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On some mappings in topological spaces

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Abstract. In this paper, we continue studying the properties of γ-semi-continuous and γ -semi-open functions introduced in [5], [10]. We also introduce γ^*-irresolute functions and γ -pre-semi-open (closed) functions and discuss their properties.

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Key words: γ-closed (open), γ-closure , γ^*-semi-closed (open), γ^*-semi-closure, γ-regular, γ^*-semi-interior, γ-semi-continuous functions, γ-semi-open (closed) functions, γ^*-irresolute functions, γ-pre-semi-open(closed) functions.

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1. Introduction

A. Csaszar [7,8] defined generalized open sets in generalized topological spaces. In 1975, Maheshwari and Prasad [13] introduced concepts of semi- T_1-spaces and semi-R_0-spaces. In 1979, S. Kasahara [11] defined an operation α on topological spaces. In 1992 (1993), B. Ahmad and F.U. Rehman [1], [15] introduced the notions of γ-interior, γ-boundary and γ-exterior points in topological spaces. They also studied properties and characterizations of (γ,β)-continuous mappings introduced by H. Ogata [14]. In 1999 (resp. 2005), B. Ahmad and S. Hussain introduced the concept of γ^*-regular spaces (resp. γ_0-compact, γ-normal spaces) and explored their many interesting properties. They initiated and discussed the concepts of γ^*-semi-open sets which generalizes γ-open sets introduced and discussed by H. Ogata [14], γ^*-semi-closed sets, γ^*-semi-closure, γ^*-semi-interior point in topological spaces [5], [9]. In 2006, they introduced Λ_s^{γ}-set and $\Lambda^{s^{\gamma}}$-set by using γ^*-semi-open sets. Moreover, they introduced the γ-semi-continuous functions and γ-semi-open (closed) functions in topological spaces and established several interesting properties.

In this paper, we continue studying the properties of γ-semi-continuous and γ-semi-open functions introduced by B. Ahmad and S. Hussain [5], [10]. We also introduce γ^*-irresolute

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functions and γ-pre-semi-open (closed) functions and discuss their properties.

Hereafter, we shall write spaces in place of topological spaces in the sequel.

2. Preliminaries

We recall some definitions and results used in this paper to make it self-contained.

Definition [11]. Let (X,τ) be a space. An operation $\gamma : \tau \to$ P(X) is a function from τ to the power set of X such that V $\subseteq V^\gamma$, for each V $\in \tau$, where V^γ denotes the value of γ at V. The operations defined by γ(G) = G, γ(G) = cl(G) and γ(G) = intcl(G) are examples of operation γ.

Definition [14]. Let A$\subseteq$ X. A point x$\in$ A is said to be γ-interior point of A iff there exists an open nbd N of x such that $N^\gamma \subseteq$ A and we denote the set of all such points by int_γ(A). Thus

$$int_\gamma \text{ (A)} = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A\} \subseteq A.$$

Note that A is γ-open [14] iff A =int_γ(A). A set A is called γ- closed [14] iff X-A is γ-open.

Definition [14]. A point x$\in$ X is called a γ-closure point of A$\subseteq$ X, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by cl_γ(A). A subset A of X is called γ-closed, if $cl_\gamma(A)$ $\subseteq$A. Note that cl_γ(A) is contained in every γ-closed superset of A.

Definition [14]. An operation γ on τ is said be regular, if for any open nbds U,V of x $\in$ X, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition [14]. An operation γ on τ is said to be open, if for every nbd U of each $x \in X$, there exists γ-open set B such that $x \in B$ and $U^\gamma \supseteq B$.

We defined [9] γ^*-semi-open sets using γ-open sets in the sense of H. Ogata [14] as :

Definition [9]. A subset A of a space (X,τ) is said to be a γ^*-semi-open set, if there exists a γ -open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ^*-semi-open sets is denoted by $SO_{\gamma^*}(X)$.

