EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 4, No. 4, 2011, 435-447
ISSN 1307-5543 – www.ejpam.com

On Certain Subclasses of Meromorphically p-Valent Functions Associated with Integral Operators

M. K. Aouf*, A. Shamandy, A. O. Mostafa and F. Z. El-Emam

Faculty of Science Mansoura university Mansoura, 35516, Egypt

Abstract. The object of the present paper is to introduce and study new classes of meromorphically p-valent functions associated with the integral operators $P^{\alpha}_{\beta,p}$ and $Q^{\alpha}_{\beta,p}$.

Key Words and Phrases: Meromorphic functions; Hadamard product; p-valent functions; differential subordination; integral operators.

1. Introduction

For any integer $m > -p$, let $\sum_{p,m}$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} a_k z^k (p \in \mathbb{N} = \{1,2,3,\ldots\}), \tag{1}$$

which are analytic and $p-$valent in the punctured unit disk $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = U \backslash \{0\}$. For convenience, we write $\sum_{p,1-p} = \sum_p$.

If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$. If $g(z)$ is univalent in U, then the following equivalence, (cf., e.g.,[3] and [7]):

$$f(z) \prec g(z)\ (z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in \sum_{p,m}$ given by (1) and $g(z) \in \sum_{p,m}$ defined by

$$g(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} b_k z^k \ \ (m > -p, p \in \mathbb{N}), \tag{2}$$

*Corresponding author.

Email addresses: mkaouf127@yahoo.com (M. Aouf), shamandy16@hotmail.com (A. Shamandy), adelaeg254@yahoo.com (A. Mostafa), fatma_elemam@yahoo.com (F. El-Emam)

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=\frac{1}{z^p}+\sum_{k=m}^{\infty}a_k b_k z^k=(g*f)(z). \tag{3}$$

We now define the integral operators $P^{\alpha}_{\beta,p}, Q^{\alpha}_{\beta,p}:\sum_{p,m}\to\sum_{p,m}$ as follows:

$$\begin{aligned} P^{\alpha}_{\beta,p}f(z) &= \frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{1}{z^{\beta+p}}\int_0^z t^{\beta+p-1}\left(\log\frac{z}{t}\right)^{\alpha-1}f(t)dt \\ &= \frac{1}{z^p}+\sum_{k=m}^{\infty}\left(\frac{\beta}{k+\beta+p}\right)^{\alpha}a_k z^k \\ &= \left(\frac{1}{z^p}+\sum_{k=m}^{\infty}\left(\frac{\beta}{k+\beta+p}\right)^{\alpha}z^k\right)*f(z) \\ &\quad (\alpha,\beta>0;p\in\mathbb{N};f\in\textstyle\sum_{p,m}), \end{aligned} \tag{4}$$

and $P^{0}_{\beta,p}f(z)=P^0f(z)=f(z)$ $(\alpha=0;\beta>0)$,

$$\begin{aligned} Q^{\alpha}_{\beta,p}f(z) &= \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)}\frac{1}{z^{\beta+p}}\int_0^z t^{\beta+p-1}\left(1-\frac{t}{z}\right)^{\alpha-1}f(t)dt \\ &= \frac{1}{z^p}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=m}^{\infty}\frac{\Gamma(k+\beta+p)}{\Gamma(k+\beta+\alpha+p)}a_k z^k \\ &= \left(\frac{1}{z^p}+\frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)}\sum_{k=m}^{\infty}\frac{\Gamma(k+\beta+p)}{\Gamma(k+\beta+\alpha+p)}z^k\right)*f(z) \\ &\quad (\alpha>0;\beta>-1;p\in\mathbb{N};f\in\textstyle\sum_{p,m}), \end{aligned} \tag{5}$$

and $Q^{0}_{\beta,p}f(z)=Q^0f(z)=f(z)$ $(\alpha=0;\beta>-1)$,

$$\begin{aligned} J_{\beta,p}f(z) &= \frac{\beta}{z^{\beta+p}}\int_0^z t^{\beta+p-1}f(t)dt \\ &= \frac{1}{z^p}+\sum_{k=m}^{\infty}\frac{\beta}{k+\beta+p}a_k z^k \\ &= \left(\frac{1}{z^p}+\sum_{k=m}^{\infty}\frac{\beta}{k+\beta+p}z^k\right)*f(z) \end{aligned} \tag{6}$$

