EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 5, No. 4, 2012, 492-510
ISSN 1307-5543 – www.ejpam.com

Stringy and Orbiforld Cohomology of Wreath Product Orbifolds

Tomoo Matsumura

Department of Mathematical Sciences, ASARC, KAIST, 291 Daehak-ro Yuseong-gu, Daejeon 305-701, South Korea

---

Abstract. Let $[X/\mathsf{G}]$ be an orbifold which is a global quotient of a compact almost complex manifold X by a finite group G. Let Σ_n be the symmetric group on n letters. Their semidirect product $\mathsf{G}^n \rtimes \Sigma_n$ is called the *wreath product* of G and it naturally acts on the n-fold product X^n, yielding the orbifold $[X^n/(G^n \rtimes \Sigma_n)]$. Let $\mathscr{H}(X^n, \mathsf{G}^n \rtimes \Sigma_n)$ be the stringy cohomology [7, 10] of the $(\mathsf{G}^n \rtimes \Sigma_n)$-space X^n. We prove that the space G^n-invariants of $\mathscr{H}(X^n, \mathsf{G}^n \rtimes \Sigma_n)$ is isomorphic to the algebra $H_{orb}([X/\mathsf{G}])\{\Sigma_n\}$ introduced by Lehn and Sorger [14], where $H_{orb}([X/\mathsf{G}])$ is the Chen-Ruan orbifold cohomology of $[X/\mathsf{G}]$. We also prove that, if X is a projective surface with trivial canonical class and Y is a crepant resolution of X/G, then the Hilbert scheme of n points on Y, denoted by $Y^{[n]}$, is a crepant resolution of $X^n/(\mathsf{G}^n \rtimes \Sigma_n)$. Furthermore, if $H^*(Y)$ is isomorphic to $H_{orb}([X/\mathsf{G}])$ as Frobenius algebras, then $H^*(Y^{[n]})$ is isomorphic to $H^*_{orb}([X^n/(\mathsf{G}^n \rtimes \Sigma_n)])$ as rings. Thus we verify a special case of the cohomological hyper-Kähler resolution conjecture due to Ruan [22].

2010 Mathematics Subject Classifications: 14N35 14A20 14E15 14J81 14L30

Key Words and Phrases: Orbifold, Gromov-Witten Theory, Frobenius Algebra, Symmetric Product, Wreath Product

---

1. Introduction

The *stringy cohomology* $\mathscr{H}(X, \mathsf{G})$ of an almost complex manifold X with an action of a finite group G was first introduced by Fantechi-Göttsche [7] and studied further by Jarvis-Kaufmann-Kimura [10, 11]. It is a G-Frobenius algebra [23, 12] which is a G-equivariant generalization of Frobenius algebras and the space of its G-invariants is the Chen-Ruan orbifold cohomology $H^*_{orb}([X/\mathsf{G}])$ introduced in [4].

Let $\mathscr{W}$ be an orbifold and $\pi : Y \to W$ be a hyper-Kähler resolution of the coarse moduli space W of $\mathscr{W}$. Ruan's cohomological hyper-Kähler resolution conjecture [22] predicts that the ordinary cohomology ring of Y is isomorphic to the orbifold cohomology ring of $\mathscr{W}$ over $\mathbb{C}$-coefficients. This is a special case of the crepant resolution conjecture of Ruan [22] and Bryan-Graber [3].

Email address: `tomoomatsumura@kaist.ac.kr`

Among the examples which support the cohomological hyper-Kähler resolution conjecture, the symmetric product is perhaps the most fascinating. The symmetric group on n-letters, Σ_n, naturally acts on the n-fold product Y^n of a manifold Y, yielding the *symmetric product orbifold* $[Y^n/\Sigma_n]$. If Y is a projective surface with trivial canonical class, then the Hilbert scheme of n points on Y, denoted by $Y^{[n]}$, is a hyper-Kähler resolution of the quotient space Y^n/Σ_n [1]. Fantechi and Göttsche [7] showed that the ring of Σ_n-invariants of $\mathscr{H}(Y^n,\Sigma_n)$ is isomorphic to $H^*(Y^{[n]})$ over $\mathbb{C}$. Their proof proceeds by showing that $\mathscr{H}(Y^n,\Sigma_n)$ is isomorphic to the algebra $H^*(X)\{S_n\}$ defined by Lehn and Sorger [14], i.e.

$$\mathscr{H}(Y^n,\Sigma_n)\cong H^*(Y)\{\Sigma_n\} \implies H^*_{orb}([Y^n/\Sigma_n])\cong H^*(Y)\{\Sigma_n\}^{\Sigma_n}\cong H^*(Y^{[n]})$$

where the last isomorphism is due to [14] (see also [24, 18, 16]).

In this paper, we consider a generalization of the algebra isomorphism on the left-hand side of the arrow above, namely, replace Y by an orbifold $[X/\mathsf{G}]$ and $H^*(Y)$ by $H^*_{orb}([X/\mathsf{G}])$. The symmetric group Σ_n naturally acts on the n-fold product G^n and their semidirect product $\mathsf{G}^n\rtimes\Sigma_n$ is called the *wreath product* of G. It naturally acts on the n-fold product X^n, yielding the orbifold $[X^n/(\mathsf{G}^n\rtimes\Sigma_n)]$. This orbifold is called the *wreath product orbifold* of a G-space X. The linear structure of the orbifold cohomology of a wreath product orbifold has been studied in a sequence of papers by Qin, Wang and Zhou, cf. [18, 25, 26] through a careful analysis of the fixed point loci. However, one of the goals of this paper is to analyze the multiplication in stringy cohomology and in Chen-Ruan orbifold cohomology of a wreath product orbifold. The multiplication in the special case when $X=\mathbb{C}^2$ and G is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ has been studied in [6, 19].

The main result of this paper is Theorem 4 which proves that, when X is compact, there is a canonical Σ_n-Frobenius algebra isomorphism

$$\mathscr{H}(X^n,\mathsf{G}^n\rtimes\Sigma_n)^{\mathsf{G}^n}\cong H^*_{orb}([X/\mathsf{G}])\{\Sigma_n\}.$$

When G is a trivial group, this isomorphism reduces to the isomorphism defined by Fantechi and Göttsche [7]. This result means that $\mathscr{H}(X^n,\mathsf{G}^n\rtimes\Sigma_n)^{\mathsf{G}^n}$ gives a geometric construction of the *second quantization* [Definition 8.16, 13] of an orbifold $[X/\mathsf{G}]$.

There are two results that play key roles in our proof of the main theorem. One is the formula (1) proved in [11] for the obstruction bundle of the stringy cohomology. Since their definition avoids any construction of complex curves, admissible covers, or moduli spaces, it greatly simplifies the analysis of the obstruction bundle and allows us to write the obstruction bundle of $[X^n/(\mathsf{G}^n\rtimes\Sigma_n)]$ in terms of the ones of $[X/G]$ and $[X^n/\Sigma_n]$. The other result is Theorem 6.5 of [13] which states that there is a unique product structure on a *normalized, special* Σ_n-Frobenius algebra (reviewed in Appendix). Lemma 3 which computes the obstruction bundle in a certain case using the formula (1) is necessary to apply Theorem 6.5 of [13] and prove our main theorem. The direct and geometric proof of Theorem 4 in the case of an abelian group G is also available in [17].

In order to relate our result to Ruan's conjecture, we prove that, if X/G is an even dimensional Gorenstein variety and Y is a crepant resolution of X/G, then the natural map $Y^n/\Sigma_n\longrightarrow X^n/(\mathsf{G}^n\rtimes\Sigma_n)$ is crepant (Theorem 5). This implies that, if Y is a projective surface with the trivial canonical class, then $Y^{[n]}$ is a crepant resolution of $X^n/(\mathsf{G}^n\rtimes\Sigma_n)$, i.e.

the composition $Y^{[n]} \longrightarrow Y^n/\Sigma_n \longrightarrow X^n/(\mathsf{G}^n \rtimes \Sigma_n)$ is a crepant resolution (conjectured in [25, p.20]). Together with Theorem 4 and the result in [14], we obtain a verification of the cohomological hyper-Kähler resolution conjecture in a special case: if $H^*(Y) \cong H^*_{orb}([X/G])$, then

$$H^*_{orb}([X^n/(\mathsf{G}^n \rtimes \Sigma_n)]) \cong H^*_{orb}([X/\mathsf{G}])\{\Sigma_n\}^{\Sigma_n} \cong H^*(Y)\{\Sigma_n\}^{\Sigma_n} \cong H^*(Y^{[n]}).$$

When $X = \mathbb{C}^2$ and G is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, it is proved in a completely different way [6].

The structure of the rest of the paper is as follows. In Section 2, we review the definition of a G-Frobenius algebra and show that, if $\mathscr{H}$ is an $(K \rtimes L)$-Frobenius algebra, then the K-invariants of $\mathscr{H}$ form an L-Frobenius algebra. Also we review the construction of stringy and orbifold cohomology following [11]. In Section 3, we study wreath product orbifolds and compute the obstruction bundles for the cases that we need to prove the main theorem. In Section 4, we prove the main theorem. In Section 5, we prove the crepantness of the map $Y^n/\Sigma_n \longrightarrow X^n/(\mathsf{G}^n \rtimes \Sigma_n)$ and apply our main theorem to verify the spacial case of the cohomological hyper-Kähler resolution conjecture. In the Appendix, we review the construction of Lehn-Sorger's algebras and the uniqueness theorem of Kaufmann.

Unless otherwise specified, we assume throughout the paper that *all groups are finite and all group actions are left actions*. Also, unless otherwise specified, all of the vector spaces are finite dimensional and over $\mathbb{Q}$, and all coefficient rings for cohomology and K-theory are $\mathbb{Q}$.

2. G-Frobenius Algebras and Semidirect Products

Recall the definition of a G-Frobenius algebra for a group G from [11] Section 3.

