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Mathematical Modeling of Condensing on Metric Spaces

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Abstract. This work aims to present an energy based model of the collective dynamics where local interactions between particles of a collection causes all particles to reorganize in new positions. The self-organizing phenomenon involved by singular local moves of individual particles leads to condensing. This model is analyzed on metric spaces, simulated on a finite and in a continuous Euclidean subsets.

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1. Introduction

In real live condensing of populations shows a very interesting emergence phenomena. Living in a population obligate all members to adapt their moves to the collective dynamics of all Individuals [1]. This group dynamics is one of the challenges to study emergence, for instance flocking of flocks and swarming of fishes [3–5, 8, 9]. Another interesting natural and complex example is the collective Motion of Ants. They live in large populations and show a complicated and strict division of labor for the individual ant, which on the one hand is not determined by the genetic structure of the single ant and on the other hand makes the whole population react effectively to all kinds of events as if steered by some clever and experienced brain, which however does not exist. The division of labor, which makes an ant a forager, and another might be a patroller is called emergent, see for instance [2, 10]. It is very strictly and very stable, but one does not detect it as a program in the individual. How is this to be understood? This is the challenge of emergence as I see it and we will briefly discuss in how far our model models emergence. There is an emergent pattern: segregation into isolated and ε-distanced positions. However, the number of positions and the distribution of individuals onto there positions seem to be random.

An early modeling of populations are subject for example of the the Price equation [13, 14]. This describes characteristics of a population model that clumps individuals by shared

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property. That is, individuals with the same height are modeled as having the same fitness and thus each have the same number of off- spring. So in these population models, once many distinct groups collide into one similar group, they do not break apart ever again. Thus, these population models share some characteristics with our work. Unfortunately, conventional population dynamics tend to show consensus (where the consensus property might be a polymorphic mix) or limit-cycle behavior. Isolated equilibria are usually not explicit results of conventional population models. Where again, polymorphism is represented in other ways in these cases, for more details see for instance [6, 7].

Modeling of diffusion phenomena are widely studied and analyzed using several mathematical and physical techniques, for instance using partial differential equations or particle methods. The inverse problem, which is condensing, is till now an open and a very interesting case. This mixture with the collective dynamics provide many mathematical open questions. For example consensus or total condensing of opinions in social aggregation [11, 12] in the context of the opinion dynamics, is one of these unsolved problems. The Agents move simultaneously to the barycenter of all Agents in an ε neighborhood, see also [5]. The final state of this time dependent model may be consensus if all Agents meet at the same position or grouping in several ε distanced classes of Agents such that all Agents in the same class maintain the same position. In this work, we are interested to extend the barycenter dynamics presented for example by [5, 12] to an energy-based model. Observe that the barycenter of a positive measure m locally minimizes the ε-energy of a spatial position x:

$$e_\varepsilon(x,m)=\int_{d(x,y)\leq\varepsilon} d^2(x,y)m(dy), \tag{1}$$

where for the barycenter dynamics, $d(\cdot,\cdot)$ is an Euclidean distance. This observation is the starting point for our present study to generalize the barycenter dynamics. We replace the Euclidean space by an arbitrary metric space, and let the Agents move to where the local energy is minimal within an ε neighborhood. Moreover, note that the second claim does not provide the synchronized barycenter dynamics, as it is already demonstrated by two Agents and Euclidean metric: Two Agents may decrease the energy to zero by jumping either to the same place, or to different places if the distance exceeds ε. Since the energy minimizing points are in general not unique on metric spaces, it is important to note that our dynamics, because of the second claim, is not a deterministic one. Furthermore, the convergence of the process of condensing sequences is not guaranteed. This fact can be seen in the case of two Agents, they may exchange their position forever, with periodic local energy. Therefore, in order to prove the convergence, let us consider that the Agents do not move simultaneously but one at a time in an arbitrary order. By doing so, they decrease the global ε-energy:

$$E_\varepsilon(m)=\int_X e_\varepsilon(x,m)m(dx)=\int_X\int_{d(x,y)\leq\varepsilon} d^2(x,y)m(dy)m(dx), \tag{2}$$

which guaranties the convergence and in fact zero energy after finitely many steps. It is also important to note that the arbitrary order of action of different Agents and the non-uniqueness of the positions minimizing the local energy introduce sources of indeterminacy.

Such indeterminacy gives the opportunity for stochastic investigations, which however are not part of the present study. Our concern in this paper is the introduction of a new class of dynamical systems together with some elementary analysis and a number of numerical simulations. It is important to note that, the arbitrary range of the reactions of the particles and the non uniqueness of the positions minimizing the local energy give a source of stochastic investigations, which we analyze in a future work.

In this work, we are interested to extend the barycenter model to an energy based model, to perform theoretical analysis for this model such as convergence theorems and to numerically validate our results in finite and continuous metric spaces. This paper is structured in two principal sections. The first one proposes the construction of condensing sequences. The second section proposes a numerical simulation of such a phenomena. Some general remarks are then listed.

