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On $\omega\beta-$Continuous Functions

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Abstract. A subset A of topological space (X,τ) is said to be $\omega\beta-$open [3] if for every $x \in A$ there exists an $\beta-$open set U containing x such that $U - A$ is a countable. In this paper, we introduce and study new class of function which is $\omega\beta-$continuous functions by using the notion of $\omega\beta-$open sets. This new class of function defines as a function $f : (X,\tau) \to (Y,\sigma)$ from a topological space (X,τ) into a topological space (Y,σ) is $\omega\beta-$Continuous function if and only if for each $x \in X$ and each open set V in (Y,σ) containing $f(x)$ there exists an $\omega\beta-$open set U containing x such that $f(U) \subseteq V$. We give some characterizations of $\omega\beta-$Continuous functions, define $\omega\beta-$irresolute and $\omega\beta-$open function. Finally, we find relationship between these type of function.

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1. Introduction

Throughout the present paper, a space mean topological space on which no separation axiom is assumed unless explicitly stated. Let A be a subset of a space (X,τ). The closure of A and interior of A in (X,τ) are denoted by $Int(A)$ and $cl(A)$, respectively. A subset A of a space (X,τ) is said to be $b-$open [4], (reps. $\beta-$open [7]) if $A \subseteq Int(cl(A)) \cup cl(Int(A))$, (resp. $A \subseteq cl(Int(cl(A)))$).

Recall that a subset A of a space (X,τ) is said to be $\omega\beta-$open [3] (resp. $\omega b-$open [9], $\omega-$open [2]) set if for every $x \in A$ there exists an $\beta-$open (resp. $b-$open, open) set U containing x such that $U - A$ is a countable. We write $\omega\beta O(X,\tau)$ (resp. $\omega bO(X,\tau)$, $\beta O(X,\tau)$, $\omega O(X,\tau)$, $bO(X,\tau)$) to denote the family of all $\omega\beta-$open (resp. $\omega b-$open, $\beta-$open, $\omega-$open, $b-$open) subsets of (X,τ).

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Definition 1. *A function $f : (X,\tau) \to (Y,\sigma)$ is called $\omega-$continuous [6] (resp. $\omega b-$continuous [9]) if for every $x \in X$ and each open set V in (Y,σ) containing $f(x)$ there exists an $\omega O(X,\tau)$ (resp. $\omega bO(X,\tau)$) set U containing x such that $f(U) \subseteq V$.*

Lemma 1. *[3] Let (X,τ) be a topological space:*

i. The union of any family of $\omega\beta O(X,\tau)$ sets is $\omega\beta O(X,\tau)$.

ii. The intersection of an $\omega\beta O(X,\tau)$ set and open set is $\omega\beta O(X,\tau)$.

Theorem 1. *[3] Let (Y,τ_Y)be a subspace of (X,τ), $A \subseteq Y$and Y is $\beta O(X,\tau)$ sets. Then $A \in \omega\beta O(X,\tau)$ if and only if $A \in \omega\beta O(Y,\tau_Y)$.*

Theorem 2. *[3] Let A be a subset of a topological space (X,τ). Then $x \in \omega\beta cl(A)$ if and only if for every $\omega\beta O(X,\tau)$ set U containing x, $A \cap U \neq \phi$.*

Theorem 3. *[5] If $f : (X,\tau) \to (Y,\sigma)$ is an open continuous function, then $f^{-1}(cl(A)) = cl(f^{-1}(A))$.*

2. $\omega\beta-$Continuous Functions

Definition 2. *A function $f : (X,\tau) \to (Y,\sigma)$ is called $\omega\beta-$continuous at a point $x \in X$, if for every open set V in (Y,σ) containing $f(x)$ there exists an $\omega\beta O(X,\tau)$ set U containing x such that $f(U) \subseteq V$. If f is $\omega\beta-$continuous at each point of X then f is said to be $\omega\beta-$continuous on X.*

Definition 3. *Let (X,τ) be any space, a set $A \subseteq X$ is said to be $\omega\beta-$neighborhood of a point x in X if and only if there exists a $\omega\beta O(X,\tau)$ set U containing x such that $U \subseteq A$.*

Theorem 4. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function, where X and Y are topological space. Then the following are equivalent:*

i. The function f is $\omega\beta-$continuous.

ii. For each open set $V \subset Y$, $f^{-1}(V)$ is $\omega\beta O(X,\tau)$.

iii. For each $x \in X$, the inverse of every neighborhood of $f(x)$ is an $\omega\beta-$neighborhood of x.

iv. For each $x \in X$ and each neighborhood N_x of $f(x)$, there is an $\omega\beta-$neighborhood V of x such that $f(U) \subseteq N_x$.

v. For each closed set $M \subset Y$, $f^{-1}(M)$ is $\omega\beta-$closed in X.

vi. For each subset $A \subset X$, $f(\omega\beta cl(A)) \subset cl(f(A))$.

vii. For each subset $B \subset Y$, $\omega\beta cl(f^{-1}(B)) \subseteq (f^{-1}(cl(B)))$.