Definition 1. *Let G be a group. A* G-graded G-module $(\mathscr{H}, \rho)$ *is a G-graded vector space $\mathscr{H} := \bigoplus_{g\in\mathsf{G}} \mathscr{H}_g$ with the structure of a left G-module by isomorphisms $\rho_g : \mathscr{H} \xrightarrow{\simeq} \mathscr{H}$ such that ρ_g takes $\mathscr{H}_h$ to $\mathscr{H}_{ghg^{-1}}$ for all g, h in G. We denote a vector in $\mathscr{H}_g$ by v_g for any $g \in \mathsf{G}$*

Definition 2. *A tuple $(\mathscr{H}, \rho, \cdot, \mathbf{1}, \eta)$ is said to be a* G-(equivariant) Frobenius algebra *provided that the following properties hold:*

i) (G-graded G-module) $(\mathscr{H}, \rho)$ is a G-graded G-module.

ii) (Self-invariance) For all g in G, $\rho_g : \mathscr{H}_g \to \mathscr{H}_g$ is the identity map.

iii) (Metric) η is a symmetric non-degenerate bilinear form on $\mathscr{H}$ s.t. $\eta(v_g, v_h) = 0$ unless $gh = 1$.

iv) (Associativity) $(\mathscr{H}, \cdot, \mathbf{1})$ is a unital associative algebra.

v) (G-graded Multiplication) $v_g \cdot v_h \in \mathscr{H}_{gh}$ for all $g, h \in \mathsf{G}$.

vi) (Braided Commutativity) $v_g \cdot v_h = \rho_g(v_h) \cdot v_g$ for all $g, h \in G$.

vii) (G-equivariance of the Multiplication) $\rho_g(v) \cdot \rho_g(w) = \rho_g(v \cdot w)$ for all g in G, and all $v, w \in \mathscr{H}$.

viii) *(G-invariance of the Metric) $\eta(\rho_g(v), \rho_g(w)) = \eta(v, w)$ for all g in G, and all $v, w \in \mathscr{H}$.*

ix) *(Invariance of the Metric) $\eta(v_1 \cdot v_2, v_3) = \eta(v_1, v_2 \cdot v_3)$ for all $v_1, v_2, v_3 \in \mathscr{H}$.*

x) *G-invariant Identity) $\rho_g(\mathbf{1}) = \mathbf{1}$ for all g in G.*

xi) *(Trace Axiom) For all a, b in G and v in $\mathscr{H}_{[a,b]}$, if L_v denotes the left multiplication by v, then the following equation is satisfied: $\mathrm{Tr}_{\mathscr{H}_a}(L_v \circ \rho_b) = \mathrm{Tr}_{\mathscr{H}_b}(\rho_{a^{-1}} \circ L_v)$.*

Remark 1.

1) *The G-Frobenius algebras are introduced in [23, 12]. Definition 2.1.1 of [12] is slightly more general than the above definition (see ii) and xi)). Our definition is obtained by setting $\chi_g = 1$ for all $g \in \mathsf{G}$.*

2) *A G-Frobenius algebra when $\mathsf{G} = \{1\}$ is a Frobenius algebra in the usual sense.*

3) *We can also define a G-Frobenius superalgebra [12] by introducing $\mathbb{Z}/2\mathbb{Z}$-grading and by introducing signs in the usual manner, c.f. Section 1.2 of [13].*

Definition 3. *A G-Frobenius algebra $\mathscr{H}$ is said to be* $\mathbb{Q}$-graded *if each $\mathscr{H}_g$ comes with a $\mathbb{Q}$-grading $\mathscr{H}_g = \bigoplus_{r \in \mathbb{Q}} \mathscr{H}_{g,r}$ and the G-action and the multiplication respect the $\mathbb{Q}$-grading and the metric η satisfies $\eta(v, w) = 0$ unless $\deg a + \deg b = d \geq 0$, i.e. $\mathscr{H}$* has degree *d. In this paper, we assume that all G-Frobenius algebras are $\mathbb{Q}$-graded.*

2.1. K-Invariants of a $\mathsf{K} \rtimes \mathsf{L}$-Frobenius Algebra

Let K and L be groups. Suppose that L acts on K from left where the action of $l \in \mathsf{L}$ on $k \in \mathsf{K}$ is denoted by $k \overset{l}{\mapsto} k^{l^{-1}}$. Let $\mathsf{K} \rtimes \mathsf{L}$ be a semidirect of groups K and L with respect to this action. We identify K with the normal subgroup $\mathsf{K} \rtimes 1$ and hence the *left* adjoint action of L on K can be identified with the given action of L on K, namely, we have $k^{l^{-1}} = lkl^{-1}$, i.e. $lk^l = kl$ or $lk = k^{l^{-1}}l$.

Let $(\mathscr{H}, \rho, \cdot, \mathbf{1}, \eta)$ be a $(\mathsf{K} \rtimes \mathsf{L})$-Frobenius algebra. Let $\pi_{\mathsf{K}} : \mathscr{H} \to \mathscr{H}$ be the *averaging map* over K:

$$\pi_{\mathsf{K}}(v) := \frac{1}{|\mathsf{K}|} \sum_{k \in \mathsf{K}} \rho_k(v).$$

The image $\pi_{\mathsf{K}}(\mathscr{H})$ is the space of K-invariants of $\mathscr{H}$, which we denote by $\mathscr{H}^{\mathsf{K}}$. The direct sum $\mathscr{H}_{[l]} := \oplus_{k \in \mathsf{K}} \mathscr{H}_{kl}$ is a K-module and so denote its K-invariants also by $\mathscr{H}^{\mathsf{K}}_{[l]} := \pi_{\mathsf{K}}(\mathscr{H}_{[l]})$. The following theorem is the starting point of this paper.

Theorem 1. *If $\mathscr{H}$ is a $(\mathsf{K} \rtimes \mathsf{L})$-Frobenius algebra, then $\mathscr{H}^{\mathsf{K}}$ is an L-Frobenius algebra.*

Proof. All of the properties except the self-invariance property and the trace axiom follow immediately from those properties of $\mathscr{H}$. The self-invariance property of $\mathscr{H}^{\mathsf{K}}$ is that, for all $l \in L$, $\rho_l : \mathscr{H}^{\mathsf{K}}_{[l]} \to \mathscr{H}^{\mathsf{K}}_{[l]}$ is the identity map. This is true because of the self-invariance property of $\mathscr{H}$. Indeed, for all $kl \in K \rtimes L$, ρ_{kl} restricted to $\mathscr{H}_{kl}$ is the identity map so that $\rho_k = \rho_{l^{-1}}$ on $\mathscr{H}_{kl}$. Let $v \in \mathscr{H}_{k_0 l}$, then

$$\rho_l \pi_{\mathsf{K}}(v) = \frac{1}{|\mathsf{K}|}\sum_{k\in\mathsf{K}} \rho_l \rho_k v = \frac{1}{|\mathsf{K}|}\sum_{k'\in\mathsf{K}} \rho_{k'}\rho_l v = \frac{1}{|\mathsf{K}|}\sum_{k'\in\mathsf{K}} \rho_{k'}\rho_{k_0^{-1}} v = \frac{1}{|\mathsf{K}|}\sum_{k''\in\mathsf{K}} \rho_{k''} v = \pi_{\mathsf{K}}(v)$$

For the trace axiom for $\mathscr{H}^{\mathsf{K}}$, we need to show $\mathrm{Tr}_{\mathscr{H}^{\mathsf{K}}_{[l_1]}}(L_{v_m} \circ \rho_{l_2}) = \mathrm{Tr}_{\mathscr{H}^{\mathsf{K}}_{[l_2]}}(\rho_{l_1^{-1}} \circ L_{v_m})$, for $l_1, l_2 \in L$ and $v_m \in \mathscr{H}^{\mathsf{K}}_{[m]}$ where $m = [l_1, l_2]$. The left-hand side is

$$\begin{aligned}\mathrm{Tr}_{\mathscr{H}^{\mathsf{K}}_{[l_1]}}(L_{v_m} \circ \rho_{l_2}) &= \mathrm{Tr}_{\mathscr{H}_{[l_1]}}(L_{v_m} \circ \rho_{l_2} \circ \pi_{\mathsf{K}}) = \frac{1}{|\mathsf{K}|}\sum_{k_1,k} \mathrm{Tr}_{\mathscr{H}_{k_1 l_1}}(L_{v_m} \circ \rho_{l_2} \circ \rho_k)\\ &= \frac{1}{|\mathsf{K}|}\sum_{k_1,k_2} \mathrm{Tr}_{\mathscr{H}_{k_1 l_1}}(L_{v_m} \circ \rho_{k_2 l_2}) = \frac{1}{|\mathsf{K}|}\sum_{k_1,k_2} \mathrm{Tr}_{\mathscr{H}_{k_2 l_2}}(\rho_{(k_1 l_1)^{-1}} \circ L_{v_m}),\end{aligned}$$

where the third equality is obtained by replacing the parameter $k^{l_2^{-1}}$ by k_2 and the fourth equality follows from the trace axiom for $\mathscr{H}$. The right-hand side is

$$\mathrm{Tr}_{\mathscr{H}^{\mathsf{K}}_{[l_2]}}(\rho_{l_1^{-1}} \circ L_{v_m}) = \frac{1}{|\mathsf{K}|}\sum_{k,k_2} \mathrm{Tr}_{\mathscr{H}_{k_2 l_2}}(\rho_{l_1^{-1}} \circ L_{v_m} \circ \rho_k) = \frac{1}{|\mathsf{K}|}\sum_{k_1,k_2} \mathrm{Tr}_{\mathscr{H}_{k_2 l_2}}(\rho_{l_1^{-1} k_1^{-1}} \circ L_{v_m}),$$

where the second equality follows from the cyclicity of the trace and by replacing the parameter $k_1^{l_1^{-1}}$ by k_1^{-1}. Thus, the trace axiom holds for the K-invariants $\mathscr{H}^{\mathsf{K}}$.

2.2. Stringy and Orbifold Cohomology

We review the definition of the stringy and Chen-Ruan orbifold cohomology, following [7] and [11]. Let X be a compact almost complex manifold of complex dimension d with an action ρ of a finite group G preserving the almost complex structure. Let $X^{g_1,\cdots,g_r}$ be the submanifold of points in X fixed by the subgroup generated by $g_1, \cdots, g_r \in \mathsf{G}$. Then

$$\mathscr{H}(X, \mathsf{G}) := \bigoplus_{g\in\mathsf{G}} H^*(X^g)$$

is naturally a G-graded G-module where $\rho_g : X^h \longrightarrow X^{ghg^{-1}}$ $(x \mapsto \rho_g x)$. The G-equivariant multiplication requires the class of the obstruction bundle in rational K-theory $K(X^{g,h}) \otimes \mathbb{Q}$ [7, 11]:

$$\mathscr{R}(g,h) = TX^{g,h} \ominus TX|_{X^{g,h}} \oplus \mathscr{S}_g|_{X^{g,h}} \oplus \mathscr{S}_h|_{X^{g,h}} \oplus \mathscr{S}_{(gh)^{-1}}|_{X^{g,h}}. \tag{1}$$

Here, the class $\mathscr{S}_g$ in $K(X^g) \otimes \mathbb{Q}$ is given by

$$\mathscr{S}_g := \bigoplus_{k=0}^{r-1} \frac{k}{r} W_{g,k} \quad \text{(S-bundle)} \tag{2}$$

where r is the order of g (i.e. $g^r = 1$), and $W_{g,k}$ is the eigenbundle of $W_g := TX|_{X^g}$ such that g acts with the eigenvalue $\exp(2\pi ki/r)$. Now the multiplication is defined by,

$$v_g \cdot v_h := \mathbf{q}_* \left[v_g|_{X^{g,h}} \cup v_h|_{X^{g,h}} \cup \mathsf{c}_{g,h} \right], \quad \mathsf{c}_{g,h} := c_{top}\left(\mathscr{R}(g,h)\right) \tag{3}$$

where $\mathbf{q} : X^{g,h} \hookrightarrow X^{gh}$ is the obvious inclusion. The G-equivariance of this multiplication follows from the G-equivariance of $\mathscr{S}$ and $\mathscr{R}$:

$$\rho_m^* \mathscr{S}_{mgm^{-1}} = \mathscr{S}_g, \quad \rho_m^* \mathscr{R}(mgm^{-1}, mhm^{-1}) = \mathscr{R}(g,h). \tag{4}$$

