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A Note on the Operator-Valued Poisson Kernel

Serap BULUT

Kocaeli University, Civil Aviation College, Arslanbey Campus

41285 İzmit, Kocaeli / Turkey

Abstract. The purpose of this paper is to give a different proof of the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} K_{r,t}(T)dt = I,$$

where $K_{r,t}(T)$ is the operator-valued Poisson kernel.

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1. Introduction

Let $\mathscr{H}$ be a Hilbert space which will be always complex and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators from $\mathscr{H}$ to $\mathscr{H}$. We write I for the identity operator on $\mathscr{H}$. For $T \in \mathscr{L}(\mathscr{H})$, we denote by $\sigma(T)$ the spectrum of T. T is called a unitary operator if it satisfies $T^*T = TT^* = I$ where T^* is the adjoint of T.

Email address: `serap.bulut@kocaeli.edu.tr`

Throughout the paper $\mathbb{D}$ will denote the open unit disc $\mathbb{D}=\{z:|z|<1\}$ in the complex plane $\mathbb{C}$.

For $re^{it}\in\mathbb{D}$, the (*scalar*) *Poisson kernel* $P_{r,t}$ is defined by

$$\begin{aligned} P_{r,t}(e^{i\theta}) &= \frac{1-r^2}{\left(1-re^{it}e^{-i\theta}\right)\left(1-re^{-it}e^{i\theta}\right)} \\ &= \frac{1}{1-re^{it}e^{-i\theta}}+\frac{1}{1-re^{-it}e^{i\theta}}-1 \\ &= \sum_{n\geq 0} r^n e^{int}e^{-in\theta}+\sum_{n\geq 0} r^n e^{-int}e^{in\theta}-1. \end{aligned}$$

It is the well-known property of the (scalar) Poisson kernel that the integral formula

$$\frac{1}{2\pi}\int_0^{2\pi} P_{r,t}(e^{i\theta})d\theta=1 \tag{1.1}$$

holds.

In [1], the author gave the definition of the *operator-valued Poisson kernel* $K_{r,t}(T)\in\mathscr{L}(\mathscr{H})$ for $T\in\mathscr{L}(\mathscr{H})$ such that $\sigma(T)\subset\overline{\mathbb{D}}$ and for $re^{it}\in\mathbb{D}$, in the following way:

$$K_{r,t}(T)=(I-re^{it}T^*)^{-1}+(I-re^{-it}T)^{-1}-I. \tag{1.2}$$

For an operator $T\in\mathscr{L}(\mathscr{H})$ and a polynomial $p(z)=\sum_{k=0}^{n}a_k z^k\in\mathbb{C}[z]_{|\overline{\mathbb{D}}}$, $p(T)\in\mathscr{L}(\mathscr{H})$ is defined by

$$p(T)=\sum_{k=0}^{n}a_k T^k.$$

Remark. T^0 is defined to be the identity operator, whatever the operator T.

On the other hand, for $0\leq r<1$, $p(rT)$ is defined by means of the operator-valued Poisson kernel as follows.

Lemma 1.1. *[1] Let $T\in\mathscr{L}(\mathscr{H})$ such that $\sigma(T)\subset\overline{\mathbb{D}}$. For all $r\in[0,1)$, we have:*

$$p(rT)=\frac{1}{2\pi}\int_0^{2\pi} p(e^{it})K_{r,t}(T)dt\,,\qquad p\in\mathbb{C}[z]_{|\overline{\mathbb{D}}}.$$

Note that in the case p identically equal to 1, we have

Main Theorem.

$$\frac{1}{2\pi}\int_0^{2\pi} K_{r,t}(T)dt = I \tag{1.3}$$

for $0 \le r < 1$ and $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$.

The purpose of this paper is to give a different proof of (1.3) independently a polynomial.

In [2] which is a motive of our present paper, a proof of (1.1) is given by using the functional equation

$$F(r^{2^n}) = F(r), \qquad n = 1, 2, \ldots$$

where

$$F(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{\left(1-re^{i\theta}\right)\left(1-re^{-i\theta}\right)} d\theta, \qquad 0 \le r < 1.$$

In this note, we use a similar method for the operator-valued Poisson kernel $K_{r,t}(T)$.