Definition [5]. A function $f : (X,\tau) \to (Y,\tau)$ is said to be γ-semi-continuous if for any γ-open set B of Y, $f^{-1}(B)$ is γ^*-semi-open in X.

Definition [5]. A function $f : X \to Y$ is said to be γ-semi-open (closed) if for each γ-open (closed) set U in X, $f(U)$ is γ^*-semi-open (closed) in Y.

Definition [5]. A set A in a space X is said to be γ^*-semi-closed if there exists a γ-closed set F such that $int_\gamma(F) \subseteq A \subseteq F$.

Proposition [5]. A subset A of a space X is γ^*-semi-closed if $X - A$ is γ^*-semi-open.

Definition [10]. A subset A of a space X is said to be γ-semi-nbd of a point $x \in X$ if there exists a γ^*-semi-open set U such that $x \in U \subseteq A$.

3. γ^*-irresolute functions

Definition 3.1. Let X and Y be spaces. A function $f : X \to Y$ is said to be γ^*-irresolute if and only if for any γ^*-semi-open subset S of Y, $f^{-1}(S)$ is γ^*-semi-open in X.

Definition 3.2[5]. An operation γ is said to be semi-regular, if for any semi-open sets U and V containing $x \in X$, there exists a semi-open set W containing x such that $U^{\gamma} \cap V^{\gamma} \supseteq W^{\gamma}$.
Theorem 3.3. A function $f : X \to Y$ is γ^*-irresolute if and only if for each x in X, the inverse of every γ-semi-nbd of $f(x)$ is a γ-semi-nbd of x, where γ is a semi regular operation .
Proof. Let $x \in X$ and A a γ-semi-nbd of $f(x)$. By definition of γ-semi-nbd , there exists $V \in SO_{\gamma^*}(Y)$ such that $f(x) \in V \subseteq A$. This implies that $x \in f^{-1}(V) \subseteq f^{-1}(A)$. Since f is γ^*-irresolute, so $f^{-1}(V) \in SO_{\gamma^*}(X)$. Hence $f^{-1}(A)$ is a γ-semi-nbd of x. This proves the necessity.

Conversely, let $A \in SO_{\gamma^*}(Y)$. Put $B = f^{-1}(A)$. Let $x \in B$, then $f(x) \in A$. Clearly, A (being γ^*-semi-open) is a γ-semi-nbd of $f(x)$. So by hypothesis, $B = f^{-1}(A)$ is a γ-semi-nbd of x. Hence by definition , there exists $B_x \in SO_{\gamma^*}(X)$ such that $x \in B_x \subseteq B$. Thus $B = \bigcup_{x \in B} B_x$. Since γ is semi regular, therefore it follows that B is γ^*-semi-open in X [5]. Therefore f is γ^*-irresolute.
Remark 3.4. γ-semi-nbd of x may be replaced by γ^*-semi-open nbd of x in Theorem 3.3.
Theorem 3.5. A function $f : X \to Y$ is γ^*-irresolute if and only if for each x in X and each γ-semi-nbd A of $f(x)$, there is a γ-semi-nbd B of x such that $f(B) \subseteq A$, where γ is a semi regular operation.
Proof. Let $x \in X$ and A a γ-semi-nbd of $f(x)$. Then there exists $O_{f(x)} \in SO_{\gamma^*}(Y)$ such that $f(x) \in O_{f(x)} \subseteq A$. It follows that $x \in f^{-1}(O_{f(x)}) \subseteq f^{-1}(A)$. By hypothesis, $f^{-1}(O_{f(x)}) \in SO_{\gamma^*}(X)$. Let $B = f^{-1}(A)$. Then it follows that B is γ-semi-nbd of x and $f(B) = f f^{-1}(A) \subseteq A$. This proves the necessity.