The metric η of $\mathscr{H}(X,\mathsf{G})$ is defined by

$$\eta(v_g, w_{g^{-1}}) := \int_{X^g} v_g \cup \iota^* w_{g^{-1}}, \text{ and } \eta(v_g, w_h) = 0 \text{ if } gh \neq 1 \tag{5}$$

where $\iota : X^g \longrightarrow X^{g^{-1}}$ is the identity map. The orbifold $\mathbb{Q}$-grading is given by

$$\deg^{\mathbb{Q}}(v_g) := |v_g| + 2\,\mathrm{age}(g), \text{ where } \mathrm{age}(g) := \mathrm{rk}\mathscr{S}_g \tag{6}$$

where $|v_g|$ is the ordinary degree of the cohomology class v_g. The following summarizes the algebraic structure of $\mathscr{H}(X,\mathsf{G})$:

Theorem 2 ([7, 10, 11]). *$(\mathscr{H}(X,\mathsf{G}), \cdot, \mathbf{1}, \eta, \rho, \deg^{\mathbb{Q}})$ is a $\mathbb{Q}$-graded G-Frobenius (super-) algebra of degree $2\dim_{\mathbb{C}} X$. It is called the* stringy cohomology *of G-manifold X.*

The G-invariants of the stringy cohomology is isomorphic as a Frobenius algebra to the orbifold cohomology of Chen-Ruan [4], i.e. $\mathscr{H}(X,\mathsf{G})^{\mathsf{G}} = H^*_{orb}([X/G])$. Here the metric η_{CR} on $H^*_{orb}([X/G])$ is given by

$$\eta_{CR}(v,w) := \frac{1}{|\mathsf{G}|}\eta(v,w) \text{ for } v,w \in \mathscr{H}(X,\mathsf{G})^{\mathsf{G}}. \tag{7}$$

Remark 2. *If $\mathsf{G} = \mathsf{K} \rtimes \mathsf{L}$, then we have an action of L on an orbifold $[X/\mathsf{K}]$. For the action of a group on an orbifold, see [15] or [21] for example. By Theorem 1, we have an L-Frobenius algebra $\mathscr{H}(X, \mathsf{K} \rtimes \mathsf{L})^{\mathsf{K}}$ which should play the role of the stringy cohomology of L-orbifold $[X/\mathsf{K}]$. In general, when L acts on an orbifold $\mathscr{X}$, it should be possible to define its stringy cohomology $\mathscr{H}(\mathscr{X},\mathsf{L})$ analogously and to show that it is an L-Frobenius algebra. Then our main result in this paper should be easily generalized for the symmetric product of a global quotient orbifold $[X/\mathsf{H}]$ where H is a Lie group, namely $\mathscr{H}([X/\mathsf{H}]^n, \Sigma_n) \cong H_{orb}([X/\mathsf{H}])\{\Sigma_n\}$ with the help of the explicit formula for the obstruction bundle of $[X/\mathsf{H}]$ in [5], or if H is a torus, [2, 9].*

Remark 3. *In the case of the wreath product orbifold $[X^n/\mathsf{G}^n \rtimes \Sigma_n]$ that we study, the main result of this paper implies that $\mathscr{H}(X^n, \mathsf{G}^n \rtimes \Sigma_n)^{\mathsf{G}_n}$ gives a geometric construction of the second quantization of an orbifold $[X/\mathsf{G}]$ (see [13]).*

It is convenient to generalize the formula (3) to the multi-product:

Lemma 1. *Let* $g_1, \cdots, g_r \in \mathsf{G}$, $\mathbf{q} : Z := X^{g_1,\cdots,g_r} \longrightarrow X^{g_1\cdots g_r}$. *Then*

$$v_{g_1} \cdot \cdots \cdot v_{g_r} = \mathbf{q}_* \left(v_{g_1}|_Z \cup \cdots \cup v_{g_r}|_Z \cup \mathsf{c}(g_1, \cdots, g_r) \right),$$

where

$$\begin{aligned}\mathscr{R}(g_1, \cdots, g_r) &:= TZ \ominus TX|_Z \oplus \mathscr{S}_{g_1} \oplus \cdots \oplus \mathscr{S}_{g_r} \oplus \mathscr{S}_{(g_1\cdots g_r)^{-1}} \\ \mathsf{c}(g_1, \cdots, g_r) &:= c_{top}(\mathscr{R}(g_1, \cdots, g_r)).\end{aligned}$$

The proof of the associativity of the product in [11] can be easily generalized to the proof of this lemma (c.f. Proposition 5.3 of [17]).

3. The Wreath Product Orbifold

In this section, we review the wreath product orbifold to fix the notation (cf. Section 1 of [25]) and then compute the obstruction bundle in certain cases.

Notation 1. *The set of conjugacy classes of* G *is denoted by* $\bar{\mathsf{G}}$. *For all* $\alpha \in \mathsf{G}$, *let* $\mathscr{Z}_{\mathsf{G}}(\alpha)$ *be the centralizer of* α *in* G. *The subgroup generated by the subset* $\{\alpha_k\}_{k=1,\cdots,r}$ *of* G *is denoted by* $\langle \alpha_1, \cdots, \alpha_r \rangle$. *For a finite set* J, *let* G^J *be the set of maps,* $\mathrm{Map}(J, \mathsf{G}) \cong G^{|J|}$ *and let* $g_i := g(i)$ *for* $g \in \mathsf{G}^J$ *and* $i \in J$. *If* $g \in \mathsf{G}^J$, *then* $\bar{g} \in \bar{\mathsf{G}}^J$ *is defined by* $(\bar{g})_i := \bar{g}_i \in \mathsf{G}$. *Let* $\Delta^J : \mathsf{G} \to \mathsf{G}^J$ *be the diagonal map and let* $\Delta^J_{\mathsf{G}} := \Delta^J(\mathsf{G})$. *The same notation is applied to any set, i.e. if* X *is a manifold, then* $X^J := \mathrm{Map}(J, X)$, $x_i := x(i)$ *for* $x \in X^J$ *and* $\Delta^J_X := \Delta^J$ *where* $\Delta^J : X \to X^J$ *is the diagonal map.*

Definition 4 (Wreath Product and Wreath Product Orbifold)**.** *Fix a finite set* I *of cardinality* n *and let* Σ_I *be the permutation group of the set* I. *For all* $\sigma, \tau \in \Sigma_I$, *let* $I_\sigma := I/\langle\sigma\rangle$ *be the set of orbits in* I *under the action of the subgroup* $\langle\sigma\rangle$ *and similarly let* $I_{\sigma,\tau} := I/\langle\sigma,\tau\rangle$. *Let* $|\sigma|$ *be the minimum number of transpositions to express* σ *and then* $|\sigma| = n - |I_\sigma|$.

The natural left action of Σ_I *on* G^I *is given by* $\sigma : g_i \mapsto g_{\sigma^{-1}(i)}$ *for all* $\sigma \in \Sigma_I$ *and* $g \in \mathsf{G}^I$. *The semidirect product* $\mathsf{G}^I \rtimes \Sigma_I$ *is called the* wreath product *of* G. *Let* X *be a compact almost complex manifold with a left action* ρ *of* G. *There is a natural left action of the wreath product* $\mathsf{G}^I \rtimes \Sigma_I$ *on* X^I, *which we also denote by* ρ. *Namely, for* $g\sigma \in \mathsf{G}^I \rtimes \Sigma_I$, $\rho_{g\sigma}(x) \in X^I$ *is defined by*

$$(\rho_{g\sigma}(x))_i := \rho_{g_i}(x_{\sigma^{-1}(i)}).$$

Thus, we have an orbifold $[X^I/G^I \rtimes \Sigma_I]$ *which we call the* wreath product orbifold *associated to* $[X/\mathsf{G}]$.

Definition 5 (Cycle product)**.** *For each* $a \in I_\sigma$, *choose a representative* $i_a \in a$. *For each* $\sigma \in \Sigma_I$, *define a map* $\theta^\sigma : \mathsf{G}^I \longrightarrow \mathsf{G}^{I_\sigma}$ $(g \mapsto \theta^\sigma_g)$ *where*

$$(\theta^\sigma_g)_a := g_{\sigma^{|a|-1}(i_a)} g_{\sigma^{|a|-2}(i_a)} \cdots g_{\sigma^0(i_a)}, \quad \forall a \in I_\sigma. \tag{8}$$

θ^σ_g *is a* cycle product *of* g *with respect to* σ. The map θ^σ depends on the choice of representatives $\{i_a\}$, *but if we choose different representatives, then each* $(\theta^\sigma_g)_a$ *is conjugated by some element in* G. *Hence* $\bar{\theta^\sigma_g} \in \bar{\mathsf{G}}^{I_\sigma}$ *is independent of the choice of representatives* $\{i_a\}$.

Now we compute the orbits of the action of G^I by conjugation on $\mathsf{G}^I \rtimes \Sigma_I$.

Proposition 1. *Choose a representative i_a in a for each $a \in I_\sigma$. For $g\sigma \in \mathsf{G}^I \rtimes \Sigma_I$, let $\mathfrak{g} := \theta_g^\sigma \in \mathsf{G}^{I_\sigma}$. The orbit of $g\sigma$ under the action of G^I by conjugation is given by*

$$\mathscr{O}_{\mathfrak{g}}\sigma := \left\{ g'\sigma \in \mathsf{G}^I\sigma \;\middle|\; \bar{\theta}_{g'}^{\sigma} = \bar{\mathfrak{g}} \right\}. \tag{9}$$

Proof. Define $\epsilon_{\mathfrak{g}} \in \mathsf{G}^I$ by $(\epsilon_{\mathfrak{g}})_i := \mathfrak{g}_a$ if $i = i_a$ for some $a \in I_\sigma$ and otherwise $(\epsilon_{\mathfrak{g}})_i = 1$. Then it is easy to check $(v_g^\sigma)^{-1} \cdot g\sigma \cdot v_g^\sigma = \epsilon_{\mathfrak{g}}\sigma$ where v_g^σ is given by

$$(v_g^\sigma)_{\sigma^m(i_a)} := g_{\sigma^m(i_a)} g_{\sigma^{m-1}(i_a)} \cdots g_{\sigma^0(i_a)}, \quad m = 0, \cdots, |a| - 1. \tag{10}$$

Therefore $g\sigma$ and $\epsilon_{\mathfrak{g}}\sigma$ are in the same orbit. On the other hand, for $\mathfrak{g}, \mathfrak{g}'$ in G^{I_σ}, $f \in \mathsf{G}^I$ satisfies $\epsilon_{\mathfrak{g}}\sigma = f^{-1}\epsilon_{\mathfrak{g}'}\sigma f$ if and only if $f \in \prod_{a\in I_\sigma} \Delta_G^a$ and $\mathfrak{g}_a = f_{i_a}^{-1}\mathfrak{g}'_a f_{i_a}$. Thus, $g\sigma$ and $g'\sigma$ are in the same orbit if and only if $\bar{\theta_g^\sigma} = \bar{\theta_{g'}^\sigma}$.

