrary N, $N \to GF(N)$ is a quasi-isomorphism.

Proposition 5. *Let K_p^* be a projective system of complexes of modules over a ring $R : \ldots \to K_{p+1}^* \to K_p^* \to \ldots$. Suppose that K_p^* satisfies the Mittag-Leffler condition (ML) and that each $H^a(K_p^*)$ satisfies the ML condition, for instance if it satisfies the descending chain condition (dcc), (eg., if each is a finite dimensional vector space over the field $R = k$). Then*

$$(\varprojlim_p)H^a(K_p^*) = H^a(\varprojlim_p K_p^*).$$

Lemma 6. *For all sufficiently large i, the projective systems*

$$p \to F_i G(\sigma^{\le p} M) \quad \text{and} \quad p \to H^a(F_i G(\sigma^{\le p} M))$$

satisfy the ML condition when M is bounded below, therefore

$$\varprojlim_p H^a(F_i G(\sigma^{\le p} M)) = H^a(F_i G(M)) = H^a(\varprojlim_p F_i G(\sigma^{\le p} M)).$$

Proof. Since M is bounded below we know that $\sigma^{\le p} M$ is a bounded complex. By choosing i large enough we know that $F_i G(\sigma^{\le p})M \to \sigma^{\le p} M$ is a quasi-isomorphism. Note that the i that works depends only on the lower bound of the complex $\sigma^{\le p} M$ and this is the same for all p, so let us fix an i. Thus $p \to H^a(F_i G(\sigma^{\le p} M))$ is ML since for large p (in fact $p > a$), these values are constant and equal to $H^a(M)$.

For $F_i G(\sigma^{\le p})M$ note that in any degree, the transition maps

$$F_i G(\sigma^{\le p+1} M)^j = U_{i+j} \otimes \prod_{r=0}^{p+1-j} \mathrm{Hom}((A^!)^r, M^{j+r}) \to F_i G(\sigma^{\le p} M)^j$$

induced by the natural projection

$$\prod_{r=0}^{p+1-j} \mathrm{Hom}((A^!)^r, M^{j+r}) \to \prod_{r=0}^{p-j} \mathrm{Hom}((A^!)^r, M^{j+r})$$

are surjective, hence we have the ML condition satisfied.

Lemma 7. *For all sufficiently large i, the projective systems*

$$p \to GF(\sigma^{\le p} N) \quad \text{and} \quad p \to H^a(GF(\sigma^{\le p} N))$$

satisfy the ML condition for any N, therefore

$$\varprojlim_p H^a(GF(\sigma^{\le p} N)) = H^a(GF(N)) = H^a(\varprojlim_p GF(\sigma^{\le p} N)).$$

Proof. The second of the two projective systems clearly satisfies the ML condition since $H^a(GF(\sigma^{\le p} N)) = H^a(\sigma^{\le p} N)$ and since each $\sigma^{\le p} N$ is bounded above, and since for all i the projective system is constant for large values of p and it equals $H^a(N)$. Now to show that the first system satisfies the ML condition observe that

$$(GF\sigma^{\le p} N)^j = \prod_{r \ge 0} \mathrm{Hom}((A^!)^r, U \otimes (\sigma^{\le p} N)^{j+r})$$

and since $\sigma^{\le p} N$ equals N^{j+r} when $j + r \le p$ and zero otherwise we have

$$(GF\sigma^{\le p} N)^j = \prod_{r \ge 0}^{p-j} \mathrm{Hom}((A^!)^r, U \otimes N^{j+r})$$

so the natural projection

$$\prod_{r\geq 0}^{p-j}\operatorname{Hom}((A^!)^r, U\otimes N^{j+r}) \to \prod_{r\geq 0}^{p-j-1}\operatorname{Hom}((A^!)^r, U\otimes N^{j+r})$$

is clearly surjective thus showing that the ML condition is satisfied.

7. Equivalences of Categories

Let $K_\Lambda(A^!, d, c)$ be the category $Com_\Lambda(A^!, d, c)$ with morphisms being chain homotopy equivalence classes of maps. Similarly we may define the category $K_\Lambda(U)$.

Definition 7. *The null system, N, of $K_\Lambda(A^!, d, c)$ is defined to be all of the complexes, X, such that $F(X)$ is acyclic. The null system, N, of $K_\Lambda(U)$ is defined to be all of the complexes, Y, such that $G(Y)$ is acyclic.*

We define $D_\Lambda(A^!, d, c)$ to be the category $K_\Lambda(A^!, d, c)/N$ where N is the null system of $K_\Lambda(A^!, d, c)$. We also define $D_\Lambda(U)$ to be the category $K_\Lambda(U)/N$ where N is the null system of $K_\Lambda(U)$.

Theorem 3 (The Main Result)**.** *The adjunction F and G given in Proposition 3 induces an equivalence of categories between the quotient categories $D_\Lambda(A^!, d, c)$ and $D_\Lambda(U)$.*

Proof. The proof of this is exactly the same as in [3].