Conversely, let $U \in SO_{\gamma^*}(Y)$. Take $O = f^{-1}(U)$. Let $x \in O$, then $f(x) \in U$. Thus U is a γ-semi-nbd of $f(x)$. So by hypothesis, there exists a γ-semi-nbd V_x of x such that $f(V_x) \subseteq U$. Thus it follows that $x \in V_x \subseteq f^{-1}f(V_x) \subseteq f^{-1}(U) = O$. So V_x is a γ-semi-nbd of x, which implies there exists an $O_x \in SO_{\gamma^*}(X)$ such that $x \in O_x \subseteq O$. Thus $O = \bigcup_{x \in o} O_x$. Since γ is semi regular, then it follows that O is γ^*-semi-open in X. Thus f is γ^*-irresolute.
Definition 3.6[10]. Let X be a space. $A \subseteq X$ and $p \in X$. Then p is called a γ^*-semi-limit point of A if $U \cap (A - \{p\}) \neq \phi$, for any γ^*-semi-open set U containing p. The set of all γ^*-semi-limit points of A is called a γ^*-semi-derived set of A and is denoted by $sd_{\gamma^*}(A)$.

Clearly if $A \subseteq B$ then $sd_{\gamma^*}(A) \subseteq sd_{\gamma^*}(B)$ ..... (*)

Denote $\Gamma(X)$, the set of all monotone operators on X. Then we have:
Definition 3.7[9]. Let A be a subset of space X and $\gamma \in \Gamma(X)$. The intersection of all γ^*-semi-closed sets containing A is called γ^*-semi-closure of A and is denoted by $scl_{\gamma^*}(A)$.
Remark 3.8. From the definition 3.6, it follows that p is a γ^*-semi-limit point of A if and only if $p \in scl_{\gamma^*}(A - \{p\})$.
Theorem 3.9[10]. For any $A, B \subseteq X$, the γ^*-semi-derived sets have the following properties:
(1) $scl_{\gamma^*}(A) = A \cup sd_{\gamma^*}(A)$.
(2) $\bigcup_i sd_{\gamma^*}(A_i) = sd_{\gamma^*}(\bigcup_i A_i)$.
(3) $sd_{\gamma^*}(sd_{\gamma^*}(A)) \subseteq sd_{\gamma^*}(A)$.
(4) $scl_{\gamma^*}(sd_{\gamma^*}(A)) = sd_{\gamma^*}(A)$.

In terms of γ^*-semi-derived sets, we use Theorem 3.9 and characterize γ^*-irresolute functions as:

Theorem 3.10. A function $f : X \to Y$ is γ^*-irresolute if and only if

$$f(sd_{\gamma^*}(A)) \subseteq scl_{\gamma^*}(f(A)), \text{ for all } A \subseteq X .$$

Proof. Let $f : X \to Y$ be γ^*-irresolute. Let $A \subseteq X$, and $x \in sd_{\gamma^*}(A)$. Assume that $f(x) \notin f(A)$ and let V denote a γ-semi-nbd of $f(x)$. Since f is γ^*-irresolute, so by Theorem 3.5, there exists a γ-semi-nbd U of x such that $f(U) \subseteq V$. From $x \in sd_{\gamma^*}(A)$, it follows that $U \cap A \neq \phi$; there exists, therefore, at least one element $a \in U \cap A$ such that $f(a) \in f(A)$ and $f(a) \in V$. Since $f(x) \notin f(A)$, we have $f(a) \neq f(x)$. Thus every γ-semi-nbd of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently, $f(x) \in sd_{\gamma^*}(f(A))$.This proves the necessity.

Conversely, suppose that f is not γ^*-irresolute. Then by Theorem 3.5, there exists $x \in X$ and a γ-semi-nbd V of $f(x)$ such that every γ-semi-nbd U of x contains at least one element $a \in U$ for which $f(a) \notin V$. Put $A = \{a \in X : f(a) \notin V\}$. Since $f(x) \in V$, therefore $x \notin A$ and hence $f(x) \notin f(A)$. Since $f(A) \cap (V - f(x)) = \phi$, therefore $f(x) \notin sd_{\gamma^*}(f(A))$. It follows that $f(x) \in f(sd_{\gamma^*}(A)) - (f(A) \cup sd_{\gamma^*}(f(A))) \neq \phi$, which is a contradiction to the given condition. This proves sufficiency.