Remark 4. *From the above proof, it is clear that $\mathscr{Z}_{\mathsf{G}^I}(\epsilon_{\mathfrak{g}}\sigma) = \prod_{a\in I_\sigma} \Delta^a_{\mathscr{Z}_G(\mathfrak{g}_a)}$.*

Lemma 2 ([Lemma 4 and 5, 25])**.** *For $g\sigma \in \mathsf{G}^I \rtimes \Sigma_I$, we have $(X^I)^{g\sigma} = \rho_{v_g^\sigma}\left(\prod_{a\in I_\sigma} \Delta^a_{X^{\mathfrak{g}_a}}\right)$ where $\mathfrak{g} := \theta_g^\sigma$.*

Proof. When $g\sigma = \epsilon_{\mathfrak{g}}\sigma$, it follows from [Lemma 4, 25]. In general, it follows from the definition (10) of v_g^σ. Indeed, $(X^I)^{g\sigma} = (X^I)^{v_g^\sigma \cdot \epsilon_{\mathfrak{g}}\sigma \cdot (v_g^\sigma)^{-1}} = \rho_{v_g^\sigma}\left((X^I)^{\epsilon_{\mathfrak{g}}\sigma}\right)$.

3.1. The Obstruction Bundle of the Wreath Product Orbifold

Now we will compute the obstruction bundle of the wreath product orbifold in *certain* cases. Theorem 3 is crucial because it roughly says that the S-bundles for the wreath product $[X^I/\mathsf{G}^I \rtimes \Sigma_I]$ can be written in terms of the S-bundles of $[X/\mathsf{G}]$ and $[X^I/\Sigma_I]$. We use Lemma 3 in the proof of our main theorem, in particular, in the proof of Proposition 3.

Theorem 3. *Let $\sigma \in \Sigma_I, \mathfrak{g} \in \mathsf{G}^{I_\sigma}$. Let $\epsilon_{\mathfrak{g}} \in \mathsf{G}^I$ be the element defined in the proof of Proposition 1. We have*

$$\mathscr{S}_{\epsilon_{\mathfrak{g}}\sigma} = \prod_{a\in I_\sigma} \left(\Delta_*^a \left(\mathscr{S}_{\mathfrak{g}_a} \oplus \frac{|a|-1}{2} TX|_{X^{\mathfrak{g}_a}} \right) \right)$$

where $\mathscr{S}_{\mathfrak{g}_a} \in K(X^{\mathfrak{g}_a})$ is the S-bundle with respect to the action of G on X and $\Delta^a : X^{\mathfrak{g}_a} \cong \Delta^a_{X^{\mathfrak{g}_a}}$.

Proof. Without loss of generality, we can assume $I = \{1, \cdots, n\}$ and $\sigma = (12\cdots n)$. Let $\epsilon_{\mathfrak{g}} = (\mathfrak{g}, 1, \cdots, 1) \in \mathsf{G}^I$. Let ρ be the natural left action of Σ_I on $V := \mathbb{C}^n$ so that $\rho_\sigma(\mathbf{e}_j) = \mathbf{e}_{\sigma^{-1}(j)}$ where $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ is the standard basis of V. As a $\langle\epsilon_{\mathfrak{g}}\sigma\rangle$-equivariant vector bundle, $TX^I|_{(X^I)^{\epsilon_{\mathfrak{g}}\sigma}}$ can be identified to $(T\Delta_X \otimes V)|_{\Delta_{X^{\mathfrak{g}}}}$. Explicitly, $\epsilon_{\mathfrak{g}}\sigma$ acts on $\mathbf{u}\otimes\mathbf{v} \in T_p\Delta_X \otimes V$ as $\rho_{\epsilon_{\mathfrak{g}}\sigma}(\mathbf{u}\otimes\mathbf{v}) = \rho_{\mathfrak{g}}\mathbf{u} \otimes v_1\mathbf{e}_n + \sum_{j=2}^n \mathbf{u} \otimes v_j\mathbf{e}_{j-1}$ where $\mathbf{v} = \sum_j v_j\mathbf{e}_j$. Let r be the order of $\mathfrak{g} \in \mathsf{G}$ and let $T\Delta_X|_{\Delta_{X^{\mathfrak{g}}}} = \bigoplus_{l=0}^{r-1} U_l$ be the eigenbundle decomposition of the diagonal action of $\mathfrak{g}$

where the eigenvalue on U_l is $e^{2\pi i\frac{l}{r}}$. On the other hand, the eigenvalues of the action of σ on V are $e^{2\pi i\frac{k}{n}}, k=0,\cdots,n-1$ and the corresponding eigenvectors are $\mathbf{v}_k := \sum_{j=1}^n \left(e^{2\pi i\frac{k}{n}}\right)^j \mathbf{e}_j$. For each $l=0,\cdots,r-1$, define $\mathbf{v}_{k,l} = \sum_{j=1}^n \left(e^{2\pi i\left(\frac{l}{rn}+\frac{k}{n}\right)}\right)^j \mathbf{e}_j$ and then $\{v_{k,l}, k=0,\cdots,n-1\}$ forms a basis of V. Thus we have the following decomposition

$$TX^I|_{(X^I)^{\epsilon_{\mathfrak{g}}\sigma}} = \bigoplus_{k=0}^{n-1}\left(\bigoplus_{l=0}^{r-1} U_l \otimes V_{k,l}\right). \tag{11}$$

where $V_{k,l}$ is the 1-dimensional subspace spanned by $v_{k,l}$. This turns out to be the eigenbundle decomposition of the action of $\epsilon_{\mathfrak{g}}\sigma$ where the eigenvalue of $U_l \otimes V_{k,l}$ is $e^{2\pi i\left(\frac{l}{nr}+\frac{k}{n}\right)}$. Indeed, if $\mathbf{u}_l \otimes \mathbf{v}_{k,l} \in U_l \otimes V_{k,l}$,

$$\begin{aligned}
\rho_{\epsilon_{\mathfrak{g}}\sigma}(\mathbf{u}_l \otimes \mathbf{v}_{k,l}) &= e^{2\pi i\left(\frac{l}{rn}+\frac{k}{n}\right)}\left(e^{2\pi i\frac{l}{r}}\right)\mathbf{u}_l \otimes \mathbf{e}_n + \sum_{j=2}^n \left(e^{2\pi i\left(\frac{l}{rn}+\frac{k}{n}\right)}\right)^j \mathbf{u}_l \otimes \mathbf{e}_{j-1}\\
&= e^{2\pi i\left(\frac{l}{rn}+\frac{k}{n}\right)}\left\{\left(e^{2\pi i\frac{l}{r}}\right)\mathbf{u}_l \otimes \mathbf{e}_n + \sum_{j=2}^n \left(e^{2\pi i\left(\frac{l}{rn}+\frac{k}{n}\right)}\right)^{j-1} \mathbf{u}_l \otimes \mathbf{e}_{j-1}\right\}\\
&= e^{2\pi i\left(\frac{l}{rn}+\frac{k}{n}\right)}\cdot \mathbf{u}_l \otimes \mathbf{v}_{k,l}
\end{aligned}$$

Thus we have

$$\mathscr{S}_{\epsilon_{\mathfrak{g}}\sigma} = \bigoplus_{k=0}^{n-1}\bigoplus_{l=0}^{r-1}\left(\frac{l}{nr}+\frac{k}{n}\right)U_l\otimes V_k^l = \bigoplus_{l=0}^{r-1}\frac{l}{r}U_l \oplus \bigoplus_{k=0}^{n-1}\frac{k}{n}T\Delta_X|_{\Delta_{X^{\mathfrak{g}_a}}} = \Delta_*\left(\mathscr{S}_{\mathfrak{g}} \oplus \frac{n-1}{2}TX|_{X^{\mathfrak{g}}}\right),$$

where the second equality follows from forgetting the group action and identifying $V_{k,l}$ with $\mathbb{C}$.

Corollary 1. *Theorem 3 leads to the following formula obtained in [26] through the direct calculation:*

$$\mathrm{age}(g\sigma) := \mathrm{rk}\mathscr{S}_{\epsilon_{\mathfrak{g}}\sigma} = \frac{\dim_{\mathbb{C}} X\cdot|\sigma|}{2} + \sum_{a\in I_\sigma}\mathrm{age}(\mathfrak{g}_a)$$

where $\mathrm{age}(\mathfrak{g}_a)$ *is the age of* $\mathfrak{g}_a$ *with respect to the action* G *on* X*.*

We need the following lemma to compute the action of the untwisted sector of $\mathscr{H}(X^I, \mathsf{G}^I \rtimes \Sigma_I)^{\mathsf{G}^I}$.

Lemma 3. *For every* $h \in \mathsf{G}^I$ *and* $\mathfrak{g} \in \mathsf{G}^{I_\sigma}$*, let* $Z_a := X^{\mathfrak{g}_a, h_i, i\in a}$ *for* $a \in I_\sigma$*, then*

$$(X^I)^h \cap (X^I)^{\epsilon_{\mathfrak{g}}\sigma} = \prod_{a\in I_\sigma}\Delta^a_{Z_a} \text{ and } \mathscr{R}(h,\epsilon_{\mathfrak{g}}\sigma) = \prod_{a\in I_\sigma}\Delta^a_*\mathscr{R}(h_{\sigma^{|a|-1}(i_a)},\cdots,h_{\sigma^0(i_a)},\mathfrak{g}_a)$$

Proof. The first statement follows immediately from Lemma 2. For the second claim, without loss of generality we can assume that $I = \{1, \cdots, n\}$, $\sigma = (12\cdots n)$ and $\epsilon_{\mathfrak{g}} = (\mathfrak{g}, 1, \cdots, 1)$. Since $(h\epsilon_{\mathfrak{g}}\sigma)^{-1} = (h_2^{-1}, \cdots, h_n^{-1}, \mathfrak{g}^{-1}h_1^{-1})\sigma^{-1}$, we have $m \cdot (h\epsilon_{\mathfrak{g}}\sigma)^{-1} \cdot m^{-1} = \epsilon_{(h_n\cdots h_2h_1\mathfrak{g})^{-1}}\sigma^{-1}$ for some $m \in \langle h_1, \cdots, h_n, \mathfrak{g}\rangle^I$. Now compute

$$\begin{aligned}\mathscr{R}(h, \epsilon_{\mathfrak{g}}\sigma) =& T\Delta_Z \ominus TX^I|_{\Delta_Z} \oplus \Delta_*\left(\mathscr{S}_{h_1}|_Z \oplus \cdots \oplus \mathscr{S}_{h_n}|_Z\right) \oplus \Delta_*\left(\mathscr{S}_{\mathfrak{g}}|_Z \oplus \frac{n-1}{2}TX|_Z\right)\\ &\oplus \rho_m^*\Delta_*\left(\mathscr{S}_{(h_n\cdots h_2h_1\mathfrak{g})^{-1}}|_Z \oplus \frac{n-1}{2}TX|_Z\right)\end{aligned}$$

where $Z := X^{g,h_1,\cdots,h_n}$. Since m_i fixes Z, $\rho_m|_{\Delta_Z} = \mathrm{id}$ and therefore

$$\begin{aligned}\mathscr{R}(h, \epsilon_{\mathfrak{g}}\sigma) =& \Delta_*\left(TZ \ominus TX|_Z \oplus \mathscr{S}_{h_1}|_Z \oplus \cdots \oplus \mathscr{S}_{h_n}|_Z \oplus \mathscr{S}_{\mathfrak{g}}|_Z \oplus \mathscr{S}_{(h_n\cdots h_2h_1\mathfrak{g})^{-1}}|_Z\right)\\ =& \Delta_*\mathscr{R}(h_n, \cdots, h_1, \mathfrak{g}).\end{aligned}$$

The next lemma is only related to Proposition 4.