2. Discrete Energy Function

Let X be a finite set of $\mathscr{R}^n$. A non-negative measure m on X is represented by a function $m : X \mapsto [0, \infty)$ and we denote by $M_+(X)$ the set of all positive measures on X with discrete support. A measure $m \in M_+(X)$ is given as

$$m = \sum_{x \in X} m(x)\delta_x = \sum_{x \in S(m)} m(x)\delta_x \tag{3}$$

where δ_x denotes the Kronecker symbol and by $S(m)$ we denote the support of m given as $S(m) := \{y \in X | m(y) > 0\}$.

Definition 1. *Let a pair $(a, a^*) \in X \times X$ operates on the set $M_+(X)$ as following*

$$m \mapsto m^* = (a, a^*, m) \tag{4}$$

$$m^*(x) := \begin{cases} m(x); \text{if } x \notin \{a, a^*\}, \\ 0; & \text{if } x = a, \\ m(a) + m(a^*); & \text{if } x = a^*. \end{cases}$$

The mapping above is a mass translating map. Where, the move of a to a^* means that the mass point of a^* will be adjusted by a new mass, namely the mass of a. The fussing operator (4) has different interpretations, namely if the Agents a with identity like a bird or a fish, so the moving action represents for example the grouping phenomena either by forming a massive group or several ε distanced subgroups. But if the considered Agent is identicalness the fusing characteristic explains the physical fusion of masses.

One of our main concerns is to define and to analyze the local and the global energies already mentioned by equations (1) and (2) with respect to a positive discrete measure.

Definition 2. *For a fixed real number $\varepsilon > 0$ and a given n points metric metric space, the ε-energy of mass point $a \in X$ with respect to a positive measure m is*

$$e_\varepsilon(a, m) = \sum_{d(a,y) \leq \varepsilon} m(y) d^2(a, y), \tag{5}$$

and the global ε-energy of a positive measure m in $M_+(X)$ is given as:

$$E_\varepsilon(m) = \sum_{d(x,y)\leq\varepsilon} m(x)m(y)d^2(x,y). \tag{6}$$

To observe the energy evolution of both the local and the global energies according to the moves presented in Definition 1, we propose the following example:

Example 1. *We consider a three points metric space as subset of the real line. We assume that two neighbors points have a distance of one and we define a measure m by the masses punted in the three points. We denote the mass of m in each point by the the numbers given on the Figures 1. For $\varepsilon = 1$ let us consider three moves, namely (a), (b) and (c).*

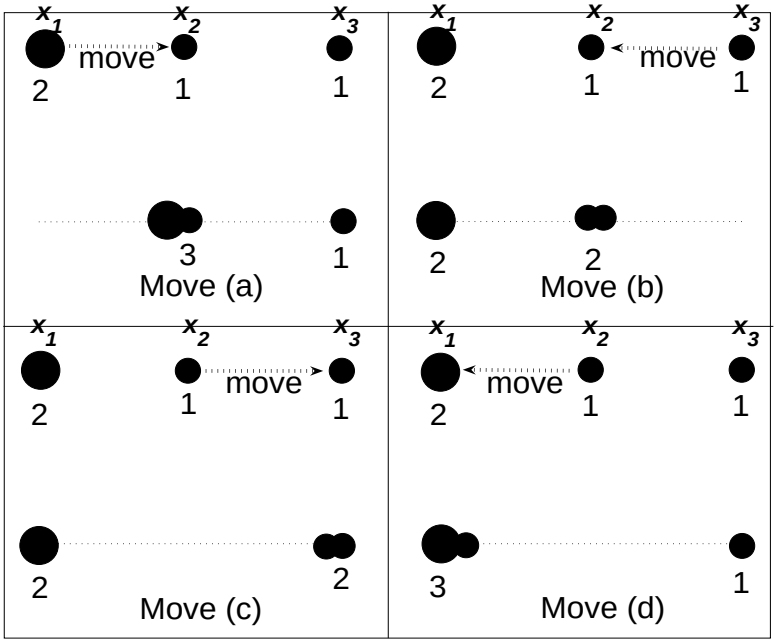


Figure 1: Examples of moves

Show that the resulting masses are given as

$$\begin{aligned}
\textit{Initial mass distribution:} &\quad m = 2\delta_1 + \delta_{x_2} + \delta_{x_3} \\
\textit{Move (a):} &\quad m \longrightarrow m_a^* = 3\delta_{x_2} + \delta_{x_3}. \\
\textit{Move (b):} &\quad m \longrightarrow m_b^* = 2\delta_{x_1} + 2\delta_{x_2}. \\
\textit{Move (c):} &\quad m \longrightarrow m_c^* = 2\delta_{x_1} + 2\delta_{x_3}. \\
\textit{Move (d):} &\quad m \longrightarrow m_d^* = 3\delta_{x_1} + \delta_{x_3}.
\end{aligned}$$

Figure 1 presents three example of moves:

(a) The mass 2 *in* x_1 *moves to the mass* 1 *on the middle (i.e. posted at* x_2*). i.e.* $m \to m_a^*$ *with conserving global energy* $E(m) = E(m_a^*) = 6$*. Consequently, the total energy does not change* $E(m) = E(m_a^*) = 6$*.*

(b) The mass 1 *(in* x_3*) on the left moves to the mass one on the middle (i.e.* x_2*), hence,* $m \to m_b^*$ *with increasing global energy* $E(m) = 6 < E(m_b^*) = 8$*. Consequently, the total energy increases* $E(m_b^*) = 8$*:*

(c) The mass 1 *in* x_2 *in the middle moves to the mass* 1 *to the right one. Which causes a vanishing of the total energy:* $m \to m_c^*$ *with decreasing global energyv*$E(m) = 6 > E(m_c^*) = 0$*.*

(d) The mass 1 *in* x_2 *in the middle moves to the mass* 1 *to the left one.* $m \to m_d^*$ *with decreasing global energy* $E(m) = 6 > E(m_d^*) = 0$*.*

In our model we are looking for a local rule for a move of particles, which causes an decreasing of the global energy.