$$\tag{7}$$

where $\Gamma(\alpha)$ is the familiar Gamma function. We write $P^{\alpha}_{1,p}f(z)=P^{\alpha}_{p}f(z)$ and $P^{\alpha}_{\beta,1}f(z)=P^{\alpha}_{\beta}f(z)$, where $P^{\alpha}_{p}f(z), Q^{\alpha}_{\beta,p}:\sum_{p,0}\to\sum_{p,0}$ were investigated by Aqlan et al. [2], $P^{\alpha}_{\beta}f(z):\sum_{1,1}\to\sum_{1,1}$ was investigated by Lashin [6]and$J_{\beta,p}:\sum_{p,m}\to\sum_{p,m}$ was investigated by many authors (see for example [1], [5], [13] and [16]). From (4), (5) and (6), we can see that

$$J_{\beta,p}f(z) = P^{1}_{\beta,p}f(z) = Q^{1}_{\beta,p}f(z)\ (\beta > 0),$$

$$z\left(P^{\alpha}_{\beta,p}f(z)\right)^{'} = \beta P^{\alpha-1}_{\beta,p}f(z) - (\beta+p)P^{\alpha}_{\beta,p}f(z)\ (\alpha \geq 0; \beta > 0) \tag{8}$$

and

$$z\left(Q^{\alpha}_{\beta,p}f(z)\right)^{'} = (\beta+\alpha-1)Q^{\alpha-1}_{\beta,p}f(z) - (\beta+\alpha+p-1)Q^{\alpha}_{\beta,p}f(z)(\alpha \geq 0; \beta > -1. \tag{9}$$

By using the integral operators $P^{\alpha}_{\beta,p}f(z)$ and $Q^{\alpha}_{\beta,p}f(z)$, we define two subclasses of $\sum_{p,m}$ as follows:

Definition 1. *For fixed parameters A, B $(-1 \leq B < A \leq 1)$, a function $f(z) \in \sum_{p,m}$ is said to be in the class $\sum^{P}_{p,m}(\beta,\alpha,\lambda,A,B)$ if*

$$-\frac{z^{p+1}}{p}\left\{(1-\lambda)\left(P^{\alpha}_{\beta,p}f(z)\right)^{'} + \lambda\left(P^{\alpha-1}_{\beta,p}f(z)\right)^{'}\right\} \prec \frac{1+Az}{1+Bz}(z \in U), \tag{10}$$

where $\alpha \geq 0$, $\beta > 0$, $p \in \mathbb{N}$ and $\lambda \geq 0$.

Definition 2. *For fixed parameters A, B $(-1 \leq B < A \leq 1)$, a function $f(z) \in \sum_{p,m}$ is said to be in the class $\sum^{Q}_{p,m}(\beta,\alpha,\lambda,A,B)$ if*

$$-\frac{z^{p+1}}{p}\left\{(1-\lambda)\left(Q^{\alpha}_{\beta,p}f(z)\right)^{'} + \lambda\left(Q^{\alpha-1}_{\beta,p}f(z)\right)^{'}\right\} \prec \frac{1+Az}{1+Bz}(z \in U), \tag{11}$$

where $\alpha \geq 0$, $\beta > -1$, $p \in \mathbb{N}$ and $\lambda \geq 0$.

We note that $\sum^{P}_{1,1}(\beta,\alpha,\lambda,A,B) = \sum^{P}_{\beta,\alpha}(\lambda,A,B)$ and $\sum^{Q}_{1,1}(\beta,\alpha,\lambda,A,B) = \sum^{Q}_{\beta,\alpha}(\lambda,A,B)$ (see Lashin [6]).

In this paper, we obtain some properties of the classes $\sum^{P}_{p,m}(\beta,\alpha,\lambda,A,B)$ and $\sum^{Q}_{p,m}(\beta,\alpha,\lambda,A,B)$. Our results generalize the work of Lashin [6].

2. Preliminaries

To derive our main results, we shall need the following lemmas.

Lemma 1 ([4] see also [7]). *Let the function $h(z)$ be analytic and convex (univalent) in U with $h(0) = 1$ and $\phi(z)$ given by*

$$\phi(z) = 1 + c_{p+m}z^{p+m} + c_{p+m+1}z^{p+m+1} + \ldots. \tag{12}$$

If

$$\phi(z) + \frac{z\phi^{'}(z)}{\gamma} \prec h(z)\ (\text{Re}(\gamma) \geq 0, \gamma \neq 0; z \in U), \tag{13}$$

then

$$\phi(z) \prec \psi(z) = \frac{\gamma}{p+m} z^{-\frac{\gamma}{p+m}} \int_0^z t^{\frac{\gamma}{p+m}-1} h(t)dt \prec h(z) \ (z \in U),$$

and $\psi(z)$ is the best dominant of (13).