Lemma 4. *Let $\sigma \in \Sigma$ and let $\tau = (ij)$ be a transposition. Suppose that σ, τ are transversal, i.e. $|\sigma| + |\tau| = |\sigma\tau|$ and let $i \in a$ and $j \in b$ for $a, b \in I_\sigma$. Then*

$$(X^I)^\sigma \cap (X^I)^{g\tau g^{-1}} = \left(\prod_{c\in I_\sigma\setminus\{a,b\}} \Delta_X^c\right) \times \rho_{g'}\left(\Delta_X^{a\cup b}\right)$$

where $g' \in \mathsf{G}^{a\cup b}$ is given by $g'_l = g_i$ if $l \in a$ and $g'_l = g_j$ if $l \in b$. Moreover, $\mathrm{rk}\mathscr{R}(\sigma, g\tau g^{-1}) = 0$ and so $\mathsf{c}(\sigma, g\tau g^{-1}) = 1$.

Proof. The first statement is straightforward. Without loss of generality, we can assume that $I_{\sigma,\tau} = \{I\}$. Also we can assume that $g = g'$, since $g\tau g^{-1} = g'\tau(g')^{-1}$. Since $g\sigma g^{-1} = \sigma$, we have $\mathscr{R}(\sigma, g\tau g^{-1}) = \rho_{g*}\mathscr{R}(\sigma, \tau)$. However, from Corollary 1 we can compute that $\mathrm{rk}\mathscr{R}(\sigma, \tau) = 0$ and thus $\mathsf{c}(\sigma, g\tau g^{-1})$=1.

Remark 5. *If G is abelian, the general computation of the obstruction bundle is available in [17].*

4. Lehn-Sorger's Algebras and G^I-Invariants of Stringy Cohomology

In this section, we prove our main theorem. Since our G-Frobenius algebras are special G-Frobenius algebras, we can use the structure theorems in [12] and [13]. For the summary of definitions and theorems, please see the appendix. Let $\mathscr{H}(X^I, \mathsf{G}^I \rtimes \Sigma_I)$ be the stringy cohomology of the $(G^I \rtimes \Sigma_I)$-space X^I reviewed in Section 2.2 and let $H^*_{orb}([X/\mathsf{G}])\{\Sigma_I\}$ be the Lehn-Sorger algebra associated to $H^*_{orb}([X/\mathsf{G}])$ reviewed in Section 5. By Proposition 1, the G^I-invariants of $\mathscr{H}(X^I, \mathsf{G}^I \rtimes \Sigma_I)$ is

$$\mathscr{H}(X^I, \mathsf{G}^I \rtimes \Sigma_I)^{\mathsf{G}^I} = \bigoplus_{\sigma\in\Sigma_I}\bigoplus_{\bar{\mathfrak{g}}\in\bar{\mathsf{G}}^{I_\sigma}} \mathscr{H}_{\bar{\mathfrak{g}},\sigma}, \quad \mathscr{H}_{\bar{\mathfrak{g}},\sigma} := \left(\bigoplus_{g\sigma\in\mathscr{O}_{\mathfrak{g}}\sigma} H^*((X^I)^{g\sigma})\right)^{\mathsf{G}^I}.$$

On the other hand, the Lehn-Sorger algebra associated to $H^*_{orb}([X/\mathsf{G}])$ is

$$H^*_{orb}([X/\mathsf{G}])\{\Sigma_I\} = \bigoplus_{\sigma\in\Sigma_I}\bigoplus_{\bar{\mathfrak{g}}\in\bar{\mathsf{G}}^{I_\sigma}} \mathscr{A}_{\bar{\mathfrak{g}},\sigma}, \quad \mathscr{A}_{\bar{\mathfrak{g}},\sigma} := \left(\bigoplus_{\mathfrak{g}'\in\bar{\mathfrak{g}}} H^*((X^{I_\sigma})^{\mathfrak{g}'})\right)^{\mathsf{G}^{I_\sigma}} .$$

Proposition 2. *There is a canonical isomorphism of graded Σ_I-graded Σ_I-modules which preserves the metric:*

$$\Phi : H^*_{orb}([X/\mathsf{G}])\{\Sigma_I\} \xrightarrow{\simeq} \mathscr{H}(X^I, \mathsf{G}^I \rtimes \Sigma_I)^{\mathsf{G}^I}.$$

Proof. Choose $\{i_a \in a\}_{a\in I_\sigma}$. Consider the following isomorphisms:

$$H^*((X^{I_\sigma})^{\mathfrak{g}})^{\mathscr{Z}_{\mathsf{G}^{I_\sigma}}(\mathfrak{g})} \cong \mathscr{A}_{\bar{\mathfrak{g}},\sigma}, \quad \mathbf{x} \mapsto \mathsf{L}_{\mathbf{x}} := \sum_{\mathfrak{f}\in\mathsf{G}^{I_\sigma}} \rho_{\mathfrak{f}*}(\mathbf{x});$$

$$H^*((X^I)^{\epsilon_{\mathfrak{g}}\sigma})^{\mathscr{Z}_{\mathsf{G}^I}(\epsilon_{\mathfrak{g}}\sigma)} \cong \mathscr{H}_{\bar{\mathfrak{g}},\sigma}, \quad \mathbf{v} \mapsto \mathsf{F}_{\mathbf{v}} := \sum_{f\in\mathsf{G}^I} \rho_{f*}(\mathbf{v}).$$

From Lemma 2 and Remark 4, we also have $\mathscr{Z}_{\mathsf{G}^{I_\sigma}}(\mathfrak{g}) = \prod_{a\in I_\sigma} \mathscr{Z}_{\mathsf{G}}(\mathfrak{g}_a) \cong \mathscr{Z}_{\mathsf{G}^I}(\epsilon_{\mathfrak{g}}\sigma)$ and $(X^{I_\sigma})^{\mathfrak{g}} = \prod_{a\in I_\sigma} X^{\mathfrak{g}_a} \cong (X^I)^{\epsilon_g\sigma}$, which imply

$$H^*((X^{I_\sigma})^{\mathfrak{g}})^{\mathscr{Z}_{\mathsf{G}^{I_\sigma}}(\mathfrak{g})} \cong H^*((X^I)^{\epsilon_{\mathfrak{g}}\sigma})^{\mathscr{Z}_{\mathsf{G}^I}(\epsilon_{\mathfrak{g}}\sigma)}$$

where ($\mathbf{x} \mapsto \Delta_{\mathbf{x}}$). Define Φ by $\Phi(\mathsf{L}_{\mathbf{x}}) := \mathsf{F}_{\Delta_{\mathbf{x}}}$. Because of the summations over G^{I_σ} and G^I, Φ is independent of the choices we made. The Σ_I-equivariance is clear from the commutative diagram

$$\begin{tikzcd}
\prod_{a\in I_\sigma} X^{\mathfrak{g}_a} \arrow[r, "\cong"'] \arrow[d, "\cong"'] & \prod_{a\in I_\sigma} \Delta^a_{X^{\mathfrak{g}_a}} \arrow[d, "\cong"'] \\
\prod_{\tau(a)\in I_{\tau\sigma\tau^{-1}}} X^{\mathfrak{g}_{\tau(a)}} \arrow[r, "\cong"'] & \prod_{\tau(a)\in I_{\tau\sigma\tau^{-1}}} \Delta^{\tau(a)}_{X^{\mathfrak{g}_{\tau(a)}}}
\end{tikzcd}$$

It follows from Eq. (13), Corollary 1 and Eq. (6) that Φ preserves the $\mathbb{Q}$-grading.

Since the component of $\mathscr{H}(X^I, \mathsf{G}^I \rtimes \Sigma_I)^{\mathsf{G}^I}$ graded by the identity permutation (the untwisted sector) is exactly the Frobenius algebra $H^*(X^I, \mathsf{G}^I)^{\mathsf{G}^I} = H^*_{orb}([X/\mathsf{G}])^{\otimes I}$, this yields the following proposition.

Proposition 3. *The isomorphism Φ is an isomorphism as $H^*_{orb}([X/\mathsf{G}])^{\otimes I}$-modules. In particular, $\mathscr{H}(X^I, \mathsf{G}^I \rtimes \Sigma_I)^{\mathsf{G}^I}$ is a special Σ_I-Frobenius algebra. Furthermore, Φ preserves the metric.*

Proof. We will show that the actions of $H^*_{orb}([X/\mathsf{G}])^I$ on $\mathscr{A}_{\bar{\mathfrak{g}},\sigma}$ and on $\mathscr{H}_{\bar{\mathfrak{g}},\sigma}$ are identified by Φ. Without loss of generality, we can assume $I = \{1, \cdots, n\}$ and $\sigma = (12\cdots n)$. Let $x_g \in H^*(X^g)^{\mathscr{Z}_{\mathsf{G}}(g)}$ for every $g \in \mathsf{G}$ and let $\mathsf{L}_{x_g} := \sum_{\mathfrak{k}\in\mathsf{G}} \rho_{\mathfrak{k}}(x_g) \in \mathsf{A}_{\bar{g},\sigma}$ and $\Phi(\mathsf{L}_{x_g}) = \sum_{k\in\mathsf{G}^I} \rho_k(\Delta_{x_g}) \in \mathscr{H}_{\bar{g},\sigma}$. We need to show that the Lehn-Sorger product

$$\mathsf{P}_{LS} := \left(\sum_{f_1,\cdots,f_n\in\mathsf{G}} \rho_{f_1}(x_{h_1}) \otimes \cdots \otimes \rho_{f_n}(x_{h_n})\right) \cdot \left(\sum_{\mathfrak{k}\in\mathsf{G}} \rho_{\mathfrak{k}}(x_g)\right)$$

corresponds via Φ to the stringy product

$$\mathsf{P}_{ST} := \left(\sum_{f_1,\cdots,f_n\in\mathsf{G}} \rho_{f_1}(x_{h_1})\otimes\cdots\otimes\rho_{f_n}(x_{h_n})\right)\cdot\left(\sum_{k\in\mathsf{G}^I}\rho_k(\Delta_{x_g})\right).$$

Let $Z := X^{g,f_1h_1f_1^{-1},\cdots,f_nh_nf_n^{-1}}$, $\mathbf{q} : Z \hookrightarrow X^{f_nh_nf_n^{-1}\cdots f_1h_1f_1^{-1}g}$ and $\Delta_{\mathbf{q}} : \Delta_Z \hookrightarrow \Delta_{X^{f_nh_nf_n^{-1}\cdots f_1h_1f_1^{-1}g}}$. Then P_{LS} is computed as follows.