Proof. (i → ii) Let V be open in Y and $x \in f^{-1}(V)$ then $f(x) \in V$, by (i), there exists an $\omega\beta O(X,\tau)$ set U_x in X containing x and $f(U_x) \subseteq V$. Then $x \in U_x \subseteq f^{-1}(V)$ and hence $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$. By Lemma 1(i), $f^{-1}(V) \in \omega\beta O(X,\tau)$, which implies that f is $\omega\beta-$continuous.
(ii → iii) For $x \in X$, let V be the neighborhood of $f(x)$ then $f(x) \in W \subseteq V$, where W is open in Y. By (ii), $f^{-1}(W) \in \omega\beta O(X,\tau)$, and $x \in f^{-1}(W) \subseteq f^{-1}(V)$. Then by Definition 3, $f^{-1}(V)$ is $\omega\beta-$neighborhood of x.
(iii → iv) For $x \in X$ and N_x be a neighborhood of $f(x)$. Then $V = f^{-1}(N_x)$ is an $\omega\beta-$neighborhood of x and $f(V) = f(f^{-1}(N_x)) \subset N_x$.
(iv → v) For any $x \in X - f^{-1}(M)$, $f(x) \in Y - M$. Since M is closed, the set $Y - M$ is neighborhood of $f(x)$, hence there is a $\omega\beta-$neighborhood V of x such that $f(V) \subset Y - M$, there exists an $\omega\beta O(X,\tau)$ set U_x in X containing x and $U_x \subseteq V \subseteq X - f^{-1}(M)$, take $(X - f^{-1}(M)) = \bigcup_{x \in f^{-1}(Y-M)} U_x$. By Lemma 1(i), the set $(X - f^{-1}(M)) \in \omega\beta O(X,\tau)$, which implies $f^{-1}(M)$ is $\omega\beta C(X,\tau)$.
(v → vi) Let $A \subseteq X$, since $cl(f(A))$ is a closed set in Y by (vi), $f^{-1}(cl(f(A)))$ is an $\omega\beta C(X,\tau)$ set containing A, then $f(\omega\beta cl(A)) \subset cl(f(A))$.
(vi → vii) Let $B \subset Y$. By (vi), $f(\omega\beta cl(f^{-1}(B))) \subseteq cl(B)$, so $\omega\beta cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.
(vii → i) Suppose on the contrary that f is not $\omega\beta-$continuous. So there exist $x \in X$ and $V \in \sigma$ with $f(x) \in V$ such that for all $\omega\beta O(X,\tau)$ sets U with $x \in U$ and $f(U) \not\subset (V)$ i.e. $f(U) \cap (Y - V) \neq \phi$. Therefore, by Theorem 2, $x \in \omega\beta cl(f^{-1}(Y - V))$ and so by (vii), $f(x) \in cl(Y - V)$, thus for all open sets V in (Y,σ) containing $f(x)$, the set $V \cap (Y - V) \neq \phi$, a contradiction. Therefore, f is $\omega\beta-$continuous.

Definition 4. *For any subset A of a topological space (X,τ) the frontier of A, denoted by $\omega\beta F_r(A)$, is define as $\omega\beta cl(A) \cap \omega\beta cl(X - A)$.*

Theorem 5. *Let (X,τ), (Y,σ) be a topological space and $f : (X,\tau) \to (Y,\sigma)$ be a function. Then $X - \omega\beta c(f) = \cup\{\omega\beta F_r(f^{-1}(V)) : V \in \sigma, f(x) \in V, x \in X\}$ where $\omega\beta c(f)$ denotes the set of points at which f is $\omega\beta-$continuous.*

Proof. Let $x \in X - \omega\beta c(f)$. Then for every $\omega\beta O(X,\tau)$ set U containing x there exists open sets V in (Y,σ) containing $f(x)$ such $f(U) \not\subset V$, Hence $U \cap (X - f^{-1}(V)) \neq \phi$ for every $\omega\beta O(X,\tau)$ set U containing x. Therefore, by Theorem 2 $x \in \omega\beta cl(X - f^{-1}(V))$. Then $x \in f^{-1}(V) \cap \omega\beta cl(X - f^{-1}(V)) \subseteq \omega\beta F_r(f^{-1}(V))$. Hence, $X - \omega\beta c(f) \subseteq \cup\{\omega\beta F_r(f^{-1}(V)), V \in \sigma, f(x) \in V, x \in X\}$. Conversely, let $x \notin X - \omega\beta c(f)$. Then for each open sets V in (Y,σ) containing $f(x)$, $f^{-1}(V)$ is $\omega\beta O(X,\tau)$ containing x, thus for every $V \in \sigma$ containing $f(x)$, $x \in \omega\beta Int(f^{-1}(V))$ and hence $x \notin \omega\beta F_r(f^{-1}(V))$. So $\cup\{\omega\beta F_r(f^{-1}(V)) : V \in \sigma, f(x) \in V, x \in X\} \subseteq X - \omega\beta c(f)$.

Corollary 1. *A function $f : (X,\tau) \to (Y,\sigma)$ is $\omega\beta-$continuous if and only if $f^{-1}(\mathrm{int}(G)) \subseteq \omega\beta\, \mathrm{int}(f^{-1}(G))$, for any subset $G \subseteq Y$.*

Proof. NECESSITY. Let G be any subset of Y. Since f is $\omega\beta-$continuous, $f^{-1}(\mathrm{int}(G))$ is $\omega\beta O(X,\tau)$ set. As $f^{-1}(\mathrm{int}(G)) \subseteq f^{-1}(G)$, then $f^{-1}(\mathrm{int}(G)) \subseteq \omega\beta\, \mathrm{int}(f^{-1}(G))$.