We denote by $P(\gamma)$ the class of functions $\varphi(z)$ given by

$$\varphi(z) = 1 + b_1 z + b_2 z^2 + \ldots, \tag{14}$$

which are analytic in U and satisfy the following inequality:

$$\text{Re}(\varphi(z)) > \gamma (0 \leq \gamma < 1; z \in U).$$

Lemma 2 ([9])**.** *Let the function $\varphi(z)$, given by (14) be in the class $P(\gamma)$. Then*

$$\text{Re}(\varphi(z)) \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \ (0 \leq \gamma < 1; \ z \in U).$$

Lemma 3 ([12])**.** *If $\varphi_j \in P(\gamma_j)$ $(0 \leq \gamma_j < 1; \ j = 1,2)$, then*

$$\varphi_1 * \varphi_2 \in P(\gamma_3) \ (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2).$$

The result is the best possible.

For real or complex numbers a, b and c $(c \neq 0, -1, -2, \ldots)$, the Gauss hypergeometric function ${}_2F_1$ is defined in U by

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{15}$$

where $(x)_k$ denotes the Pochhammer symbol given by

$$(x)_k = \begin{cases} x(x+1)(x+2)\ldots(x+k-1) & (k \in \mathbb{N}, x \in \mathbb{C}) \\ 1 & (k = 0, x \in \mathbb{C}\backslash\{0\}). \end{cases}$$

We note that the series defined by (15) converges absolutely for $z \in U$ and hence represents an analytic function in the open unit disk U (see [14]).

Lemma 4 ([14])**.** *For real or complex numbers a, b and c $(c \neq 0, -1, -2, \ldots)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} \ {}_2F_1(a,b;c;z) \ (\text{Re}(c) > \text{Re}(b) > 0); \tag{16}$$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a} \ {}_2F_1(a, c-b; c; \frac{z}{z-1}); \tag{17}$$

$$ {}_2F_1(a,b; \frac{a+b+1}{2}; \frac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+1}{2})\Gamma(\frac{b+1}{2})}; \tag{18}$$

$$ {}_2F_1(1,1;2; \frac{z}{z+1}) = \frac{z+1}{z} \ln(1+z) \ (z \neq 0). \tag{19}$$

3. Main Results

Unless otherwise mentioned, we assume throughout this paper that

$$m > -p,\ p \in \mathbb{N},\ \alpha \geq 0,\ \lambda > 0 \text{ and } -1 \leq B < A \leq 1.$$

Theorem 1. *If* $f \in \sum_{p,m}^{P}(\beta, \alpha, \lambda, A, B)$ $(\beta > 0)$, *then*

$$-\frac{z^{p+1}\left(P_{\beta,p}^{\alpha} f(z)\right)'}{p} \prec q_1(z) \prec \frac{1+Az}{1+Bz}\ (z \in U), \tag{20}$$

where the function $q_1(z)$ *given by*

$$q_1(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{\beta}{\lambda(p+m)}+1;\frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{\beta}{\beta+\lambda(p+m)}Az & (B = 0), \end{cases}$$

is the best dominant of (20). Furthermore,

$$\text{Re}\left(-\frac{z^{p+1}\left(P_{\beta,p}^{\alpha} f(z)\right)'}{p}\right) > \rho\ (z \in U), \tag{21}$$

where

$$\rho(\beta, p, \lambda, A, B) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\ {}_2F_1(1,1;\frac{\beta}{\lambda(p+m)}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\beta}{\beta+\lambda(p+m)}A & (B = 0). \end{cases}$$

The result is the best possible.

Proof. Setting

$$\phi(z) = -\frac{z^{p+1}\left(P_{\beta,p}^{\alpha} f(z)\right)'}{p}\ (z \in U). \tag{22}$$

Then the function $\phi(z)$ is of the form (12) and is analytic in U. Differentiating (22), and with the aid of the identity (8) we get

$$\begin{aligned} \phi(z) + \frac{\lambda z \phi'(z)}{\beta} &= -\frac{z^{p+1}}{p}\left\{(1-\lambda)\left(P_{\beta,p}^{\alpha} f(z)\right)' + \lambda\left(P_{\beta,p}^{\alpha-1} f(z)\right)'\right\} \\ &\prec \frac{1+Az}{1+Bz}\ (z \in U). \end{aligned} \tag{23}$$