Theorem 3.11. Let $f : X \to Y$ be one-to-one function. Then f is γ^*-irresolute if and only if

$$f(sd_{\gamma^*}(A)) \subseteq sd_{\gamma^*}(f(A)), \text{ for all } A \subseteq X.$$

Proof. Let $A \subseteq X$, $x \in sd_{\gamma^*}(A)$ and V be a γ-semi-nbd of $f(x)$. Since f is γ^*-irresolute, so by Theorem 3.5, there exists a γ-semi-nbd U of x such that $f(U) \subseteq V$. But $x \in sd_{\gamma^*}(A)$ gives there exists an element $a \in U \cap A$ such that $a \neq x$. Clearly $f(a) \in f(A)$ and since f is one to one, $f(a) \neq f(x)$. Thus every γ-semi-nbd V of $f(x)$ contains an element $f(a)$ of $f(A)$ different from $f(x)$. Consequently $f(x) \in sd_{\gamma^*}(f(A))$. Therefore we have $f(sd_{\gamma^*}(A)) \subseteq sd_{\gamma^*}(f(A))$. This proves the necessity.

Sufficiency follows from Theorem 3.10. This completes the proof.

H. Ogata [14] defined the notion of γ-T_2 spaces in topological spaces, we generalize this notion and define:

Definition 3.12. A space X is said to be γ-semi-T_2 , if for each two distinct points $x, y \in X$ there exist $U, V \in SO_{\gamma^*}(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Theorem 3.13. If $f : X \to Y$ is a γ^*-irresolute injection and Y is γ-semi-T_2 , then X is γ-semi-T_2 .

Proof. Let x_1 and x_2 be two distinct points of X. Since f is injective, therefore $f(x_1) \neq f(x_2)$. Since Y is γ-semi-T_2 , there exist $V_1, V_2 \in SO_{\gamma^*}(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \phi$. Then $x_1 \in f^{-1}(V_1)$, $x_2 \in f^{-1}(V_2)$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Since f is γ^*-irresolute, so $f^{-1}(V_1), f^{-1}(V_2) \in SO_{\gamma^*}(X)$. This proves that X is γ-semi-T_2 .

4. γ-pre-semi-open functions

Definition 4.1. Let X and Y be spaces. Then a function $f : X \to Y$ is said to be γ-pre-semi-open if and only if for each $A \in SO_{\gamma^*}(X)$, $f(A) \in SO_{\gamma^*}(Y)$.

Remark 4.2. The class of γ-pre-semi-open functions is a subclass of class of γ-semi-open functions defined in [5].

Note that if $f : X \to Y$ and $g : Y \to Z$ be any two γ-pre-semi-open functions, then the composition $gof : X \to Z$ is a γ-pre-semi-open function.

The following theorem is easy to prove:

Theorem 4.3. A function $f : X \to Y$ is γ-pre-semi-open if and only if for each $x \in X$ and for every $A \in SO_{\gamma^*}(X)$ such that $x \in A$, there exists $B \in SO_{\gamma^*}(Y)$ such that $f(x) \in B$ and $B \subseteq f(A)$.

Theorem 4.4. A function $f : X \to Y$ is γ-pre-semi-open if and only if for each $x \in X$ and for every γ-semi-nbd U of x in X, there exists a γ-semi-nbd V of $f(x)$ in Y such that $V \subseteq f(U)$.

Proof. Let U be a γ-semi-nbd of $x \in X$. Then by definition there exists $W \in SO_{\gamma^*}(X)$ such that $x \in W \subseteq U$. Then $f(x) \in f(W) \subseteq f(U)$. Since f is γ-pre-semi-open , therefore $f(W) \in SO_{\gamma^*}(Y)$. Hence $V = f(W)$ is a γ-semi-nbd of $f(x)$ and $V \subseteq f(U)$.