SUFFICIENCY. Let $x \in X$ and $V \in \sigma$ with $f(x) \in V$. Then $x \in f^{-1}(V)$ and so by assumption $x \in \omega\beta\,\mathrm{Int}(f^{-1}(V))$. There exists an $\omega\beta O(X,\tau)$ such that $x \in U \subseteq f^{-1}(V)$. Hence $f(x) \in f(U) \subseteq V$ and the result follows.

Note that if X is a countable set then every function $f : (X,\tau) \to (Y,\sigma)$ is $\omega\beta-$continuous. The following diagram follows immediately from the definitions in which none of the implications is reversible.

$$\begin{array}{ccccc} \text{continuous} & \rightarrow & b-\text{continuous} & \rightarrow & \beta-\text{continuous} \\ \downarrow & & \downarrow & & \downarrow \\ \omega-\text{continuous} & \rightarrow & \omega b-\text{continuous} & \rightarrow & \omega\beta-\text{continuous} \end{array}$$

Example 1. *Let $X = R$ with the topology $\tau = \tau_u$ and $Y = \{0,1\}$ with the topology $\sigma = \{\phi, Y, \{0\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} - \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

Then f is $\omega\beta-$continuous but it is neither continuous nor $\omega-$continuous.

Example 2. *Let $X = \{1,2,3\}$ with the topology $\tau = \{X, \phi, \{1\}, \{2\}, \{1,2\}\}$ and $Y = \{a,b\}$ with the topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} b & x = \{1,2\} \\ a & x = 3 \end{cases}$$

Then f is not $\beta-$continuous, but it can be easily seen that f is $\omega\beta-$continuous.

Example 3. *Let $X = R$ with the topology $\tau = \tau_u$ and $Y = \{a,b\}$ with the topology $\sigma = \{\phi, Y, \{a\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} a & x \in [0,2) \cap \mathbb{R} - \mathbb{Q} \\ b & x \in [0,2) \cap \mathbb{Q} \end{cases}$$

Then f is $\omega\beta-$continuous, but it is not $\omega b-$continuous.

Proposition 1. *If $f : (X,\tau) \to (Y,\sigma)$ is an $\omega\beta-$continuous function and A is an open set in X, then the restriction $f|_A : (A,\tau_A) \to (Y,\sigma)$ is $\omega\beta-$continuous.*

Proof. Since f is an $\omega\beta-$continuous, for any open set V in Y, $f^{-1}(V)$ is a $\omega\beta O(X,\tau)$ set. Hence by Lemma 1(ii), $f^{-1}(V) \cap A$ is a $\omega\beta O(X,\tau)$ since A is an open set. Therefore, by Theorem 1, $(f|_A)^{-1}(V) = f^{-1}(V) \cap A$ is $\omega\beta O(A,\tau_A)$ sets, which implies that $f|_A$ is $\omega\beta-$continuous function.

Observe that the above theorem is not true if A were taken to be $\beta O(X,\tau)$ sets or $\omega O(X,\tau)$, as it shown in the next examples.

Example 4. *Let $X = R$ with the topology $\tau = \tau_{coc}$ and $Y = \{0,1\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1 & x \in (0,1] \\ 0 & x \notin (0,1] \end{cases}$$

It can be easily seen that f is $\omega\beta-$continuous. We take $A = (0,1]$. Then $A \in \beta O(X,\tau)$ and $f|_A$ is not $\omega\beta-$continuous since $(f|_A)^{-1}(1) = \{1\} \notin \omega\beta O(A, \tau_A)$.

Example 5. *Let $X = R$ with the topology $\tau = \tau_u$ and $Y = \{0,1\}$ with the topology $\sigma = \{\phi, Y, \{1\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1 & x = \sqrt{2} \\ 0 & x \in \mathbb{Q} \end{cases}$$

It can be easily seen that f is $\omega\beta-$continuous. We take $A = \mathbb{R} - \mathbb{Q}$. Then $A \in \omega O(X,\tau)$ and $f|_A$ is not $\omega\beta-$continuous since $(f|_A)^{-1}(Y) = \{\sqrt{2}\} \notin \omega\beta O(A, \tau_A)$.

Definition 5. *[7] A cover $\upsilon = \{U_\alpha : \alpha \in \Delta\}$ of subset of X is called a $\beta O(X,\tau)$ cover if U_α is $\beta O(X,\tau)$ for each $\alpha \in \Delta$.*

Now we prove the following proposition.