$$\begin{aligned}\mathsf{P}_{LS} &= \sum_{f_1,\cdots,f_n,\mathfrak{k}\in\mathsf{G}}\left(\rho_{f_1}(x_{h_1})\cdot\cdots\cdot\rho_{f_n}(x_{h_n})\cdot\rho_{\mathfrak{k}}(x_g)\right) = \sum_{\mathfrak{k}\in\mathsf{G}}\rho_{\mathfrak{k}}\left(\sum_{f_1,\cdots,f_n\in\mathsf{G}}\left(\rho_{f_n}(x_{h_n})\cdot\cdots\cdot\rho_{f_1}(x_{h_1})\cdot x_g\right)\right)\\ &= \sum_{\mathfrak{k}\in\mathsf{G}}\rho_{\mathfrak{k}}\left(\sum_{f_1,\cdots,f_n\in\mathsf{G}}\mathbf{q}_*\left(\rho_{f_n}(x_{h_n})|_Z\cup\cdots\cup\rho_{f_1}(x_{h_1})|_Z\cup x_g|_Z\cup \mathsf{c}(f_nh_nf_n^{-1},\cdots,f_1h_1f_1^{-1},g)\right)\right)\end{aligned}$$

where the first equality follows from the definition and the fact that $\mathsf{gd}(1,\sigma)=0$, the second equality follows by the G-equivariance and the commutativity of the multiplication in $\mathscr{H}(X,\mathsf{G})^{\mathsf{G}}$ and replacing $\mathfrak{k}^{-1}f_i$ by f_i and the third equality follows from Lemma 1. On the other hand, P_{ST} is computed as follows.

$$\begin{aligned}\mathsf{P}_{ST} &= \sum_{k\in\mathsf{G}^I}\rho_k\left(\sum_{f_1,\cdots,f_n\in\mathsf{G}}\left(\rho_{f_1}(x_{h_1})\otimes\cdots\otimes\rho_{f_n}(x_{h_n})\right)\cdot\Delta_{x_g}\right)\\ &= \sum_{k\in\mathsf{G}^I}\rho_k\left(\sum_{f_1,\cdots,f_n\in\mathsf{G}}\Delta_{\mathbf{q}*}\left(\rho_{f_1}(x_{h_1})\otimes\cdots\otimes\rho_{f_n}(x_{h_n})\big|_{\Delta_Z}\cup\Delta_{x_g}|_{\Delta_Z}\cup\Delta^*\mathsf{c}(f_nh_nf_n^{-1},\cdots,f_1h_1f_1^{-1},g)\right)\right)\end{aligned}$$

where the second equality follows from Lemma 3. Now it is clear that $\Phi(P_{LS}) = P_{ST}$.

Since the metric of a special G-Frobenius algebra is completely determined by the Frobenius algebra structure on the untwisted sector and its action (see [Theorem 4.1, 12] or Appendix), the isomorphism also preserves the metric.

So far, we have proved that Φ is a G-equivariant isomorphism of special Σ_I-reconstruction data. Thus, by Theorem 4.1 of [12], if the associated graded cocycles coincide, then Φ is a G-Frobenius algebra isomorphism. From Theorem 6.5 of [13] and the fact that the graded cocycle of a Lehn-Sorger's algebra is normalized [13, p.77,], it suffices to show:

Proposition 4. *Let $\mathbf{1}_\sigma$ be the identity of the Frobenius algebra $H_{orb}([X/\mathsf{G}])^{\otimes I_\sigma}$. The graded cocycle defined by the special Σ_I-Frobenius algebra structure on $\mathsf{H}(X^I,\mathsf{G}^I\rtimes\Sigma_I)^{\mathsf{G}^I}$ with the cyclic generators $\{\Phi(\mathbf{1}_\sigma)\}$ is normalized.*

Proof. The graded cocycle γ is defined by $\Phi(\mathbf{1}_\sigma)\cdot\Phi(\mathbf{1}_\tau) = \gamma_{\sigma,\tau}\Phi(\mathbf{1}_{\sigma\tau})$. If 1_σ is the identity element of the ordinary cohomology ring $H^*((X^I)^\sigma)$, then

$$\Phi(\mathbf{1}_\sigma) = \frac{1}{|\mathsf{G}^{I_\sigma}|}\sum_{f\in\mathsf{G}^I}\rho_f\left(1_\sigma\right).$$

Let σ, τ be transversal and $|\tau| = 1$. Without loss of generality, we can assume $I_{\sigma,\tau} = \{I\}$ and therefore, let $\tau = (ij)$ and $I_\sigma = \{a, b\}$ where $i \in a$ and $j \in b$. We compute

$$\begin{aligned}
\Phi(\mathbf{1}_\sigma) \cdot \Phi(\mathbf{1}_\tau) &= \left(\frac{1}{|\mathsf{G}^{I_\sigma}|} \sum_{f \in \mathsf{G}^I} \rho_f\left(1_\sigma\right) \right) \cdot \left(\frac{1}{|\mathsf{G}^{I_\tau}|} \sum_{g \in \mathsf{G}^I} \rho_g\left(1_\tau\right) \right) \\
&= \frac{1}{|\mathsf{G}|^{|I_\sigma|+|I_\tau|}} \sum_{f \in \mathsf{G}^I} \rho_f \left(\sum_{g \in \mathsf{G}^I} 1_\sigma \cdot \rho_g(1_\tau) \right) \\
&= \frac{1}{|\mathsf{G}|^{n+1}} \sum_{f \in \mathsf{G}^I} \rho_f \left(|\mathsf{G}|^{I_\tau} \sum_{g \in \mathsf{G}^I / \left(\prod_{c \in I_\tau} \Delta^c_{\mathsf{G}} \right)} 1_\sigma \cdot \rho_g(1_\tau) \right) \\
&= \frac{|\mathsf{G}|^{n-1}}{|\mathsf{G}|^{n+1}} \sum_{f \in \mathsf{G}^I} \rho_f \left(\sum_{g \in \Delta^a_{\mathsf{G}} \times \Delta^b_{\mathsf{G}} / \Delta_{\mathsf{G}}} \rho_g(1_\sigma \cdot 1_\tau) \right) \\
&= \frac{1}{|\mathsf{G}|} \sum_{f \in \mathsf{G}^I} \rho_f\left(1_{\sigma\tau}\right) = \Phi(\mathbf{1}_{\sigma\tau})
\end{aligned}$$

where the fourth equality follows from $\mathsf{G}^I / \left(\prod_{c \in I_\tau} \Delta^c_{\mathsf{G}} \right) \cong \Delta^a_{\mathsf{G}} \times \Delta^b_{\mathsf{G}} / \Delta_{\mathsf{G}}$ and the G^I-equivariance of the stringy product. The last equality follows from $1_\sigma \cdot 1_\tau = 1_{\sigma\tau}$ which holds because of transversality. Thus $\gamma_{\sigma,\tau} = 1$.

Theorem 4. *The canonical isomorphism Φ is an isomorphism of Σ_I-Frobenius algebras.*

Proof. Since the Lehn-Sorger algebra is always normalized (see Appendix), Propositions 2, 3 and 4 imply that Φ is an isomorphism of normalized special Σ_I-reconstruction data. Therefore by Theorem 4.1 of [12] and Theorem 6.5 of [13], we can conclude that Φ preserves the Σ_I-Frobenius algebra structures.

5. Hilbert Schemes and Wreath Products Orbifolds

In this section, we will relate the wreath product orbifold associated to a G-variety X to the Hilbert scheme of n-points on Y when Y is a crepant resolution of X/G. Throughout the section, all vector spaces are over $\mathbb{C}$ and we will work in the algebraic category.

Definition 6. *Let W be a normal variety over $\mathbb{C}$ and let $\mathscr{L}$ be a rank 1, torsion free, coherent sheaf of $\mathscr{O}_W$-module over W. $\mathscr{L}$ is called* divisorial *[20] if and only if any torsion free coherent sheaf of $\mathscr{O}_W$-module, $\mathscr{M}$, such that $\mathscr{L} \subset \mathscr{M}$ and* $\mathrm{Supp}(\mathscr{M}/\mathscr{L})$ *has codimension ≥ 2, coincides with $\mathscr{L}$.*

Remark 6. *Let $\mathscr{L}$ be divisorial. If $W^0 \subset W$ is a non-singular open subvariety such that $W \backslash W^0$ has codimension ≥ 2, then $\mathscr{L}|_{X^0}$ is invertible and $\mathscr{L} = j_*(\mathscr{L}|_{X^0})$ [20], where $j : W^0 \hookrightarrow W$*

denotes the canonical inclusion. Let K_W be the canonical divisor of W. By Proposition (7) in [20], the canonical sheaf $\omega_W := \mathscr{O}(K_W)$ of W is divisorial. Hence, we have $\omega_W = j_*\omega_{W^0}$ since $\omega_W|_{W^0} = \omega_{W^0}$.

Definition 7. *Let W and Y be normal varieties. A birational morphism $\pi : Y \to W$ is* crepant *if $\omega_Y \cong \pi^*\omega_W$. A normal variety W is* Gorenstein *if and only if all of the local rings are Cohen-Macaulay and K_W is Cartier.*

Lemma 5. *Let W and Y be Gorenstein varieties. If $\pi : Y \to W$ is birational, then π^*K_X is divisorial.*

Proof. Let $\dim W = \dim Y = n$. Since K_W is Cartier, π^*K_W is also Cartier. Hence, $\pi_*\omega_W$ is torsion-free and of rank 1. Let $\mathscr{M}$ be a torsion-free sheaf such that $\pi^*\omega_W \subset \mathscr{M}$ and the dimension of $\operatorname{Supp}\mathscr{M}/\pi^*\omega_W$ is at most $n-2$. Let $L := K_Y - \pi^*K_W$. L is Cartier and $\mathscr{L} := \mathscr{O}(L)$ is an invertible sheaf. It follows that $\pi^*\omega_W \otimes \mathscr{L} \cong \omega_Y \subset \mathscr{M} \otimes \mathscr{L}$ and $\dim(\operatorname{Supp}((\mathscr{M}\otimes\mathscr{L})/\omega_Y)) \le \dim(\operatorname{Supp}\mathscr{M}/\pi^*\omega_W) \le n-2$. Since $H^{n-1}((\mathscr{M}\otimes\mathscr{L})/\omega_Y) = 0$, we have $H^n(\mathscr{M}\otimes\mathscr{L}) \cong H^n(\omega_Y) \cong \mathbb{C}$ by Serre duality. Hence, there exists an element in $\operatorname{Hom}(\mathscr{M}\otimes\mathscr{L}, \omega_Y)$ which gives a splitting of the short exact sequence $0 \to \omega_Y \to \mathscr{M}\otimes\mathscr{L} \to (\mathscr{M}\otimes\mathscr{L})/\omega_Y \to 0$. However, since $\mathscr{M}\otimes\mathscr{L}$ is torsion-free, $(\mathscr{M}\otimes\mathscr{L})/\omega_Y = 0$.