Conversely, let $U \in SO_{\gamma^*}(X)$ and $x \in U$. Then U is a γ-semi-nbd of x. So by hypothesis, there exists a γ-semi-nbd V of $f(x)$ such that $f(x) \in V \subseteq f(U)$. That is, $f(U)$ is a γ-semi-nbd of $f(x)$. Thus $f(U)$ is a γ-semi-nbd of each of its points. Thus $f(U)$ is γ^*-semi-open. Hence f is γ-pre-semi-open. This completes the proof.

Definition 4.5[9]. Let A be a subset of a space X and $\gamma \in \Gamma(X)$. The union of all γ^*-semi-open sets of X contained in A is called γ^*-semi-interior of A and is denoted by $sint_{\gamma^*}(A)$.

Theorem 4.6. A function $f : X \to Y$ is γ-pre-semi-open if and only if

$$f(sint_{\gamma^*}(A)) \subseteq sint_{\gamma^*}(f(A)), \text{ for all } A \subseteq X \ .$$

Proof. Let $x \in sint_{\gamma^*}(A)$. Then there exists $U \in SO_{\gamma^*}(X)$ such that $x \in U \subseteq A$. So $f(x) \in f(U) \subseteq f(A)$. Since f is γ-pre-semi-open , therefore $f(U)$ is γ^*-semi-open in Y. Hence $f(x) \in sint_{\gamma^*}(f(A))$. Thus $f(sint_{\gamma^*}(A)) \subseteq sint_{\gamma^*}(f(A))$. This proves necessity.

Conversely let $U \in SO_{\gamma^*}(X)$. Then by hypothesis, $f(U) = f(sint_{\gamma^*}(U)) \subseteq sint_{\gamma^*}(f(U)) \subseteq f(U)$ or $f(U) \subseteq sint_{\gamma^*}(f(U)) \subseteq f(U)$. This implies $f(U)$ is γ^*-semi-open in Y. So f is γ-pre-semi-open.

We use Theorem 4.6 and prove:

Theorem 4.7. A function $f : X \to Y$ is γ-pre-semi-open if and only if

$$sint_{\gamma^*}(f^{-1}(A)) \subseteq f^{-1}(sint_{\gamma^*}(A)), \text{ for all } A \subseteq X \ .$$

Proof. Let A be any subset of Y. Clearly, $sint_{\gamma^*}(f^{-1}(A))$ is γ^*-semi-open in Y. Also $f(sint_{\gamma^*}(f^{-1}(A)) \subseteq f(f^{-1}(A)) \subseteq A$. Since f is γ-pre-semi-open, by Theorem 4.6 we have
$f(sint_{\gamma^*}(f^{-1}(A)) \subseteq sint_{\gamma^*}(A)$. Therefore,
$sint_{\gamma^*}(f^{-1}(A)) \subseteq f(f^{-1}(sint_{\gamma^*}(f^{-1}(A)))) \subseteq f^{-1}(sint_{\gamma^*}(A))$ or
$sint_{\gamma^*}(f^{-1}(A)) \subseteq f^{-1}(sint_{\gamma^*}(A))$. This proves necessity.

Conversely, let $B \subseteq X$. By hypothesis, we obtain
$sint_{\gamma^*}(B) \subseteq sint_{\gamma^*}(f^{-1}f(B)) \subseteq f^{-1}(sint_{\gamma^*}f(B))$. This implies that $f(sint_{\gamma^*}(B)) \subseteq f(sint_{\gamma^*}(f^{-1}f(B))) \subseteq f(f^{-1}(sint_{\gamma^*}f(B))) \subseteq sint_{\gamma^*}(f(B))$. Consequently, $f(sint_{\gamma^*}(B)) \subseteq sint_{\gamma^*}(f(B))$, for all $B \subseteq X$. By Theorem 4.6 , f is γ-pre-semi-open.