Proposition 2. *Let $f : (X,\tau) \to (Y,\sigma)$ be any function and $\mathrm{A} = \{A_\alpha : \alpha \in \Delta\}$ be a $\beta O(X,\tau)$ cover of X. If the restriction, $f|_{A_\alpha} : (A_\alpha, \tau_{A_\alpha}) \to (Y,\sigma)$ is $\omega\beta-$continuous for each $\alpha \in \Delta$, then f is $\omega\beta-$continuous.*

Proof. Let V be any open set in Y. Since $f|_{A_\alpha}$ is $\omega\beta-$continuous, then for each $\alpha \in \Delta$, we have$(f|_A)^{-1}(V) = f^{-1}(V) \cap A_\alpha \in \omega\beta O(A_\alpha, \tau_{A_\alpha})$. So by Theorem 1, $f^{-1}(V) \cap A_\alpha \in \omega\beta O(X,\tau)$ for each $\alpha \in \Delta$. Take $f^{-1}(V) = \bigcup_{\alpha\in\Delta} (f^{-1}(V) \cap A_\alpha)$. By Lemma 1(i) $f^{-1}(V) \in \omega\beta O(X,\tau)$.

Corollary 2. *Let $f : (X,\tau) \to (Y,\sigma)$ be any function and $\mathrm{A} = \{A_\alpha : \alpha \in \Delta\}$ a open cover of X. If the restriction, $f|_{A_\alpha} : (A_\alpha, \tau_{A_\alpha}) \to (Y,\sigma)$ is $\omega\beta-$continuous for each $\alpha \in \Delta$, then f is $\omega\beta-$continuous.*

The composition $g \circ f : (X,\tau) \to (Z,\rho)$ of a continuous function $f : (X,\tau) \to (Y,\sigma)$ and an $\omega\beta-$continuous function $g : (Y,\sigma) \to (Z,\rho)$ is not necessarily $\omega\beta-$continuous function as the following example shows. Thus, the composition of $\omega\beta-$continuous functions need not be $\omega\beta-$continuous.

Example 6. *Let $X = \mathbb{R}$ with the topology $\tau = \tau_{coc}$, Y={1,2} with the topology $\sigma = \{\phi, Y, \{1\}\}$ and $Z = \{a,b\}$ with the topology $\rho = \{\phi, Z, \{a\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} - \mathbb{Q} \\ 2 & x \in \mathbb{Q} \end{cases}$$

and $g:(X,\sigma)\rightarrow(Y,\rho)$ *be the function defined by*

$$g(x)=\begin{cases} a & x=2 \\ b & x=1 \end{cases}$$

Then f *is continuous (hence* $\omega\beta-$*continuous) and* g *is* $\omega\beta-$*continuous. However* $g\circ f$ *is not* $\omega\beta-$*continuous, because* $(g\circ f)^{-1}(\{a\})=\mathbb{Q}\notin\omega\beta O(X,\tau)$*.*

Proposition 3. *If* $f:(X,\tau)\rightarrow(Y,\sigma)$ *is* $\omega\beta-$*continuous and* $g:(Y,\sigma)\rightarrow(Z,\rho)$ *is continuous, then* $g\circ f:(X,\tau)\rightarrow(Z,\rho)$ *is* $\omega\beta-$*continuous.*

Proof. Let $x\in X$ and $V\in\rho$ with $(g\circ f)(x)\in V$ and $f(x)\in Y$, since g is continuous, there exists open sets W in (Z,ρ) with $f(x)\in W$ and $g(W)\subseteq V$. Moreover f is $\omega\beta-$continuous there exists $\omega\beta O(X,\tau)$ say U containing x such that $f(U)\subseteq W$. Now $(g\circ f)(U)\subseteq g(W)\subseteq V$.

We note that Proposition 3 is not true if g is assumed to be only $\omega-$continuous or $\beta-$continuous as it is shown in the next example.

Example 7. *Consider* $X=\mathbb{R}$ *with the topology* $\tau=\tau_{coc}$*,* $Y=\{a,b,c\}$ *with the topology* $\sigma=\{\phi,Y,\{a\},\{b\},\{a,b\}\}$ *and* $Z=\{1,2,3,4\}$ *with the topology* $\rho=\{\phi,Z,\{1\},\{1,2\},\{1,2,3\}\}$*. Let* $f:(X,\tau)\rightarrow(Y,\sigma)$ *be the function define by*

$$f(x)=\begin{cases} a & x\in\mathbb{R}-\mathbb{Q} \\ c & x\in\mathbb{Q} \end{cases}$$

and $g:(Y,\sigma)\rightarrow(Z,\rho)$ *be the function define by*

$$g(x)=\begin{cases} 1 & x=a \\ 3 & x=b \\ 2 & x=c \end{cases}$$

Then f *is* $\omega\beta-$*continuous,* g *is* $\omega-$*continuous and* $\beta-$*continuous function but* $g\circ f$ *is not* $\omega\beta-$*continuous since* $(g\circ f)^{-1}(2)=\mathbb{Q}\notin\omega\beta O(X,\tau)$*.*

Corollary 3. *If* $f:(X,\tau)\rightarrow\prod_{\alpha\in\Delta}X_\alpha$ *is an* $\omega\beta-$*continuous function from a space* (X,τ) *into a product space* $\prod_{\alpha\in\Delta}X_\alpha$*, then* $P_\alpha\circ f$ *is* $\omega\beta-$*continuous for each* $\alpha\in\Delta$*, where* P_α *is the projection function from the product space* $\prod_{\alpha\in\Delta}X_\alpha$ *onto the space* X_α *for each* $\alpha\in\Delta$*.*