Definition 3. *A pair of masses* $(m, m^*) \in M_+(X) \times M_+(X)$ *is called an* ε*-move, if there is a pair of mass points* $(a, a^*) \in X \times X$ *such that:*

(i) $m^* = (a, a^*, m)$*,*

(ii) $d(a, a^*) \leq \varepsilon$*, (Neighborhood condition)*

(iii) $e_\varepsilon(a^*, m^*) < e_\varepsilon(a, m)$*. (Energy-Minimizing condition)*

Figure 2 presents an illustration of an admissible move based on the energy, where the particle moves to a new position with only one neighbors:

Definition 4. *If every pair* (m^i, m^{i+1}) *of a nonnegative measures is an* ε*-move according to Definition 3, then the sequence* $(m^i)_{i>0} \subset M_+(X)$ *will be called* ε*-condensing.*

Clearly for every $a, a^* \in S(m)$ if $d(a, a^*) \leq \varepsilon$, then either (a, a^*, m) or (a^*, a, m) is an ε-move. Therefore, whenever $E_\varepsilon(m) > 0$ there is an ε-move (m, m^*). Thus, for every finite m with nonvanishing energy, there is an ε-condensing sequence $m^1, m^2, \ldots$. Our theorem says that such a sequence is finite.

Remark 1. *Note that the resulting measure of a condensing sequence depends not only on the initial measure, but also on the reactions order of the particles. Hence, we introduce a random range for ordering of particle reactions. This gives an interrelating source of stochastic investigations, which are not subject of our present paper. Moreover, the simultaneous displacement sequences are studied in another context in literature by using synchronous communication, moves and reactions, for example, we refer to the models studied in [5, 11].*

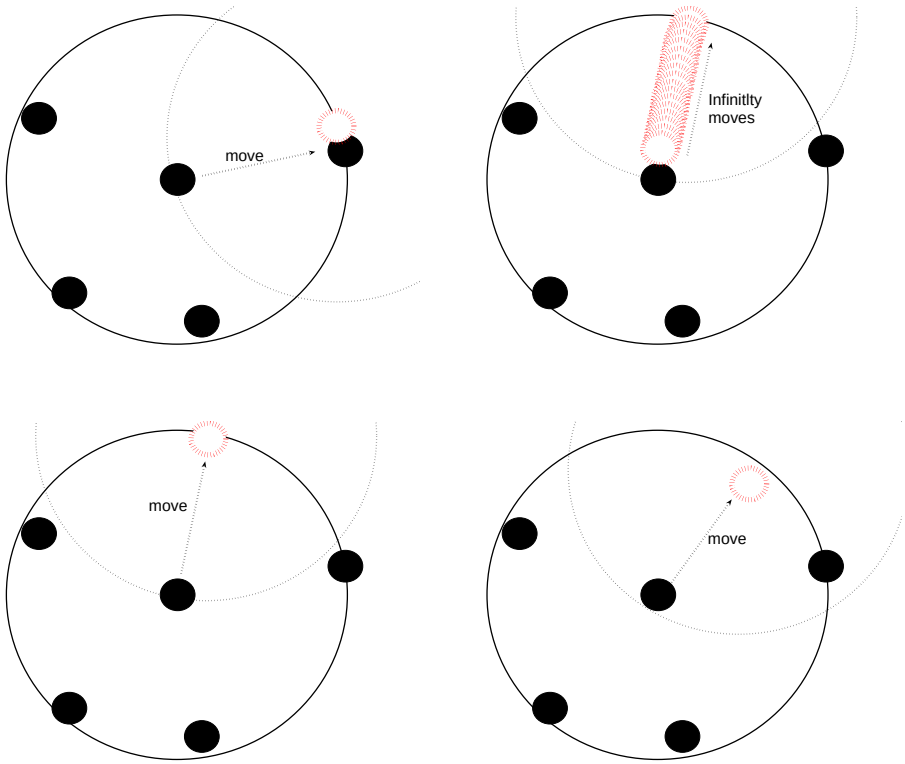


Figure 2: Example of four energy based admissible moves (from the left to the right) (a), (b), (c) and (d). The dotted circles represent the destination of the moving particle.