We use Theorem 4.7 and prove:
Theorem 4.8. A function $f : X \to Y$ is γ-pre-semi-open if and only if

$$f^{-1}(scl_{\gamma^*}(A)) \subseteq scl_{\gamma^*}(f^{-1}(A)), \text{ for all } A \subseteq X .$$

Proof. Let $A \subseteq Y$. By Theorem 4.7 , $sint_{\gamma^*}(f^{-1}(Y-A)) \subseteq f^{-1}(sint_{\gamma^*}(Y-A))$. This implies that $sint_{\gamma^*}(X-f^{-1}(A)) \subseteq f^{-1}(sint_{\gamma^*}(Y-A))$. As $sint_{\gamma^*}(A) = X - scl_{\gamma^*}(X-A)$ [5], therefore $X - scl_{\gamma^*}(f^{-1}(A)) \subseteq f^{-1}(Y - scl_{\gamma^*}(A))$. or $X - scl_{\gamma^*}(f^{-1}(A)) \subseteq X - f^{-1}(scl_{\gamma^*}(A))$. Hence $f^{-1}(scl_{\gamma^*}(A)) \subseteq scl_{\gamma^*}(f^{-1}(A))$. This proves necessity.

Conversely, let $A \subseteq Y$. By hypothesis, $f^{-1}(scl_{\gamma^*}(Y-A)) \subseteq scl_{\gamma^*}(f^{-1}(Y-A))$. This implies $X - scl_{\gamma^*}(f^{-1}(Y-A)) \subseteq X - f^{-1}(scl_{\gamma^*}(Y-A))$. Hence $X - scl_{\gamma^*}(X - f^{-1}(A)) \subseteq f^{-1}(Y-(scl_{\gamma^*}(Y-A)))$. This gives $sint_{\gamma^*}(f^{-1}(A)) \subseteq f^{-1}(sint_{\gamma^*}(A))$. Now from Theorem 4.7, it follows that f is γ-pre-semi-open. This completes the proof.

Definition 4.9[14]. A function $f : (X,\tau) \to (Y,\tau)$ is said to be (γ,β) -continuous, if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U such that $x \in U$ and $f(U^{\gamma}) \subseteq V^{\beta}$, where γ and β are operations on τ and δ respectively.

Definition 4.10[14]. A function $f : (X,\tau) \to (Y,\tau)$ is said to be (γ,β)-open (closed), if for any γ-open (closed) set A of X, $f(A)$ is β-open(closed) in Y.

In [10], we proved the following Theorem:

Theorem 4.11[10]. If $f : X \to Y$ is a (γ,β)-open and (γ,β)-continuous function, then $f^{-1}(B) \in SO_{\gamma^*}(X)$, for every $B \in SO_{\beta^*}(Y)$, where β is an open operation.

We use this theorem and prove:

Theorem 4.12. Let X, Y and Z be three spaces, and let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $gof : X \to Z$ a γ-pre-semi-open function. Then

(1) If f is a (γ,β)-open and (γ,β)-continuous surjection, then g is β -pre-semi-open.

(2) If g is a (β,δ)-open and (β,δ)-continuous injection, then f is γ-pre-semi-open.

Proof.(1). Let V be an arbitrary γ^*-semi-open set in Y. Since f is a (γ,β)-open and (γ,β)-continuous, then by Theorem 4.11, $f^{-1}(V)$ is a γ^*-semi-open set in X. Also gof is γ-pre-semi-open and f is surjection, we have $g(V) = (gof)(f^{-1}(V))$ is a β^*-semi-open set in Z. This proves that g is β-pre-semi-open.