Theorem 6. *Let* X *and* Y *be a topological spaces, let* $f:(X,\tau)\rightarrow(Y,\sigma)$ *be a function and* $g:(X,\tau)\rightarrow(X\times Y,\tau\times\sigma)$ *be the graph function of* f *given by* $g(x)=(x,f(x))$ *for every point* $x\in X$*. Then* g *is* $\omega\beta-$*continuous if and only if* f *is* $\omega\beta-$*continuous.*

Proof. Assume that g is $\omega\beta-$continuous. Now $f=P_Y\circ g$ where $P_Y:X\times Y\rightarrow Y$, then f is $\omega\beta-$continuous by Corollary 3. Conversely, assume that f is $\omega\beta-$continuous. Let $x\in X$ and

W be any open set in $X \times Y$ containing $g(x)$. Then there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $g(x) = (x, f(x)) \in U \times V \subseteq W$. Since f is $\omega\beta-$continuous, there exists $\omega\beta O(X, \tau)$ sets U_1 in containing x such that $f(U_1) \subseteq V$. Put $H = U \cap U_1$. Then $H \in \omega\beta O(X, \tau)$, by Lemma 1(ii), such that $x \in H$ and $f(H) \subseteq V$. Therefore we have $g(H) \subseteq U \times V \subseteq W$. Thus g is $\omega\beta-$continuous.

Definition 6. *[8] A function $f : (X, \tau) \to (Y, \sigma)$ is called pre-semi-preopen if the image of each semi-preopen set in X is a semi-preopen set in Y.*

Theorem 7. *Let $f : (X, \tau) \to (Y, \sigma)$ be an pre-semi-preopen surjection and let $g : (Y, \sigma) \to (Z, \rho)$ such that $g \circ f : (X, \tau) \to (Z, \rho)$ is $\omega\beta-$containuous, then g is $\omega\beta-$containuous.*

Proof. At first we show if $f : (X, \tau) \to (Y, \sigma)$ be an pre-semi-preopen function and $U \in \omega\beta O(X, \tau)$, then $f(U) \in \omega\beta O(Y, \sigma)$. So let $U \in \omega\beta O(X, \tau)$ then for all $x \in U$ there exists $\beta O(X, \tau)$ sets U_1 in (X, τ) containing x and $U_1 - U \subseteq C$ where C is a countable set. Thus $f(U_1) - f(U) \subseteq f(C)$ where $f(C)$ is a countable set. This implies $f(U) \in \omega\beta O(Y, \sigma)$. Now, Let $y \in Y$ and let $V \in \rho$ with $g(y) \in V$. Choose $x \in X$ such that $f(x) = y$. Since $g \circ f$ is $\omega\beta-$continuous there exists $U \in \omega\beta O(X, \tau)$ with $x \in U$ and $g(f(U)) \subseteq V$. But f is pre-semi-preopen function therefore, by assumption, $f(U) \in \omega\beta O(Y, \sigma)$ with $f(x) \in f(U)$. So we get the result.

Corollary 4. *Let $f_\alpha : (X_\alpha, \tau_\alpha) \to (Y_\alpha, \tau_\alpha)$ be a function for each $\alpha \in \Delta$. If the product function $f = \prod_{\alpha \in \Delta} f_\alpha : \prod_{\alpha \in \Delta} X_\alpha \to \prod_{\alpha \in \Delta} Y_\alpha$ is $\omega\beta-$continuous, then f_α is $\omega\beta-$continuous.*

Proof. At first we prove that any projection function is pre-semi-preopen function. Let $U \in \beta O(X, \tau)$ hence $f(U) \subseteq f(cl(\text{int}(cl(U))))$, by using the assumption that f is open and continuous surjective, $f(U) \subseteq cl(\text{int}(cl(f(U))))$. Thus $f(U) \in \beta O(Y, \sigma)$. Now For each $\beta \in \Delta$, let $p_\beta : \prod_{\alpha \in \Delta} X_\alpha \to X_\beta$ and $q_\beta : \prod_{\alpha \in \Delta} Y_\alpha \to Y_\beta$ be the projections, then we have $q_\beta \circ f = f_\beta \circ p_\beta$ for each $\beta \in \Delta$. Since f is $\omega\beta-$continuous and q_β is continuous, by Proposition 3 $q_\beta \circ f$ is $\omega\beta-$continuous and hence $f_\beta \circ p_\beta$ is $\omega\beta-$continuous function. Since p_β is pre-semi-preopen function it follows from Theorem 7 that f_β is $\omega\beta-$continuous function.

Theorem 8. *[3] For any space X, the following properties are equivalent:*

i. X is $\beta-$Lindelőf.

ii. Every $\omega\beta O(X, \tau)$ cover of X has a countable subcover.

Proposition 4. *Let $f : (X, \tau) \to (Y, \sigma)$ be an $\omega\beta-$continuous surjective function. If X is $\beta-$Lindelőf, then Y is Lindelőf.*

Proof. Let $\{V_\alpha : \alpha \in \Delta\}$ be an open cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is $\omega\beta O(X, \tau)$ cover of X (since f is $\omega\beta-$ continuous). Since X is $\beta-$Lindelőf, by Theorem 8, X has a countable subcover, say $f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \ldots, f^{-1}(V_{\alpha_n}), \ldots$, Thus $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}, \ldots$ is a subcover of $\{V_\alpha : \alpha \in \Delta\}$ of Y. This shows that Y is Lindelőf.