2. Proof of the Main Theorem

Let $re^{it} \in \mathbb{D}$, $0 \le r < 1$ and let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. Set

$$F(rT) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} K_{r,t}(T)dt. \tag{2.1}$$

Then F is a continuous function. Also, it is obvious that $F(0) = I$ for $r = 0$.

Let us write

$$F(rT) = \frac{1}{2\pi}\int_0^{\pi} K_{r,t}(T)dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} K_{r,x}(T)dx.$$

Making the substitution $x = t + \pi$ in the second integral, and using (1.2), we obtain

$$\begin{aligned} F(rT) &= \frac{1}{2\pi}\int_0^{\pi} \left[(I - re^{it}T^*)^{-1} + (I - re^{-it}T)^{-1} - I\right] dt \\ &+ \frac{1}{2\pi}\int_0^{\pi} \left[(I + re^{it}T^*)^{-1} + (I + re^{-it}T)^{-1} - I\right] dt. \end{aligned}$$

Hence we get

$$\begin{aligned} F(rT) &= \frac{1}{2\pi}\int_0^{\pi} \left[(I - re^{it}T^*)^{-1} + (I + re^{it}T^*)^{-1}\right] dt \\ &+ \frac{1}{2\pi}\int_0^{\pi} \left[(I - re^{-it}T)^{-1} + (I + re^{-it}T)^{-1}\right] dt \\ &- \frac{1}{2\pi}\int_0^{\pi} 2I dt. \end{aligned} \tag{2.2}$$

On the other hand, we have the equalities

$$(I - re^{it}T^*)^{-1} + (I + re^{it}T^*)^{-1} = 2\left(I - r^2e^{2it}T^{*2}\right)^{-1} \tag{2.3}$$

and

$$(I - re^{-it}T)^{-1} + (I + re^{-it}T)^{-1} = 2\left(I - r^2e^{-2it}T^2\right)^{-1}. \tag{2.4}$$

Thus, by (2.3) and (2.4), (2.2) is of the form

$$F(rT) = \frac{1}{\pi}\int_0^{\pi} \left[\left(I - r^2e^{2it}T^{*2}\right)^{-1} + \left(I - r^2e^{-2it}T^2\right)^{-1} - I\right] dt.$$

Making the substitution $\phi = 2t$ in the above integral, we find

$$F(rT) = \frac{1}{2\pi}\int_0^{2\pi} \left[\left(I - r^2e^{i\phi}T^{*2}\right)^{-1} + \left(I - r^2e^{-i\phi}T^2\right)^{-1} - I\right] d\phi.$$

By (1.2), we get

$$F(rT) = \frac{1}{2\pi}\int_0^{2\pi} K_{r^2,\phi}(T^2)d\phi. \tag{2.5}$$

In view of (2.1) and (2.5), we obtain

$$F(rT) = F(r^2T^2). \tag{2.6}$$

By repeated applications of (2.6), we see that

$$F(rT) = F((rT)^{2^n}), \qquad n = 1, 2, \ldots.$$

Since $\|rT\| < 1$, we have

$$F(rT) = \lim_{n\to\infty} F((rT)^{2^n}) = F(0) = I.$$

Thus the proof is completed.

3. Results

Corollary 3.1. *Note that*

$$F(rT^*) = I.$$

Lemma 3.2. *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. If T is invertible then*

$$K_{r^{-1},t}(T^{-1}) = -K_{r,-t}(T)$$

for $0 < r < 1$.

Corollary 3.3. *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$.*

(i) If T is invertible then

$$F(r^{-1}T^{-1}) = -F(rT^*)$$

for $0 < r < 1$.

(ii) If T is a unitary operator then

$$F(r^{-1}T^{-1}) = -F(rT^{-1})$$

for $0 < r \neq 1$.

When we consider the Corollary 3.3, we have the following

Theorem 3.4. *Let $T \in \mathscr{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$.*

(i) If T is invertible then

$$\frac{1}{2\pi}\int_0^{2\pi} K_{r^{-1},t}(T^{-1})dt = -I$$

for $0 < r < 1$.

(ii) If T is a unitary operator then

$$\frac{1}{2\pi}\int_0^{2\pi} K_{r,t}(T^{-1})dt = -I$$

for $r > 1$.

References

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