(2). Since g is injective so for every subset A of X, $f(A) = g^{-1}(g(f(A)))$. Let U be an arbitrary γ^*-semi-open set in X. Then $(gof)(U)$ is γ^*-semi-open. Since g is a (β,δ)-open and (β,δ)-semi-continuous, therefore by above Theorem 4.11, $f(U)$ is γ^*-semi-open in Y. This shows that f is γ-pre-semi-open.

Theorem 4.13. Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $gof : X \to Z$ is γ^*-irresolute. Then

(1) If g is a β-pre-semi-open injection, then f is γ^*-irresolute.

(2) If f is a γ-pre-semi-open surjection, then g is β^*-irresolute.

Proof.(1). Let $U \in SO_{\beta^*}(Y)$. Since g is β-pre-semi-open, then $g(U) \in SO_{\delta^*}(Z)$. Also gof is γ^*-irresolute, and therefore; $(gof)^{-1}(g(U)) \in SO_{\gamma^*}(X)$. Since g is injective $(gof)^{-1}(g(U)) = (f^{-1}og^{-1})g(U) = f^{-1}(g^{-1}g(U)) = f^{-1}(U)$. Consequently, $f^{-1}(U)$ is γ^*-semi-open in X. This

proves that f is γ^*-irresolute. This proves (1).
(2). Let $V \in SO_{\delta^*}(Z)$. Since gof is γ^*-irresolute, then $(gof)^{-1}(V) \in SO_{\gamma^*}(X)$. Also f is γ-pre-semi-open, so $f(gof)^{-1}(V)$ is β^*-semi-open in Y. Since f is surjective, we obtain $fo(gof)^{-1}(V) = fo(f^{-1}og^{-1})(V) = (fof^{-1})og^{-1}(V) = g^{-1}(V)$. It follows that $g^{-1}(V) \in SO_{\gamma^*}(Y)$. This proves that g is a β^*-irresolute function.

5. γ-pre-semi-closed functions

Definition 5.1. A function $f : X \to Y$ is γ-pre-semi-closed if and only if the image set $f(A)$ is γ^*-semi-closed, for each γ^*-semi-closed subset A of X.

It is obvious that the composition of two γ-pre-semi-closed mappings is a γ-pre-semi-closed mapping.

Theorem 5.2. A function $f : X \to Y$ is γ-pre-semi-closed if and only if

$$scl_{\gamma^*}f(B) \subseteq f(scl_{\gamma^*}(B)), \text{ for every subset B of X.}$$

Proof. Suppose f is γ-pre-semi-closed and let $B \subseteq X$. Since f is γ-pre-semi-closed , therefore $f(scl_{\gamma^*}(B))$ is γ^*-semi-closed in Y. Since $f(B) \subseteq f(scl_{\gamma^*}(B))$, we obtain $scl_{\gamma^*}(f(B)) \subseteq f(scl_{\gamma^*}(B))$. This proves necessity.

Conversely, suppose A is a γ^*-semi-closed set in X. By hypothesis, we obtain $f(A) \subseteq scl_{\gamma^*}(f(A)) \subseteq f(scl_{\gamma^*}(A)) = f(A)$. Hence $f(A) = scl_{\gamma^*}(f(A))$. Thus $f(A)$ is γ^*-semi-closed set in Y. This proves that f is γ-pre-semi-closed.

Theorem 5.3. A function $f : X \to Y$ is γ-pre-semi-closed if and only if

$$int_\gamma(cl_\gamma(f(B))) \subseteq f(scl_{\gamma^*}(B)), \text{ for every subset B of X.}$$

Proof. Suppose f is γ-pre-semi-closed and B is any subset of X. Then $f(scl_{\gamma^*}(B))$ is γ^*-semi-closed in Y. This implies that there exists a γ-closed subset A of Y such that $int_\gamma(A) \subseteq f(scl_{\gamma^*}(B)) \subseteq A$. This gives that $int_\gamma cl_\gamma(f(scl_{\gamma^*}(B))) \subseteq int_\gamma(A) \subseteq f(scl_{\gamma^*}(B))$. Then $int_\gamma(cl_\gamma(f(B)) \subseteq int_\gamma(cl_\gamma(f(scl_{\gamma^*}(B))))$ gives $int_\gamma(cl_\gamma(f(B))) \subseteq f(scl_{\gamma^*}(B))$. This proves necessity.