Corollary 5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a $\beta-$continuous (or $\omega-$continuous) surjective function. If X is $\beta-$Lindelőf, then Y is Lindelőf.*

3. $\omega\beta-$Irresolute Functions

Definition 7. *A function $f:(X,\tau)\rightarrow(Y,\sigma)$ is called $\omega\beta-$irresolute if the inverse image of each $\omega\beta O(Y,\sigma)$ set is an $\omega\beta O(X,\tau)$ set.*

Note that every $\omega\beta-$irresolute function is $\omega\beta-$continuous but the converse is not true, which is shown by the following example.

Example 8. *Let $X=\mathbb{R}$ with the topologies $\tau=\tau_{coc}$ and let $Y=\{1,2\}$ with the topology $\sigma=\{\phi,Y,\{2\}\}$. Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be the function defined by*

$$f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 2 & x\in\mathbb{R}-\mathbb{Q}\end{cases}$$

Then f is $\omega\beta-$continuous but not $\omega\beta-$irresolute since $f^{-1}(\{1\})=\mathbb{Q}\notin\omega\beta O(X,\tau)$.

Theorem 9. *Let $f:(X,\tau)\rightarrow(Y,\sigma)$ be a function. Then the following conditions are equivalent:*

i. The function f is $\omega\beta-$irresolute.

ii. For each $x\in X$ and $V\in\omega\beta O(Y,\sigma)$ containing $f(x)$, there exists $U\in\omega\beta O(X,\tau)$ containing x and $f(U)\subseteq V$.

iii. For each $x\in X$, the inverse of every $\omega\beta-$neighbourhood of $f(x)$ is $\omega\beta-$ neighbourhood of x.

iv. For each $x\in X$ and $\omega\beta-$neighbourhood V of $f(x)$, there exists $\omega\beta-$neighbourhood U of x such that $f(U)\subseteq V$.

Proof. (i $\rightarrow$ ii) Assume $x\in X$ and V is $\omega\beta O(Y,\sigma)$ containing $f(x)$, since f is $\omega\beta-$irresolute then $f^{-1}(V)\in\omega\beta O(X,\tau)$ containing x and hence $f(f^{-1}(V))\subseteq V$.
(ii $\rightarrow$ iii) Assume $x\in X$ and V is $\omega\beta-$neighbourhood of $f(x)$, by Definition 3 there exists $V_1\in\omega\beta O(Y,\sigma)$ such that $f(x)\in V_1\subseteq V$, there exists $U\in\omega\beta O(X,\tau)$ containing x and $f(U)\subseteq V_1$, $x\in U\subseteq f^{-1}(V_1)\subseteq f^{-1}(V)$. Hence by use Definition 3, $f^{-1}(V)$ is $\omega\beta-$neighbourhood of x.
(iii $\rightarrow$ iv) Let V is $\omega\beta-$neighbourhood of $f(x)$, by (iii), $f^{-1}(V)$ is $\omega\beta-$neighbourhood of x and $f(f^{-1}(V))\subseteq V$.
(iv $\rightarrow$ i) For each $x\in X$, let $V\in\omega\beta O(Y,\sigma)$ containing $f(x)$. Put $A=f^{-1}(V)$, let $x\in A$. Then $f(x)\in V$. Since $V\in\omega\beta O(Y,\sigma)$ then V is a $\omega\beta-$neighbourhood of $f(x)$. So by hypothesis, $A=f^{-1}(V)$ is $\omega\beta-$neighbourhood of x. Hence by Definition 3 there exists $A_x\in\omega\beta O(X,\tau)$ such that $x\in A_x\subseteq A$. Thus, by Lemma 1(i) $A=\bigcup_{x\in A}A_x$ is $\omega\beta O(X,\tau)$ set. Therefore, f is $\omega\beta-$irresolute.

Theorem 10. *The following conditions are equivalent for a function $f:(X,\tau)\rightarrow(Y,\sigma)$:*

i. f is $\omega\beta-$irresolute.

ii. For each $\omega\beta C(Y,\sigma)$ subset C of Y, $f^{-1}(C)$ is $\omega\beta C(X,\tau)$.

iii. For each subset A of X, $f(\omega\beta cl(A)) \subseteq \omega\beta cl(f(A))$.