Lemma 1. *Let $m \in M_+(X), a, a^* \in X$ such that $d(a,a^*) \leq \varepsilon$. Then*

$$E_\varepsilon(m) - E_\varepsilon(m^*) = 2m(a)\big[e_\varepsilon(a,m) - e_\varepsilon(a^*,m) + m(a)d^2(a,a^*)\big], \tag{7}$$

Proof. To simplify, we use the following notation

$$I_m := \sum_{d(x,y)\leq\varepsilon;\{x,y\}\cap\{a,a^*\}=\emptyset} m(x)m(y)d(x,y)^2 \tag{8}$$

By computing the energy of m we get

$$\begin{aligned} E_\varepsilon(m) =& \sum_{d(x,y)\leq\varepsilon} m(x)m(y)d^2(x,y) \\ =& I_m + 2m(a)\sum_{d(a,x)\leq\varepsilon} m(y)d^2(a,y) + 2m(a^*)\sum_{d(a^*,y)\leq\varepsilon} m(y)d^2(a^*,y) \\ & - 2m(a)m(a^*)d^2(a,a^*) \\ =& I_m + 2m(a)e_\varepsilon(a,m) + 2m(a^*)e_\varepsilon(a^*,m) - 2m(a)m(a^*)d^2(a,a^*). \end{aligned} \tag{9}$$

Similarly for $m^* = (a, a^*, m)$, we have

$$E_\varepsilon(m^*) = I_{m^*} + 2m^*(a)e_\varepsilon(a, m^*) + 2m^*(a^*)e_\varepsilon(a^*, m^*) - 2m^*(a)m^*(a^*)d^2(a, a^*)$$

Note that $I_m = I_{m^*}$; $m^*(a) = 0$; $m^*(a^*) = m(a) + m(a^*)$ and

$$e(a^*, m^*) = e(a^*, m) - m(a)d^2(a^*, a). \tag{10}$$

Therefore

$$\begin{aligned} E_\varepsilon(m^*) =& I_m + 2(m(a) + m(a^*))e_\varepsilon(a^*, m^*) \\ =& I_m + 2\big(m(a) + m(a^*)\big)\big(e_\varepsilon(a^*, m) - m(a)d^2(a^*, a)\big), \end{aligned} \tag{11}$$

and from (9) and (11) it follows the result of the lemma. □

Lemma 2. *For $m \in M_+(X)$ let $n(m)$ be the number of elements $a \in X$ such that $m(a) > 0$. (i.e. $n(m) = |S(m)|$. If $m^1, m^2, \ldots$ is a sequence of measures on X which is singular and $\varepsilon-$condensing, then*

(i) $i \to E_\varepsilon(m^i)$ is strictly decreasing and

(ii) $i \to n(m^i)$ is non-increasing.

Proof. The first claim follows from Lemma 1. To show the second let $S(m)$ be the support of the measure m. Consider $m^* = (a, a^*, m)$. If $a \notin S(m)$ then $S(m) = S(m^*)$ and $n(m) = n(m^*)$. If $a \in S(m)$ and $a^* \in S(m)$ then $S(m^*) = S(m) \setminus \{a\}$ and $n(m^*) < n(m)$. If $a \in S(m)$, $a^* \notin S(m)$ then $S(m^*) = (S(m) \setminus \{a\}) \cup \{a^*\}$, and again $n(m) = n(m^*)$. □

Theorem 1. *Let us consider (X, d) a finite metric space. Every singularly ε-condensing sequence of $M_+(X)$ is finite.*

Proof. Let $m^1, m^2, \ldots$ be an infinite sequence of measures, which is $\varepsilon-$condensing. Because of the preceding lemma, we may assume that $i \to n(m^i)$ is constant. Hence, for every i The measure m^{i+1} is a permutation of m^i i.e. $m^{i+1} = m^i \circ \pi_i$, where $\pi_i : X \to X$ is a permutation of X. Therefore,

$$m^i = m^1 \circ \pi_1 \circ \ldots \circ \pi_{i-1}.$$

As the group of permutations of X is finite, there exist a natural numbers $i, k > 0$ such that

$$\pi^1 \circ \ldots \pi^i = \pi^1 \circ \ldots \circ \pi^{i+k},$$

and $m^{i+1} = m^{i+k+1}$, which however is impossible in view of Lemma 2. □

Remark 2. *There exist infinite non-converging condensing sequences: Consider a simultaneously condensing sequence with two mass points $m_n = m_s$, where m_n is the mass of a point in the north pole of unit circle and m_s is a mass of a point in the south pole. Note that, this metric space is not a finite metric space but to explain this example in a finite metric space, one can use only*

four points metric spaces, namely the north, the south pole and the midpoints of them on the unit circle. Here, m is given as

$$m := m_s \delta_{\frac{3\pi}{2}} + m_n \delta_{\frac{\pi}{2}}.$$

If we consider the rule of simultaneously moves (HK-model) studied by [11, 12]. An admissible moves scenario is the periodic one, namely $\frac{3\pi}{2}$ *moves to* 0 *and* $\frac{\pi}{2}$ *moves to* π*. The condensing sequence constructed above* $m^1, m^2, \ldots$ *is simultaneously condensing and does not converge. We can also construct another type of non converging condensing sequences. We believe that, in this case, non converging sequences have a periodic behavior. In the case of the existing of many positions minimizing the energy, the particle moves to one of them.*