Theorem 5. *Let W and Y be normal varieties with dimension ≥ 2. Suppose that $W\backslash W^0$ has codimension ≥ 2 and that Y^n/Σ_n and W^n/Σ_n are Gorenstein. If $\pi : Y \to W$ is a crepant resolution, then the induced map $\tilde{\pi} : Y^n/\Sigma_n \to W^n/\Sigma_n$ is crepant.*

Proof. The smooth locus of W^n/Σ_n is equal to $(W^n\backslash\Delta_W^\star)/\Sigma_n$ where $\Delta_W^\star$ is the set of points in W^n with non-trivial isotropy. Let $\mathscr{D}_Y := \pi^{-1}(\Delta_W^\star)$. Let $\bar{\pi} : Y^n\backslash\mathscr{D}_Y \to W^n\backslash\Delta_W^\star$ be the map $\pi^{\times n}$ restricted to $Y^n\backslash\mathscr{D}_Y$. Since $\pi^{\times n} : Y^n \to W^n$ is crepant, $K_{Y^n} = (\pi^{\times n})^*K_{W^n}$. Consider the commutative diagram

$$\begin{tikzcd} Y^n \arrow[d, "\pi^{\times n}"'] & Y^n\backslash\mathscr{D}_Y \arrow[l] \arrow[d, "\bar{\pi}"'] \\ W^n & W^n\backslash\Delta_W^\star \arrow[l] \end{tikzcd}$$

where the horizontal arrows are the obvious inclusions. We have

$$K_{Y^n\backslash\mathscr{D}_Y} = K_{Y^n}|_{Y^n\backslash\mathscr{D}_Y} = \left((\pi^{\times n})^*K_{W^n}\right)|_{Y^n\backslash\mathscr{D}_Y} = \bar{\pi}^*(K_{W^n}|_{W^n\backslash\Delta_W^\star}) = \bar{\pi}^*K_{W^n\backslash\Delta_W^\star}. \tag{12}$$

Consider the following commutative diagram

$$\begin{tikzcd} Y^n\backslash\mathscr{D}_Y \arrow[r, "\mathbf{q}"] \arrow[d, "\bar{\pi}"'] & (Y^n\backslash\mathscr{D}_Y)/\Sigma_n \arrow[d, "\bar{\pi}'"'] \\ W^n\backslash\Delta_W^\star \arrow[r, "\mathbf{q}'"] & (W^n\backslash\Delta_W^\star)/\Sigma_n \end{tikzcd}$$

where $\mathbf{q}$ and $\mathbf{q}'$ are the canonical projections. Since the actions of Σ_I on $Y^n \backslash \mathscr{D}_Y$ and $W^n \backslash \Delta_W^\star$ are free, Equation (12) implies that $K_{(Y^n\backslash\mathscr{D}_Y)/\Sigma_n} = \tilde{\pi}'^* K_{(W^n\backslash\Delta_W^\star)/\Sigma_n}$. Hence

$$K_{Y^n/\Sigma_n}|_{(Y^n\backslash\mathscr{D}_Y)/\Sigma_n} = \left(\tilde{\pi}^* K_{W^n/\Sigma_n}\right)|_{(Y^n\backslash\mathscr{D}_Y)/\Sigma_n}.$$

Since both K_{Y^n/Σ_n} and $\tilde{\pi}^* K_{W^n/\Sigma_n}$ are divisorial (Remark 6, Lemma 5), we obtain $K_{Y^n/\Sigma_n} = \tilde{\pi}^* K_{W^n/\Sigma_n}$.

Remark 7. *For a non-singular variety X with an action of a finite group G, the variety X/G is Gorenstein if and only if the age of α on any connected component is an integer for all $\alpha \in \mathsf{G}$. See Remark (3.2) in [20]. If $\dim X$ is even and X/G is Gorenstein, by Corollary 1, $X^n/\mathsf{G}^n \rtimes \Sigma_n$ is Gorenstein. In particular, for a non-singular variety Y with even (complex) dimension, the age of the symmetric product Y^n/Σ_n is always an integer so that Y^n/Σ_n is Gorenstein.*

If Y is a smooth projective surface, then the Hilbert-Chow morphism $Y^{[n]} \to Y^n/\Sigma_n$ is a resolution of singularities [8], which is also crepant [1]. Hence, together with Theorem 5 and Remark 7, we obtain the following statement conjectured on p.20 of [25]:

Corollary 2. *Let X be a smooth projective surface with an action of a finite group G. Suppose that X/G is Gorenstein. If $\pi : Y \to X/\mathsf{G}$ is a crepant resolution, then $Y^{[n]} \to W^n/\Sigma_n$ is a crepant resolution.*

Together with Theorem 4, we obtain the following result.

Theorem 6. *Let Y be a smooth projective surface with trivial canonical class. Let X be a smooth projective surface with an action of G such that X/G is Gorenstein. Suppose that $\pi : Y \to X/\mathsf{G}$ is a crepant resolution and that $H^*(Y) \cong H^*_{orb}([X/\mathsf{G}])$ as Frobenius algebras, then $Y^{[n]} \to X^n/\Sigma_n$ is a hyper-Kähler resolution and $H^*(Y^{[n]})$ is isomorphic as a ring to $H^*_{orb}([X^n/\mathsf{G}^n \rtimes \Sigma_n])$.*

Proof. We have $\mathscr{H}(Y^n, \Sigma_n) \cong H^*(Y)\{\Sigma_n\} \cong H^*_{orb}([X/\mathsf{G}])\{\Sigma_n\} \cong \mathscr{H}(X^n, \mathsf{G}^n \rtimes \Sigma_n)^{\mathsf{G}^n}$ where the first equality is due to [7] and the third is Theorem 4. Since $H^*(Y^{[n]}) \cong H^*(Y)\{\Sigma_n\}^{\Sigma_n}$ [14], we obtain the theorem by taking Σ_n-invariants everywhere in the above equality.

Theorem 6 is a special case of the following conjecture due to Ruan [22].

Conjecture 1 (Cohomological hyper-Kähler resolution conjecture)**.** *Suppose that $Y \to X$ be a hyper-Kähler resolution of the coarse moduli space X of an orbifold $\mathscr{X}$. The ordinary cohomology ring $H^*(Y)$ of Y is isomorphic to the Chen-Ruan orbifold cohomology ring $H^*_{orb}(\mathscr{X})$ of $\mathscr{X}$.*

Remark 8. *The conjecture in the special case of wreath product orbifolds has been verified when $X = \mathbb{C}^2$ and G is a finite subgroup of $SL_2(\mathbb{C})$ in [6]. In particular, an explicit ring isomorphism between $H^*(Y^{[n]})$ and $H^*_{orb}([X^n/G^n \rtimes \Sigma_n])$ has been established when $X = \mathbb{C}^2$ and G is a finite cyclic subgroup of $SL_2(\mathbb{C})$ by using Fock space methods in [19].*

Remark 9. *In Theorem 6, the claim still holds if we replace the ordinary cohomology and orbifold cohomology by ordinary K-theory and orbifold K-theory respectively by the results in [10].*

ACKNOWLEDGEMENTS The author is greatly indebted to his thesis advisor Takashi Kimura, who has provided constant guidance throughout the course of this project. The author would like to thank R. Kaufmann to point out his papers to improve significantly from my thesis [17]. It is also pleasure to thank D. Abramovich, A. Craw, B. Fantechi, S. Okada, F. Perroni, W. Wang for important advice and useful conversations. The author is supported by the National Research Foundation of Korea (NRF) grants funded by the Korea government (MEST) (No. 2012-0000795, 2011-0001181). He also would like to express his gratitude to the Algebraic Structure and its Application Research Institute at KAIST for providing him an excellent research environment in 2011-2012.

References

[1] A. Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782 (1984), 1983.

[2] L. A. Borisov, L. Chen, and G. G. Smith. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.*, 18(1):193–215 (electronic), 2005.

[3] J. Bryan and T. Graber. The crepant resolution conjecture. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 23–42. Amer. Math. Soc., Providence, RI, 2009.

[4] W. Chen and Y. Ruan. A new cohomology theory of orbifold. *Comm. Math. Phys.*, 248(1):1–31, 2004.

[5] D. Edidin, T. J. Jarvis, and T. Kimura. Logarithmic trace and orbifold products. *Duke Math. J.*, 153(3):427–473, 2010.

[6] P. Etingof and V. Ginzburg. Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. *Invent. Math.*, 147(2):243–348, 2002.

[7] B. Fantechi and L. Göttsche. Orbifold cohomology for global quotients. *Duke Math. J.*, 117(2):197–227, 2003.

[8] J. Fogarty. Algebraic families on an algebraic surface. *Amer. J. Math*, 90:511–521, 1968.

[9] R. Goldin, T. S. Holm, and A. Knutson. Orbifold cohomology of torus quotients. *Duke Math. J.*, 139(1):89–139, 2007.

[10] T. J. Jarvis, R. Kaufmann, and T. Kimura. Pointed admissible G-covers and G-equivariant cohomological field theories. *Compos. Math.*, 141(4):926–978, 2005.

[11] T. J. Jarvis, R. Kaufmann, and T. Kimura. Stringy K-theory and the Chern character. *Invent. Math.*, 168(1):23–81, 2007.

[12] R. M. Kaufmann. Orbifolding Frobenius algebras. *Internat. J. Math.*, 14(6):573–617, 2003.

[13] R. M. Kaufmann. Second quantized Frobenius algebras. *Comm. Math. Phys.*, 248(1):33–83, 2004.

[14] M. Lehn and C. Sorger. The cup product of Hilbert schemes for $K3$ surfaces. *Invent. Math.*, 152(2):305–329, 2003.

[15] E. Lerman and A. Malkin. Hamiltonian group actions on symplectic Deligne-Mumford stacks and toric orbifolds. *Adv. Math.*, 229(2):984–1000, 2012.

[16] W. Li, Z. Qin, and W. Wang. Ideals of the cohomology rings of Hilbert schemes and their applications. *Trans. Amer. Math. Soc.*, 356(1):245–265 (electronic), 2004.

[17] T. Matsumura. *Orbifold Cohomology of a Wreath Product Orbifold*. PhD dissertation, Boston University, Dept. Math., August 2007. `arXiv:math/0610269`.

[18] Z. Qin and W. Wang. Hilbert schemes and symmetric products: a dictionary. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 233–257. Amer. Math. Soc., Providence, RI, 2002.

[19] Z. Qin and W. Wang. Hilbert schemes of points on the minimal resolution and soliton equations. In *Lie algebras, vertex operator algebras and their applications*, volume 442 of *Contemp. Math.*, pages 435–462. Amer. Math. Soc., Providence, RI, 2007.

[20] M. Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.

[21] M. Romagny. Group actions on stacks and applications. *Michigan Math. J.*, 53(1):209–236, 2005.

[22] Y. Ruan. Stringy orbifolds. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 259–299. Amer. Math. Soc., Providence, RI, 2002.

[23] V. Turaev. Homotopy field theory in dimension 2 and group-algebras. `arXiv:math/9910010`, October 1999.

[24] B. Uribe. Orbifold cohomology of the symmetric product. *Comm. Anal. Geom.*, 13(1):113–128, 2005.

[25] W. Wang. Equivariant K-theory, wreath products, and Heisenberg algebra. *Duke Math. J.*, 103(1):1–23, 2000.

[26] W. Wang and J. Zhou. Orbifold Hodge numbers of the wreath product orbifolds. *J. Geom. Phys.*, 38(2):152–169, 2001.

Appendix

Special Σ_I-Frobenius Algebras

In this section, we recall definitions and results about special G-Frobenius algebras from [12, 13] and the construction of Lehn-Sorger's algebras [14].