Proof. (i → ii) Let C be $\omega\beta C(Y,\sigma)$ subset of Y. Then $X - f^{-1}(C) \in \omega\beta O(X,\tau)$, which implies that $f^{-1}(C)$ is $\omega\beta C(X,\tau)$.
(ii → iii) Let A be a subset of X, Since $A \subset f^{-1}(f(A))$, we have $A \subset f^{-1}(\omega\beta cl(f(A)))$. Now by (ii), $f^{-1}(\omega\beta cl(f(A)))$ is $\omega\beta C(X,\tau)$ set containing A then $\omega\beta cl(A) \subseteq f^{-1}(\omega\beta cl(f(A)))$, which implies $f(\omega\beta cl(A)) \subseteq \omega\beta cl(f(A))$.
(iii → iv) Let $B \subset Y$, by (iii) $f(\omega\beta cl(f^{-1}(B))) \subseteq \omega\beta cl(f(f^{-1}(B))) \subseteq \omega\beta cl(B)$, hence $\omega\beta cl(f^{-1}(B)) \subseteq f^{-1}(\omega\beta cl(B))$.
(iv → i) Suppose f is not $\omega\beta-$irresolute. So there exist $x \in X$ and $V \in \omega\beta O(Y,\sigma)$ with $f(x) \in V$ such that for all $\omega\beta O(X,\tau)$ set U with $x \in U$ and $f(U) \not\subset (V)$ i.e. $f(U) \cap (Y - V) \neq \phi$. Therefore, by (vii), $x \in f^{-1}(\omega\beta cl(Y - V))$. So by Theorem 2, $f(x) \in \omega\beta cl(Y - V)$. Thus for all $\omega\beta O(Y,\sigma)$ sets V containing $f(x)$, so $V \cap (Y - V) \neq \phi$, a contradiction. Therefore, f is $\omega\beta-$irresolute.

Theorem 11. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function. Then f is $\omega\beta-$irresolute if and only if $f^{-1}(\omega\beta Int(B)) \subseteq \omega\beta Int(f^{-1}(B))$.*

Proof. NECESSITY. Let B be any subset of Y. Since f is $\omega\beta-$irrrsolute, we have $f^{-1}(\omega\beta Int(B))$ is $\omega\beta O(X,\tau)$ set. As $f^{-1}(\omega\beta Int(B)) \subseteq f^{-1}(B)$, then $f^{-1}(\omega\beta Int(B)) \subseteq \omega\beta Int(f^{-1}(B))$.
SUFFICIENCY. Let $x \in X$ and $V \in \omega\beta O(Y,\sigma)$ with $f(x) \in V$. Then $x \in f^{-1}(V)$ and so by assumption $x \in \omega\beta Int(f^{-1}(V))$. There exists an $\omega\beta O(X,\tau)$ sets such that $x \in U \subseteq f^{-1}(V)$. Hence $f(x) \in f(U) \subseteq V$ and the result follows.

Proposition 5. *If $f : (X,\tau) \to (Y,\sigma)$ is $\omega\beta-$irresolute and $g : (Y,\sigma) \to (Z,\rho)$ is $\omega\beta-$continuous, then $g \circ f$ is $\omega\beta-$continuous.*

Proof. Let $x \in X$ and let V be any open set in (Z,ρ) containing $g(f(x))$. Since g is $\omega\beta-$continuous, there exists an $\omega\beta O(Y,\sigma)$ set W containing $f(x)$ such that $g(W) \subseteq V$. Put $U = f^{-1}(W)$ since f is $\omega\beta-$irresolute, then $U \in \omega\beta O(X,\tau)$ such that $x \in U$ and $g(f(U)) \subseteq g(W) \subseteq V$. Hence $g \circ f$ is $\omega\beta-$continuous.

Corollary 6. *If $f : (X,\tau) \to (Y,\sigma)$ is $\omega\beta-$irresolute and $g : (Y,\sigma) \to (Z,\rho)$ is $\omega b-$continuous, then $g \circ f$ is $\omega\beta-$continuous.*

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is said to be $\omega-$irresolute [1] if the inverse image of each $\omega O(Y,\sigma)$ set is an $\omega O(X,\tau)$.

Theorem 12. *If $f : (X,\tau) \to (Y,\sigma)$ is $\omega-$irresolute and every $\beta O(Y,\sigma)$ set is closed in the space (Y,σ) then f is $\omega\beta-$irresolute.*

Proof. Let U be any $\omega\beta O(Y,\sigma)$ set, then for all $y \in Y$, there exists $\beta O(Y,\sigma)$ sets U_1 containing x such that $U_1 - U$ is a countable, thus by assumption
$U_1 \subseteq cl(Int(cl(U_1))) \subseteq Int(U_1)$, so U_1 is open sets in (Y,σ), hence $U \in \omega O(Y,\sigma)$. Since f is $\omega-$irresolute, then $f^{-1}(U) \in \omega O(X,\tau) \subseteq \omega\beta O(X,\tau)$.

Proposition 6. *Let $f : (X,\tau) \to (Y,\sigma)$ be an open continuous function and every $\omega\beta O(Y,\sigma)$ is closed in the space (Y,σ) then f is $\omega\beta-$irresolute.*

Proof. Let $U \in \omega\beta O(Y,\sigma)$, by Theorem 3,
$\omega\beta cl(f^{-1}(U)) \subseteq cl(f^{-1}(U)) = f^{-1}(cl(U)) \subseteq f^{-1}(\omega\beta cl(U))$, hence f is $\omega\beta-$irresolute, by Theorem 10.

In [3], Aljarrah And Noorani define the $\omega\beta - T_2$ as if for each two distinct point $x, y \in X$, there exists $U, V \in \omega\beta O(X,\tau)$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Theorem 13. *If $f : (X,\tau) \to (Y,\sigma)$ is an $\omega\beta-$irresolute injective function and the space Y is $\omega\beta - T_2$, then X is $\omega\beta - T_2$.*

Proof. Let x_1 and x_2 be two distinct points of X. Since f is injective and Y is $\omega\beta - T_2$, there exist $V_1, V_2 \in \omega\beta O(Y,\sigma)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \phi$. Now $x_1 \in f^{-1}(V_1)$, $x_2 \in f^{-1}(V_2)$ and $f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Since f is $\omega\beta-$irresolute then $f^{-1}(V_1)$, $f^{-1}(V_2)$ is $\omega\beta O(X,\tau)$. Hence X is $\omega\beta - T_2$.