8. Relating Floystad's Duality to the Classical Koszul Duality

Classical Koszul duality concerns the pair of positively graded algebras A and $A^!$. Floystad's version of Koszul duality considers the case where $U = A$ for a filtered algebra U, meaning that the filtration arises from a grading of A. The case which is of interest to us is represented in the following commutative diagram

$$\begin{array}{ccc}
G_b : C(B^!) & \overset{\longleftarrow}{\longrightarrow} & C(B) : F_b \\
{\scriptstyle i}\big\downarrow & & {\scriptstyle j}\big\downarrow \\
\tilde{G} : C_{\mathbb{Z}}(A) & \overset{\longleftarrow}{\longrightarrow} & C_{\mathbb{Z}}(A^!_\bullet) : \tilde{F} \\
\Big\| & & \phi \uparrow \ \downarrow \psi \\
G : Com_{\mathbb{Z}}(U = A) & \overset{\longleftarrow}{\longrightarrow} & Com_{\mathbb{Z}}(A^!, d = 0, c = 0) : F.
\end{array}$$

In this commutative diagram $B = A^!$, $C_{\mathbb{Z}}(A)$ is the category of chain complexes of graded left A-modules and $C_{\mathbb{Z}}(A^!_\bullet)$ is the category of chain complexes of graded left $A^!$-modules. The functors F_b and G_b are the functors from [1] given by

$$(F_b M)^p_q = \bigoplus_{\substack{p=i+j \\ q=l-j}} B^!_l \otimes M^i_j \quad , \quad (G_b N)^p_q = \bigoplus_{\substack{p=i+j \\ q=l-j}} \operatorname{Hom}_k(B_{-l}, N^i_j).$$

The functors $\tilde{F}$ and $\tilde{G}$ have been constructed such that $\tilde{F} = F\psi$ and $\tilde{G} = G\phi$ making the lower half of the diagram commute. We would like to relate the following

$$\psi : C_{\mathbb{Z}}(A^!_\bullet) \leftrightarrows Com_{\mathbb{Z}}(A^!, d = 0, c = 0) : \phi$$

by defining ϕ and ψ such that they are well defined, inverses of each other, and make the diagram commute. Now note the $A^!_\bullet$ on the left hand side is a ring while the $A^!$ on the right hand side will be regarded as a CDGA with $d = c = 0$. Namely we define $(\psi A^!_\bullet) = (A^!, d = 0, c = 0)$ as the $\mathbb{Z}$-graded CDGA by the rule,

$$(\psi A^!)^i_j = \begin{cases} 0 & \text{if } i + j \neq 0 \\ A^!_i & \text{if } -j = i \geq 0. \end{cases}$$

For any complex $(M, d_M) \in C_{\mathbb{Z}}(A^!_i)$ we define a CDG-$A^!$-module $(\psi M) = (N, d_N)$ by the rule $(\psi M)^i_j = N^i_j = M^{i+j}_{-j}$ with differential $d_N(x) = (-1)^s d_M(x)$ for $x \in N^s$. One must now check all the necessary axioms to show that $\psi(M, d_M) = (N, d_N)$ is an element of $Com_{\mathbb{Z}}(A^!, d = 0, c = 0)$ for $(M, d_M) \in C_{\mathbb{Z}}(A^!_\bullet)$. Clearly $d_N^2 = 0$ since we know that $d_M^2 = 0$. We also need to check $A^!$-linearity of d_N meaning that for $x \in N^s$ and $\alpha \in (A^!)^k_{-k}$ we have

$$d_N(\alpha x) = (-1)^s \alpha d_N(x).$$

This is clearly true since both sides equal $(-1)^{s+k}\alpha d_M(x)$. This also verifies that

$$d_N(\alpha x) = d_{A^!}(\alpha)(x) + (-1)^{deg(\alpha)}\alpha d_N(x)$$

since $d_{A^!}(\alpha)(x) = 0$. Now we need to check the module structure of (N, d_N). We know that

$$A^!_p \times M^{r+s}_{-s} \subset M^{r+s}_{p-s}$$

and under ψ this corresponds to

$$(A^!)^p_{-p} \times N^r_s \subset N^{p+r}_{-p+s}$$

and we know this is true by our definition of ψ verifying that $\psi(M)$ is an $A^!$-module. Now we'd like to check that $\phi(N, d_N) = (M, d_M)$ for $(M, d_M) \in C_{\mathbb{Z}}(A^!_\bullet)$. Now let us define ϕ such that $\phi(A^!, d = 0, c = 0) = A^!_\bullet$. Namely we define $\phi(A^!, d = 0, c = 0)$ as the chain complex of graded left $A^!$-modules by the rule

$$\phi(A^!, d = 0, c = 0)^i_{-j} = (A^!_\bullet)_j$$

and we will define an $A^!$-module by the rule

$$(\phi N)^i_j = M^i_j = N^{i+j}_{-j}$$

with differential $d_M(x) = (-1)^{i+j} d_N(x)$ for $x \in M^i_j$. Clearly $d_M^2 = 0$ since $d_N = 0$. We also must show that $d_M(\alpha x) = \alpha d_M(x)$ and this is clear since

$$d_M(\alpha x) = (-1)^{i+j}\alpha(d_N(x)) = \alpha(d_M(x)).$$

Finally it remains to show that

$$A_r^! \times M_j^i \subset M_{j+r}^i$$

which, under ϕ, corresponds to

$$(A^!)_{-r}^r \otimes N_{-j}^{i+j} \subset N_{-j-r}^{i+j+r}$$

which we know is true by our definition of ϕ thus completing our verification that $\phi(A^!, d = 0, c = 0) = A_\bullet^!$.