Definition 8 ([Definition 4.1 and 4.2, 12]). *A* special G-Frobenius algebra $(\mathscr{H}, \rho, \cdot, \{\mathbf{1}_g\}, \eta)$ *is a* G*-Frobenius algebra* $(\mathscr{H}, \rho, \cdot, \mathbf{1}_e, \eta)$ *with the choice of* $\mathbf{1}_g \in \mathscr{H}_g$ *such that* $\mathscr{H}_g = \mathscr{H}_e \cdot \mathbf{1}_g$ *and* $\varphi_g(\mathbf{1}_h) = \varphi_{g,h}\mathbf{1}_{ghg^{-1}}$ *for some* $\varphi_{g,h} \in \mathbf{k}^\times$*. Let* $r_g : \mathscr{H}_e \longrightarrow \mathscr{H}_g$ *be given by* $a \mapsto a \cdot \mathbf{1}_g$ *and let* $\mathsf{I}_g := \ker g$*. Let* i_g *be a section of* r_g*. A* special G-reconstruction datum *is a collection of Frobenius algebras* $(\mathscr{H}_g, \cdot, \eta_g, \mathbf{1}_g), g \in \mathsf{G}$ *with an action* ρ *of* G *on* $\mathscr{H}_e$ *and cyclic* $\mathscr{H}_e$*-algebra structures on* $(\mathscr{H}_g, \cdot, \mathbf{1}_g)$ *such that* $\mathscr{H}_g$ *and* $\mathscr{H}_{g^{-1}}$ *are isomorphic as* $\mathscr{H}_e$*-algebras and* $\eta(\rho_g(a), \rho_g(b)) = \eta(a, b)$*.*

Lemma 6 ([Proposition 4.1, 12]). *A special* G*-Frobenius algebra* $(\mathscr{H}, \rho, \cdot, \{\mathbf{1}_g\}, \eta)$ *defines a special* G*-reconstruction datum* $\{(\mathscr{H}_g, \cdot, \eta_g, \mathbf{1}_g), g \in \mathsf{G}, \rho\}$*. The structure of a Frobenius algebra in* $\mathscr{H}_g$ *is given by* $a_g \cdot b_g := i_g(a_g) \cdot i_g(b_g) \cdot \mathbf{1}_g$ *and* $\eta_g(a_g, b_g) := \eta(i_g(a_g)\mathbf{1}_g, i_g(b_g)\mathbf{1}_{g^{-1}})$*.*

Definition 9 ([Definition 4.3 and 4.4, 12]). *Let* $\{(\mathscr{H}_g, \cdot_g, \eta_g, \mathbf{1}_g), g \in \mathsf{G}, \rho\}$ *be a special* G*-reconstruction datum. A* graded cocycle *is a map* $\gamma : \mathsf{G} \times \mathsf{G} \longrightarrow \mathscr{H}_e$*,* $(g, h) \mapsto \gamma_{g,h}$ *such that* $\gamma_{g,h}\gamma_{gh,k} \equiv \gamma_{g,hk}\gamma_{h,k} \mod \mathsf{I}_{ghk}$*. A graded cocycle* γ *is* compatible with the special G-reconstruction datum *if* $(\mathsf{I}_g + \mathsf{I}_h)\gamma_{g,h} \subset \mathsf{I}_{gh}$ *(*section independence*),* $\gamma_{g,g^{-1}} = \check{r}_g(\mathbf{1}_g)$ *(metric compatibility) and* $\gamma_{e,h} = \mathbf{1}_e \mod \mathsf{I}_h$ *where* $\eta_g^\sharp : a_g \mapsto \eta_g(a_g, \ )$ *and* $\check{r}_g := (\eta_e^\sharp)^{-1} \circ r_g^* \circ \eta_g^\sharp$*. We identify two cocycle* γ *and* γ' *if* $\gamma_{g,h} \equiv \gamma'_{g,h} \mod \mathsf{I}_{gh}$*.*

A non-abelian cocycle *is a map* $\varphi : \mathsf{G} \times \mathsf{G} \longrightarrow \mathbf{k}^\times$ *satisfying* $\varphi_{gh,k} = \varphi_{g,hkh^{-1}}\varphi_{h,k}$ *and* $\varphi_{e,g} = \varphi_{g,e} = 1$*. A graded cocycle* γ *and a non-abelian cocycle* φ *form a* compatible pair *if* $\varphi_{g,h}\gamma_{ghg^{-1},g} = \gamma_{g,h}$ *and* $\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh}$*.*

Theorem 7 (Reconstruction Theorem [Theorem 4.1, 12]). *Let* $\{(\mathscr{H}_g, \cdot_g, \eta_g, \mathbf{1}_g), g \in \mathsf{G}, \rho\}$ *be a special* G*-reconstruction datum. Then the structures of special* G*-Frobenius algebras inducing the given reconstruction datum correspond bijectively to compatible pairs of a non-abelian cocycle* φ *and a section independent cocycle* γ *compatible with the given special* G*-reconstruction datum, such that* $\varphi_{g,g} = 1$ *and* $\mathrm{Tr}_{\mathscr{H}_g}(L_c \circ \rho_h) = \mathrm{Tr}_{\mathscr{H}_h}(\rho_{g^{-1}} \circ L_c)$ *for any* $c \in \mathscr{H}_{[g,h]}$*.*

Remark 10. *Given* $\varphi_{g,h}$ *and* $\gamma_{g,h}$*, the multiplication on* $\mathscr{H} = \oplus_g \mathscr{H}_g$ *is given by* $a_g \cdot b_h := r_{gh}(i_g(a_g) \cdot_e i_h(b_h) \cdot_e \gamma_{g,h})$*, the action* ρ_g *on* $\mathscr{H}_h$ *is defined by* $\varphi_g(b_h) := r_{ghg^{-1}}(\varphi_{g,h}\rho_g(i_h(b_h)))$*, and the metric is defined by* $\eta(a_g, b_{g^{-1}}) := \eta_e(i_g(a_g) \cdot_e i_{g^{-1}}(b_{g^{-1}}) \cdot_e \gamma_{g,g^{-1}}, \mathbf{1}_e)$ *and* $\eta_e(a_g, b_h) = 0$ *if* $gh \neq e$*. Those definition is independent from the choice of sections* i_g *because of the section independence of* γ*. On the other hand, given a* G*-Frobenius algebra,* γ *and* φ *are defined by* $\mathbf{1}_g\mathbf{1}_h = \gamma_{g,h} \cdot \mathbf{1}_{gh}$ $(\gamma_{g,h} \in \mathscr{H}_e)$ *and* $\varphi_g(\mathbf{1}_h) = \varphi_{g,h}\mathbf{1}_{ghg^{-1}}$ $(\varphi_{g,h} \in \mathbf{k}^\times)$*.*

Definition 10 (Normality). *Let* I *be a finite set of cardinality* $|I| = n$*. Consider a special* Σ_I*-Frobenius algebra* $(\mathscr{H}, \rho, \cdot, \{\mathbf{1}_\sigma\}, \eta)$*. Two permutations* $\sigma, \tau \in \Sigma$ *are transversal if* $|\sigma| + |\tau| = |\sigma\tau|$*. The graded cocycle* γ *induced by a special* Σ_I*-Frobenius algebra is* normalized *if* $\gamma_{\sigma,\tau} = \mathbf{1}_e$ *for all transversal pair* σ, τ *with* $|\tau| = 1$ *[Definition 6.4, 13].*

Theorem 8 ([Theorem 6.5, 13]). *If γ is normalized, then γ is completely determined by $\gamma_{\tau,\tau} = \check{r}_\tau(\mathbf{1}_\tau)$ with $|\tau| = 1$.*

Lehn-Sorger's Algebras

Let $(\mathsf{A}, \cdot, 1, \eta)$ be a commutative graded Frobenius algebra of degree $2d$. If J is a finite set, then the tensor product $\mathsf{A}^{\otimes J}$ over J is naturally a commutative graded Frobenius algebra. If $\phi : J_1 \twoheadrightarrow J_2$ is a surjective map between finite sets, then there is an induced homomorphism $\phi_* : \mathsf{A}^{\otimes J_1} \twoheadrightarrow \mathsf{A}^{\otimes J_2}$ defined by $\bigotimes_{i \in J_1} a_i \mapsto \bigotimes_{j \in J_2} \left(\prod_{\phi(i)=j} a_i\right)$. By using the metric, we can identify a Frobenius algebra with its vector space dual, and thus we also have an induced map $\phi^* : \mathsf{A}^{\otimes J_2} \longrightarrow \mathsf{A}^{\otimes J_1}$.

Let I be a finite set of cardinality $|I| = n$. The underlying vector space for the Lehn-Sorger's algebra associated to A is

$$\mathsf{A}\{\Sigma\} := \bigoplus_{\sigma \in \Sigma_I} \mathsf{A}^{\otimes I_\sigma}$$

where I_σ is the set of orbits in I by the action of the subgroup $\langle \sigma \rangle$ generated by σ. There is an obvious Σ_I-graded Σ_I-module structure. The $\mathbb{Q}$-grading is given by

$$\deg^{\mathbb{Q}} a_\sigma := |a_\sigma| + d \cdot |\sigma|, \quad a_\sigma \in \mathsf{A}^{\otimes I_\sigma} \tag{13}$$

where $|a_\sigma|$ is the degree in $\mathsf{A}^{\otimes I_\sigma}$ and $|\sigma|$ is the minimum transpositions to express σ.

The *Euler class* $\mathfrak{e}$ of a Frobenius algebra A is defined as the image of 1 under the map $\mathsf{A} \longrightarrow \mathsf{A} \otimes \mathsf{A} \longrightarrow \mathsf{A}$ where the first map is the comultiplication and the second map is the multiplication. The *graph defect* $\mathsf{gd}(\sigma, \tau) : I_{\sigma,\tau} \longrightarrow \mathbb{Z}_{\geq 0}$ is defined by

$$\mathsf{gd}(\sigma,\tau)_c = \frac{1}{2}\left(|c| + 2 - |c/\langle\sigma\rangle| - |c/\langle\tau\rangle| - |c/\sigma\tau|\right).$$

Using these, the Lehn-Sorger's product is defined by

$$a_\sigma \cdot b_\tau = f^* \left(f_{\sigma *} a_\sigma \cdot f_{\tau *} b_\tau \cdot \left(\bigotimes_{c \in I_{\sigma,\tau}} \mathfrak{e}^{\mathsf{gd}(\sigma,\tau)_c} \right) \right)$$

where $f_\sigma : I_\sigma \twoheadrightarrow I_{\sigma,\tau}$, $f_\tau : I_\tau \twoheadrightarrow I_{\sigma,\tau}$ and $f : I_{\sigma\tau} \twoheadrightarrow I_{\sigma,\tau}$. Note that, if σ, τ are transversal, then it is straightforward computation to see that $\mathsf{gd}(\sigma,\tau) = 0$. Thus it is also easy to see that the induced graded cocycle γ is normalized.

Theorem 9 ([14, 13]). $\mathsf{A}\{\Sigma_I\}$ *is a graded special Σ_I-Frobenius algebra and its graded cocycle γ is normalized.*