Definition 8. *A space X is said to be $\omega\beta-$connected if there exist disjoint $\omega\beta O(X,\tau)$ sets A and B such that $A \cup B = X$.*

Proposition 7. *If $f : (X,\tau) \to (Y,\sigma)$ is an $\omega\beta-$irresolute surjective function and X is $\omega\beta-$connected, then Y is $\omega\beta-$connected.*

Proof. Suppose Y is not $\omega\beta-$connected. Then there exist disjoint $\omega\beta O(Y,\sigma)$ sets A and B such that $A \cup B = Y$. Since f is $\omega\beta-$irresolute surjective, $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty $\omega\beta O(X,\tau)$ sets. Moreover $f^{-1}(A) \cup f^{-1}(B) = X$. This is show that (X,τ) is not $\omega\beta-$connected, which is a contradiction. Hence (Y,σ) is $\omega\beta-$connected.

4. $\omega\beta-$Open and $\omega\beta-$Closed Functions

Definition 9. *A function $f : (X,\tau) \to (Y,\sigma)$ is called $\omega\beta-$open (resp. $\omega\beta-$closed) if the image of each open (resp. closed) set in (X,τ) is an $\omega\beta O(Y,\sigma)$ (resp. $\omega\beta C(Y,\sigma)$).*

Note that every open (closed) function is $\omega\beta-$open (resp. $\omega\beta-$closed) function, but the converse is not true, which is shown by the following example.

Example 9. *Let $X = \{a, b\}$ with the topology $\tau = \{\phi, X, \{a\}\}$ and $Y = \{1, 2, 3\}$ with the topology $\sigma = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function define by $f(x) = 3$ for all $x \in X$. Then f is $\omega\beta-$open and $\omega\beta-$closed function, but it is neither open nor closed function.*

Proposition 8. *A function $f : (X,\tau) \to (Y,\sigma)$ is $\omega\beta-$open if and only if for each $x \in X$ and each open set U of X containing x, there exists an $\omega\beta O(Y,\sigma)$ set W containing $f(x)$ such that $W \subset f(U)$.*

Theorem 14. *Let $f : (X,\tau) \to (Y,\sigma)$ be a function from space (X,τ) into a space (Y,σ). Then f is $\omega\beta-$closed if and only if $\omega\beta cl(f(A)) \subseteq f(\omega\beta cl(A))$ for each set A subset of (X,τ).*

Proof. Let f is $\omega\beta-$closed function and A any subset of X. Then $f(A) \subset f(\omega\beta cl(A)) \in \omega\beta C(Y,\sigma)$, it follows that $\omega\beta cl(f(A)) \subset f(\omega\beta cl(A))$. Conversely, assume that $B \in \omega\beta C(X,\tau)$. Then $\omega\beta cl(f(B)) \subset f(\omega\beta cl(B)) = f(B)$. Thus we obtain that $\omega\beta cl(f(B)) = f(B)$, so f is $\omega\beta-$closed function.

Proposition 9. *Let $f : (X,\tau) \to (Y,\sigma)$ be a continuous surjection function and let $g : (Y,\sigma) \to (Z,\rho)$ be such that $g \circ f : (X,\tau) \to (Z,\rho)$ is $\omega\beta-$open function, then g is $\omega\beta-$open.*

Proof. Let $y \in Y$ and let $V \in \rho$ with $g(y) \in V$. Choose $x \in X$ such that $f(x) = y$. Since $g \circ f$ is $\omega\beta-$open function, then $g(V) = g \circ f(f^{-1}(V)) \in \omega\beta O(Z,\rho)$. This is show that g is $\omega\beta-$open function.

The following examples show that the $\omega\beta-$open function is independent with $\omega\beta-$irresolute and $\omega\beta-$continuous function.

Example 10. *Let $X = \mathbb{R}$ with the topologies $\tau = \tau_{coc}$ and let $Y = \{1,2\}$ with the topology $\rho = \{\phi, Y, \{2\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} - \mathbb{Q} \\ 2 & x \in \mathbb{Q} \end{cases}$$

Then f is not $\omega\beta-$continuous, but it can easily seen that $f(x)$ is $\omega\beta-$open function.

Example 11. *Let $X = \{1,2\}$ with the topology $\tau = \{\phi, X, \{1\}\}$ and let $Y = \mathbb{R}$ with the topologies $\sigma = \tau_{coc}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the function defined by*

$$f(x) = \begin{cases} \mathbb{R} - \mathbb{Q} & x = 2 \\ \mathbb{Q} & x = 1 \end{cases}$$

Then f is not $\omega\beta-$open, but it can easily seen that f is $\omega\beta-$continuous and $\omega\beta-$irresolute function.

Example 12. *Consider the function f in the Example 8 which is $\omega\beta-$open, but not $\omega\beta-$irresolute.*

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