3. Continuous Energy Function

In this section, we shall assume that all bounded subsets of the metric space (X, d) are compact. Hence, X is locally compact and we may use Radon measures. Let $M_+(\mathscr{R}^n)$ be the set of nonnegative Radon measures on X. We shall however for simplicity deal with discrete measures and discrete time only. A measure is given as

$$m := \sum_{x \in S(m)} m(x)\delta_x,$$

where $S(m)$ denote the support of m and δ_x the Kronecker symbol. Note that $S(m)$ is discrete. For such a measure, the energy map E, defined by (6) is in general a not continuous function of m. The following example illustrates this:

For $X = \mathbb{R}$, $S(m) = \{1, 2\}$, $\varepsilon = 1$ and a given m with

$$m = \sum_{x \in \{1,2\}} m(x)\delta_x = \delta_1 + \delta_2,$$

it follows that $E(m) = 2$. Now let m_j be a sequence of positive measures defined as

$$m_j = \delta_1 + \delta_{2+\frac{1}{j}},$$

it follows

$$\lim_j m_j = m, \text{ and } E(m_j) = 0,$$

and

$$E(\lim_j m_j) = 2 \neq \lim_j E(m_j) = 0.$$

Hence, the map E with the definition (6) is not continuous in m.

In order to obtain an energy function which depends continuously on m, we extend the definition (6) to the following:

$$E(m) = \sum_{x,y} m(x)m(y)\varphi(x, y)d^2(x, y), \tag{12}$$

where the mapping φ given as

$$\varphi : X \times X \longrightarrow [0,1];\ \varphi(x,y) := \begin{cases} 1, & \text{if } d(x,y) \leq \varepsilon, \\ 0, & \text{if } d(x,y) \geq \varepsilon + \theta. \end{cases} \tag{13}$$

is a continuous function for $\varepsilon > 0$ and $\theta > 0$. These parameters will be fixed throughout this paper. The function φ will be called intensity function.

Let $M_E(X)$ be the set of discrete and nonnegative measure with $E(m) < \infty$. A pair $(a, a^*) \in X \times X$ operates on $M_E(X)$ as finite the case such that $m \mapsto m^*(x) = (a, a^*, m)$. Note that if $m \in M_E(X)$ then $m^* \in M_E(X)$. The energy of a point $a \in X$ with respect to $m \in M_E(X)$ is a map e defined as

$$e : X \times M_E(X) \longrightarrow \mathbb{R}^+;\ e(a,m) = \sum_y m(y)\varphi(a,y)N^2(a-y). \tag{14}$$

Lemma 3. *For $a, a^* \in X$, $m \in M_E(X)$ and $m^* := (a, a^*, m)$, we have*

$$E(m) - E(m^*) = 2m(a)\big[e(a,m) - e(a^*,m^*)\big]. \tag{15}$$

Proof. For simplicity, let us denote by I_m the following term:

$$I_m := \sum_{\{x,y\}\cap\{a,a^*\}=\emptyset} m(x)m(y)\varphi(x,y)d^2(x,y),$$

and easily we see that the energy of m can be written as

$$\begin{aligned} E(m) =& I_m + 2m(a)e(a,m) + 2m(a^*)e(a^*,m) \\ & - 2m(a^*)m(a)\varphi(a^*,a)N^2(a^*-a). \end{aligned} \tag{16}$$

Similarly for m^* (by replacing m by m^*), we get

$$\begin{aligned} E(m^*) =& I_{m^*} + 2m^*(a)e(a,m^*) + 2m(a^*)e(a^*,m^*) \\ & - 2m^*(a^*)m^*(a)\varphi(a^*,a)d^2(a^*,a). \end{aligned} \tag{17}$$

Note that $m^*(a) = 0$ and $I_m = I_{m^*}$. In addition we have:

$$e(a^*,m^*) = e(a^*,m) - m(a)\varphi(a^*,a)N^2(a^*-a). \tag{18}$$

Therefore, from (17)–(18) it follows: $E(m^*) = I_{m^*} + 2m^*(a)e(a,m^*)$. From (16) and the last equality, we get (15), which prove the result of the lemma. □

Definition 5 (Moves on continuous metric spaces)**.** *A pair (m, m^*) is called condensing, if there is $(a, a^*) \in S(m) \times X$ such that*

(i) $d(a, a^*) \leq \varepsilon + \theta$,

(ii) $m^* = (a, a^*, m)$ *and*

(iii) $e(a^*, m^*)\varphi(a, a^*) < e(y, m^*)\varphi(a, y)$, $d(a, y) \leq \varepsilon + \theta$, $\forall y$.

A sequence of nonnegative measures $m^1, m^2, \ldots$ is called condensing, if for every i the pair (m^i, m^{i+1}) is condensing.

It is important to note that a singular move according to Definition 5 satisfies the minimality condition: If a moves to a^* then $e(a, m) > e(a^*, m^*)$. In other words the particle a moves where the local energy smaller, in this case it moves where the energy is minimal. Our goal is to construct a condensing sequence, with vanishing energy (stable state) at the limit state. The following corollary is useful:

Corollary 1. *If $(m^i)_{i\geq 0}$ is a singularly condensing sequence, then*

$$\lim_{i\to\infty} E(m^i) = \ell \geq 0.$$

Proof. The proof follows immediately from Definition 5 and Lemma 3. □

Our main concern in the following is to prove the existence of condensing sequences converging to a measure m, such that $E(m) = 0$. Especially to define special cases of condensing sequences, which converges in finite time steps. Therefore, we define the effectively condensing sequences:

A singularly condensing sequence $m^1, m^2, \ldots$ is called effectively condensing sequence, if there exists $c > 0$ such that

(i) $m^{i+1} = (a, a^*, m^i)$ and

(ii) $E(m^i) - E(m^{i+1}) \geq c\alpha(m^i)$,

where

$$\alpha(m^i) = \max_y \{\varphi(y, a)d^2(y, a) | y \in S(m^i)\}.$$

Remark 3. *Note that by setting*

$$c = 2\min_a \{m(a) | a \in S(m^i)\},$$

and using the results above, it follows the existence of the effectively condensing sequence.