Conversely, suppose A is a γ^*-semi-closed set in X. Then by hypothesis, $int_\gamma(cl_\gamma(f(A)) \subseteq f(scl_{\gamma^*}(A)) = f(A)$. Put $B = cl_\gamma(f(A))$. Clearly, B is γ-closed in Y. This implies that $int_\gamma(B) \subseteq f(A) \subseteq B$. Hence $f(A)$ is γ^*-semi-closed in Y. This implies f is γ-pre-semi-closed.

Theorem 5.4. A bijective function $f : X \to Y$ is a γ-pre-semi-closed if and only if for each subset A of Y and each γ^*-semi-open set B in X containing $f^{-1}(A)$, there exists a γ^*-semi-open set C in Y containing A such that $f^{-1}(C) \subseteq B$, where γ is a semi regular operation.

Proof. Let $C = Y - f(X - B)$. Then $C^c = f(B^c)$. Since f is γ-pre-semi-closed, so C is γ^*-semi-open. Since $f^{-1}(A) \subseteq B$, we have $C^c = f(B^c) \subseteq f(f^{-1}(A^c)) \subseteq A^c$. Hence, $A \subseteq C$, and thus C is a γ-semi-open nbd of A. Further $B^c \subseteq f^{-1}(f(B^c)) = f^{-1}(C^c) = (f^{-1}(C))^c$. This proves that $f^{-1}(C) \subseteq B$.

Conversely, suppose F is a γ^*-semi-closed set in X. Let $y \in Y - f(F)$. Then $f^{-1}(y) \subseteq X - f^{-1}(f(F)) \subseteq X - F$ and $X - F$ is γ^*-semi-open in X. Hence by hypothesis, there exists a γ^*-semi-open set C containing y such that $f^{-1}(C) \subseteq X - F$. Since f is one-one, we have

$y \in C \subseteq Y - f(F)$. Thus $Y - f(F) = \bigcup_{y \in Y - f(F)} C$. Hence $Y - f(F)$ is γ^*-semi-open set [5]. This proves that f is γ-pre-semi-closed.

We use Theorem 4.8 and give the following characterizations:
Theorem 5.5. Let $f : X \to Y$ be a bijective function. Then the following are equivalent:
(1) f is γ-pre-semi-closed.
(2) f is γ-pre-semi-open.
(3) f^{-1} is γ^*-irresolute.
Proof. (1) $\Rightarrow$ (2) It is straightforward.
(2) $\Rightarrow$ (3). Let $A \subseteq X$. Since f is γ-pre-semi-open, by Theorem 4.9, $f^{-1}(scl_{\gamma^*}(f(A))) \subseteq scl_{\gamma^*}(f^{-1}f(A))$ implies $scl_{\gamma^*}(f(A)) \subseteq f(scl_{\gamma^*}(A))$. Thus $scl_{\gamma^*}(f^{-1})^{-1}(A)$ is contained in $(f^{-1})^{-1}(scl_{\gamma^*}(A))$, for every subset A of X. Then again by Theorem 4.8 , it follows that f^{-1} is γ^*-irresolute.
(3) $\Rightarrow$ (1). Let A be a γ^*-semi-closed set in X. Then $X - A$ is γ^*-semi-open in X. Since f^{-1} is γ^*-irresolute, $(f^{-1})^{-1}(X - A)$ is γ^*-semi-open in Y. But $(f^{-1})^{-1}(X - A) = f(X - A) = Y - f(A)$. Thus $f(A)$ is γ^*-semi-closed in Y. This proves that f is γ-pre-semi-closed. This completes the proof.

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