Now we need to verify that the diagram commutes. The lower half of the diagram commutes by definition of $\tilde{F}$ and $\tilde{G}$. More explicitly, $\tilde{F} = F\psi$ and $\tilde{G} = G\phi$ by definition. What remains is to verify that

$$F\psi(M, d_M)_q^p = (F_b M)_q^p \quad \text{and} \quad \phi G(N, d_N)_q^p = (G_b N)_q^p.$$

We know that

$$F\psi(M, d_M)_q^p = \bigoplus_{r+s=q} A_r \otimes \psi(M)_s^p = \bigoplus_{r+s=q} B_r^! \otimes M_{-s}^{p+s}$$

and we also know that

$$(F_b M)_q^p = \bigoplus_{\substack{p=i+j \\ q=l-j}} B_l^! \otimes M_j^i$$

so $F\psi(M, d_M)_q^p = (F_b M)_q^p$ as long as $i = p + s$, $j = -s$, and $l = r$ which are all clearly true since we know that $p = i + j$, $q = l - j$ and $r + s = q$. We also must check that the differential on $F\psi(M, d_M)_q^p$ matches the differential on $(F_b M)_q^p$. For $x \in N^s$ we have the differential on $F\psi(M, d_M)$ is given by

$$d_F(a \otimes n) = \Sigma a x_\alpha \otimes \check{x}_\alpha n + a \otimes d_N(n) = \Sigma a x_\alpha \otimes \check{x}_\alpha n + (-1)^s a \otimes d_M(n).$$

The differential for $F_b(M, d_M)$ with $a \otimes m \in B_M^! \otimes M_j^i$ is given by

$$d_{F_b}(a \otimes n) = (-1)^{i+j} \Sigma a \check{v}_\alpha \otimes v_\alpha m + a \otimes d_M(n).$$

Since $i + j = s$ we see that d_F and d_{F_b} differ by a sign:

$$d_F(a \otimes n) = (-1)^{i+j} d_{F_b}(a \otimes n).$$

Let us check that $G_b = \phi G$. We know that the following expression

$$(G_b N)_q^p = \bigoplus_{\substack{p=i+j \\ q=l-j}} \mathrm{Hom}_k(B_{-l}, N_j^i)$$

will be equal to

$$\phi G(N, d_N)_q^p = G(N, d_N)_{-q}^{p+q} = \bigoplus_{r \geq 0} \mathrm{Hom}_k(B_r, N_{-q-r}^{p+q+r})$$

only if $r = -l$, $j = -q - r$, and $i = p + q + r$. These identities follow immediately by definition of p and q proving that $G_b = \phi G$. To verify that the differentials agree consider the total differential on $(G_b N)^i_l$, for $b \in B = A^!$ given by

$$(d_{G_b} f)(b) = (-1)^i \Sigma \check{v}_\alpha f(v_\alpha a) + d_N(f(b))$$

and the differential on $\phi G(N, d_N)^p_q = \mathrm{Hom}_k((A^!)^r_{-r}, N^{p+q+r}_{-q-r})$ for $a \in A^!$ which is given by

$$\begin{aligned}(d_G f)(a) &= (-1)^{|f|+1} \Sigma x_\alpha f(\check{x}_\alpha a) + (-1)^{|f|+1} f(d_{A^!}(a)) + d_N(f(a)).\\ &= (-1)^{p+q+1} \Sigma x_\alpha f(\check{x}_\alpha a) + (-1)^{p+q+1} d_N(f(a))\\ &= (-1)^{r+1} \Sigma x_\alpha f(\check{x}_\alpha a) + d_N(f(a))\end{aligned}$$

We can see that the two differentials, d_{G_b} and d_G, only differ by a sign.

References

[1] A. Beilinson, V. Ginzburg, and W. Soergel. *Koszul duality patterns in representation theory*, Journal of the American Mathematical Society. **9**), no.2 473-527. 1996.

[2] I. Bernstein, I. Gelfand, and S. Gelfand. *Algebraic bundles over* $\mathbf{P}^n$ *and problems of linear algebra*, Funktsional'nyi Analiz i ego prilozheniya **12**); English translation in Functional analysis and its applications **12**, 212-214. 1978.

[3] G. Floystad. *Koszul duality and equivalences of categories*, Transactions of the American Mathematical Society. 358, p. 2373-2398. 2006.

[4] M. Kapranov. *On the derived categories of coherent sheaves on some homogeneous spaces*, Inventions Mathematicae. 92, 479-508. 1988.

[5] A. Polishchuk and L. Positselski. *Quadratic Algebras*, University Lecture Series,37. American Mathematical Society, Providence, RI, 2005.