Lemma 4. *Suppose that $(m^i)_{i\geq 0}$ is an effectively condensing.*

$$\text{if } \lim_{i\to\infty} E(m^i) = E(m), \text{ then } \alpha(m) = 0. \tag{19}$$

Proof. Consider $\eta > 0$, from

$$\lim_{i\to\infty} E(m^i) = E(m),$$

exists i_0 such that for all $i > i_0$, it follows

$$2c\alpha(m^i) \leq E(m^i) - E(m) \leq \eta, \tag{20}$$

for large i, we get $\lim_i \alpha(m^i) = 0$ and still $\alpha(m) = 0$. □

Theorem 2. *Every effectively condensing sequence of masses converges.*

Proof. Since X is compact, then $\cup_i S(m^i)$ is relative compact and there exists a subsequence m^{i_j} of m^i such that $\lim_j m^{i_j} = m$ and $\lim_j E(m^{i_j}) = E(m)$. From Lemma 4 it follows that $\lim_j \alpha(m^{i_j}) = 0$ and $\alpha(m) = 0$. Since the sequence $m^1, m^2, \ldots$ is effectively condensing, and from Definition 5 there exists k such that $\alpha(m^k) = 0$. Therefore, for all $x, y \in S(m^k)$, it follows

$$d(x,y) = 0 \text{ or } d(x,y) \geq \varepsilon + \theta \tag{21}$$

Hence, $E(m^k) = 0$ and still m^k is a collection of isolated masses with propriety (21) or a point mass $m = m(X)\delta_a$ for $a \in X$. □

Remark 4.

- *We have remarked that even if the energy of the limit measure vanishes, the results are non necessary a singleton (total condensing). The limit measure is a collection of segregated (separated) subgroups or singleton mass point. This is justified by the condition* (21)*. Total condensing of particles as physical phenomenon is subject of several studies of many scientists such as consensus dynamics of opinions. For more details see the model proposed by Hegselmann and Krause in [11].*
- *It is important to note that if $S(m) = \mathbb{Q} \cap [0,1]$ (i.e., $S(m)$ is the rational numbers on the interval $[0,1]$). Because of the assumption of one-by-one asynchronous interleaving, a single iteration of the condensing algorithm will not finish in finite time see for instance the second illustration in Figure 2. Moreover, finite-time convergence is impossible, even if the energy vanish after infinite steps So it seems like finite-time convergence is impossible while $S(m)$ is allowed to be countably infinite. By considering the following energy function:*

$$\lim_{n\to\infty} E(m^n) = \lim_{n\to\infty} \frac{1}{n} = 0, \tag{22}$$

there is no guarantee of the existence of the mass convergence.

4. Numerical Simulations

4.1. Finite Metric Space

In our simulations, we do three numerical experiments on finite metric spaces (FMS) as a subset of an Euclidean space, namely $[0,1]^2$. The finite set will be constructed as 121 points metric space a subset of a continuous metric space. The numerical simulations are listed as follows: (a) and (b) Uniform mass distribution,(c) Uniform random mass distribution in $[0,4]$ (i.e. $m(x) \in [0,4]$ for x element of the FMS subset of $[0,1]^2$). The metric used here is the Euclidean one. For simplicity, the initial measure will be defined as a positive measure $m := \sum_{x\in X} m(x)\delta_x$, such that $S(m) = X$ and $m(x) > 0$. We run our code by using an arbitrary

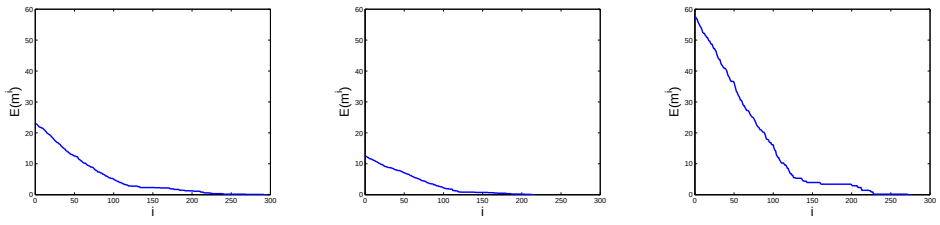
order of reactions (the array of 121 index will be permuted randomly at each iteration step). It is important to note that the positions, which minimize the energy are not unique, therefore, we choose randomly one of them. Moreover all points of the metric space are considered, namely with positive or zero mass.

Our main concern here is to observe the condensing behavior of the limit state of each simulation. Hence, if m is a limit measure of a condensing sequence, then $E_\varepsilon(m) = 0$, is equivalent either to $m(X) = m(a)$ for $a \in X$ or $d(x, y) > \varepsilon$ for all $x, y \in S(m)$. Note also both cases despond not only on the choice of ε but also of the random of the reactions and the non-uniqueness of the points minimizing of the energy function. It is also important to note that if $\varepsilon \geq diam(X)$, then $\lim_i m^i = m(X)\delta_a$ for $a \in X$. In this case we have a total collision of the particles. These limits are reached by vanishing global energy. Figure 3 shows clearly that the time despondent global energies converge to zero. It is important to note, that the limits are attained after different number of iteration as indicated by Table 1, which summarizes the results of the three simulations on the Euclidean finite metric space:

Figure 4 presents three condensing iterations in X of the three simulations (left, middle and right columns). The small dark dots represent the metric space and the large ones represent the particles. The initial measure is a collection of point masses such that each point of the grid has a positive mass. A move is only admissible on the small points (FMS). In this case the limit measure is a collection of ε-isolated mass points. It is also important to note that this plot shows in the first four rows, only the center of mass of each point mass, the weight is given as a density in the last plot of Figure 4.

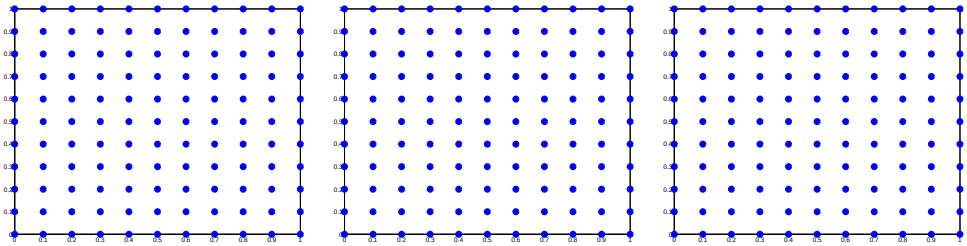
Table 1: Results of simulations (a), (b) and (c).

Parameter/Sim.	(a)	(b)	(c)
NP	121	121	121
ε	0.19	0.19	0.19
Initial state	121 masses (one)	121 masses (one)	121 masses (in U(0,4))
Final state	21 isolated masses	27 iso. masses	21 iso. masses
Number of iterations	290	273	215

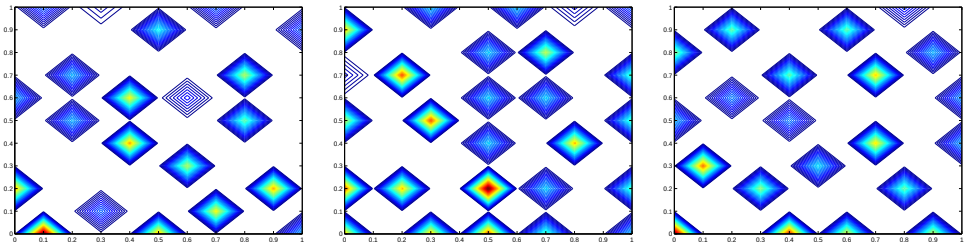


(a) (b) (c)

Figure 3: Energy functions of simulation (a), (b), and (c).



(a) Initial density (b) Initial density (c) Initial density



(a) Final density (b) Final density (c) Final density

Figure 4: Condensing in an Euclidian finite metric space of simulations (a), (b), and (c).

4.2. Condensing on the Real Line

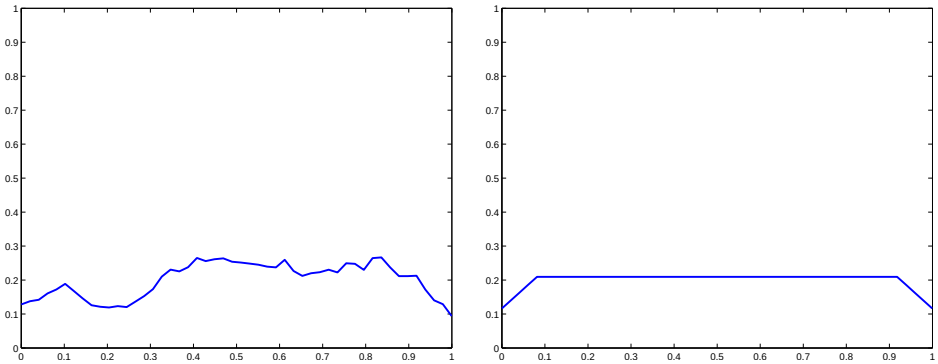
This section presents simulations on the real line. In order to compute the density of the points masses of a measure at each iteration, the space domain is the discretized into Nx uniform gridpoints. We carry out two tests of condensing sequences of two discrete measures with support $S(m) = \{x_1, \ldots, x_{50}\} \subset [0,1]$. The emergence of $S(m)$ requires the extension of the computation domain, namely even if the initial spatial positions are chosen in $[0,1]$, the final measure has not necessary positions in $[0,1]$ only. We run our code after fixing the order of reactions (the array of 50 indexes will randomly permuted). The following table summarizes the results of the simulations on the real line:

Table 2: Results of simulations in the real line.

Parameter/Sim.	(a)	(b)
ε	0.02	0.02
θ	0.01	0.01
Initial state	50 masses (one)	50 masses ($U(0,4)$)
Final state	4 isolated masses	4 isolated masses
Number of iterations	176	196

Figure 5 presents the density of a measure with 50 point masses in many condensing iterations for $\varepsilon = 0.02, \theta = 0.01$, the initial and the limit measure. The initial state is a deterministic distribution of masses and the limit measure (175 iterations) is constituted only from $\varepsilon + \theta$ isolated masses. Also, Figure 5 presents the density of measure with 50 point masses

in several condensing iterations on the real line, for $\varepsilon = 0.02, \theta = 0.01$. The first and the last Figures respectively presents the initial and the limit measures. The initial state is generated with a uniform random distribution ($U(0,4)$ (on the Figures, the density is normalized) and the final state (iter. 195) is constituted only from $(\varepsilon + \theta)$-isolated masses. The simulations in Figure 5 have different initial mass but they have the same condensing behavior. The final states are isolated point masses with different supports and different masses.



(a) Initial (b) Initial

(a) Iteration 195 (b) Iteration 175

Figure 5: Initial and final densities of condensing measures on the real line for simulations (a) and (b)

4.3. Condensing on the Real Plane

This section presents simulations on the real plane. In order to compute the density of the points masses of a measure at each iteration, the space domain is the discretized into $Nx \times Nx$ uniform two-dimensional gridpoints. We carry out two tests of condensing sequences of two measures

$$m := \sum_{x \in S(m) \subset [0,1]^2} m(x)\delta_x.$$

Where $S(m) = \{x_1, \ldots, x_{441}\}$. We run our code after fixing the order of reactions (the array of 441 index will be randomly permuted). The following Table summarizes the results of the

simulations on the real plane ($(\varepsilon,\theta)=(0.1,0.001)$):

Table 3: Results of two simulations on the real plane.

Parameter/Sim.	(a)	(b)
ε	0.1	0.1
θ	0.001	0.001
Initial state	441 masses (one)	441 masses ($U(0,4)$)
Final state	condensing	condensing
Number of iterations	more than 10000	more than 20000

Figure 6 presents the two-dimensional density of the measure. This simulation needs more the 10000 condensing iterations until the last iteration presented in the last Figure. Our code breaks down, when the total energy will be negligible (compared with a fixed tolerance). It is clearly shown in Figure 6 that the particles build a mass points with height density in the middle of the computation domain. This result will be different if we simulate another simulation, even if we use the same data for the initial measure.

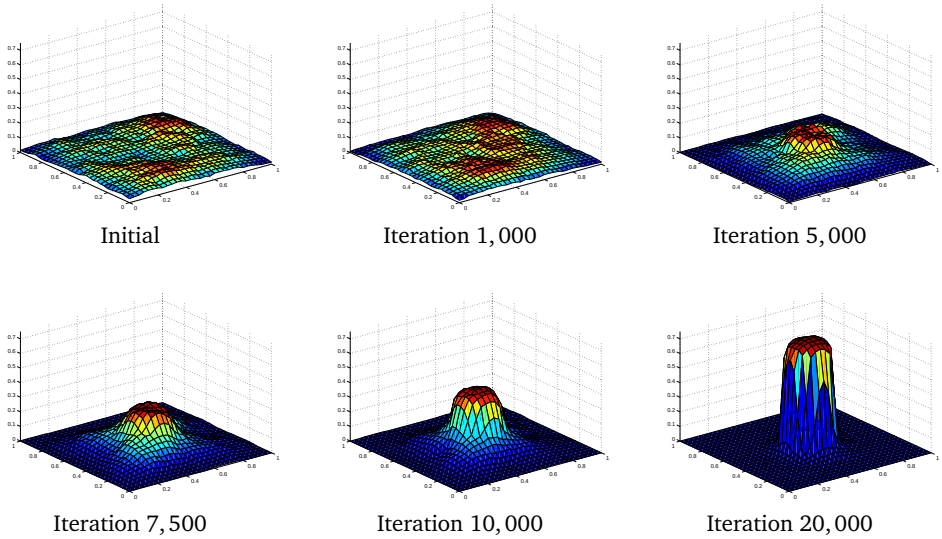


Figure 6: Densities of condensing particles on real plane of the simulation (b)

Figure 6 presents the density of six condensing iterations, we show the initial and the Final mass and some iterations. The mass is concentrated on one point on the middle of the computational domain.

5. Concluding remarks

The present work proposes a new model for condensing sequences, with special interest on the condensing process of particles. We have observed that the limit states depends not only

from the initial state but also of the condensing succession or the reaction order, therefore, they have non uniform and different distribution of mass and form ε-isolated subgroups. In one hand, we have shown how a collection of particles with a local control rule, forms an isolated distribution of masses with zero global energy. In the other hand, we have seen that the energy as local rule is in reality a global criteria for forming subgroups. However, one can easily show that the dynamics of the group is a consequence of individual moves of Agents. Moreover, it should be stressed that the stochastic behavior of our simulations is due to random choice of positions minimizing the local energy and the random range of reactions of the particles. The present study could be considered as example for explaining the concept of consensus and emergence phenomena.

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