Now, by using Lemma 1 for $\gamma = \frac{\beta}{\lambda}$, we deduce that

$$\begin{aligned}\phi(z) &\prec q_1(z) = \frac{\beta}{\lambda} z^{-\frac{\beta}{\lambda(p+m)}} \int_0^z t^{\frac{\beta}{\lambda(p+m)}-1} (\frac{1+At}{1+Bt}) dt \\ &= \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1} \ {}_2F_1(1,1;\frac{\beta}{\lambda(p+m)}+1;\frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{\beta}{\beta+\lambda(p+m)} Az & (B = 0), \end{cases}\end{aligned}$$

by change of variables followed by the use of the identities (16) and (17) (with $a = 1, b = \frac{\beta}{\lambda}$ and$c = b+1$). This proves the assertion (20) of Theorem 1. Next, to prove the assertion (21) of Theorem 1, it suffices to show that

$$\inf_{|z|<1} \{\mathrm{Re}(q_1(z))\} = q_1(-1). \tag{24}$$

Indeed, for $|z| \leq r < 1$,

$$\mathrm{Re}(\frac{1+Az}{1+Bz}) \geq \frac{1-Ar}{1-Br}.$$

Setting

$$G(s,z) = \frac{1+Asz}{1+Bsz} \text{ and } d\mu(s) = \frac{\beta}{\lambda(p+m)} s^{\frac{\beta}{\lambda(p+m)}-1} ds \ (0 \leq s \leq 1),$$

which is a positive measure on $[0,1]$, we get

$$q_1(z) = \int_0^1 G(s,z) d\mu(s),$$

so that

$$\mathrm{Re}(q_1(z)) \geq \int_0^1 \frac{1-Asr}{1-Bsr} d\mu(s) = q_1(-r) \ (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (24). The result in (21) is best possible as the function $q_1(z)$ is the best dominant of (20).

Putting $\lambda = \frac{\sigma}{1-\sigma(p+1)} \beta (0 < \sigma < \frac{1}{p+1}; \beta > 0)$ in Theorem 1, we get the following result.

Corollary 1. *If $f(z) \in \sum_{p,m}$ satisfies*

$$\frac{-z^{p+1}[\left(P^{\alpha}_{\beta,p} f(z)\right)' + \sigma z \left(P^{\alpha}_{\beta,p} f(z)\right)'']}{p[1-\sigma(p+1)]} \prec \frac{1+Az}{1+Bz} \ (z \in U),$$

then

$$-\frac{z^{p+1}\left(P^{\alpha}_{\beta,p} f(z)\right)'}{p} \prec q_2(z) \prec \frac{1+Az}{1+Bz} \ (z \in U),$$

where the function $q_2(z)$ *given by*

$$q_2(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1} \; {}_2F_1(1,1;\frac{1-\sigma(1-m)}{\sigma(p+m)};\frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{1-\sigma(p+1)}{1-\sigma(1-m)}Az & (B = 0), \end{cases}$$

is the best dominant of (20). Furthermore,

$$\text{Re}\left(-\frac{z^{p+1}\left(P^{\alpha}_{\beta,p}f(z)\right)'}{p}\right) > \rho\ (p,\sigma,A,B)\ (z \in U),$$

where

$$\rho(p,\sigma,A,B) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1} \; {}_2F_1(1,1;\frac{1-\sigma(1-m)}{\sigma(p+m)};\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{1-\sigma(p+1)}{1-\sigma(1-m)}A & (B = 0). \end{cases}$$

The result is the best possible.

Remark 1. *For* $m = \alpha = 0$ *and* $p = 1$*, Corollary 1 reduces to the recent result of Patel and Sahoo [10, Theorem 1].*

Taking $A = 1 - \frac{2\delta}{p}$ $(0 \leq \delta < p)$, $B = -1, m = 2-p$ and $\lambda = \beta$ $(\beta > 0)$ in Theorem 1 and using (18), we have the following corollary.

Corollary 2. *If* $f(z) \in \sum_{p,2-p}$ *satisfies the following inequality*

$$\text{Re}\{-z^{p+1}[(p+2)\left(P^{\alpha}_{\beta,p}f(z)\right)' + z\left(P^{\alpha}_{\beta,p}f(z)\right)'']\} > \delta\ (0 \leq \delta < p;\ z \in U),$$

then

$$\text{Re}\{-z^{p+1}\left(P^{\alpha}_{\beta,p}f(z)\right)'\} > \delta + (p-\delta)(\frac{\pi}{2}-1)\ (z \in U).$$

The result is the best possible.

Remark 2. *For* $\alpha = 0$*, Corollary 2 reduces to the recent result of Srivastava and Patel [11, Corollary 2].*

Taking $\delta = -\frac{p(\pi-2)}{4-\pi}$ in Corollary 2, we have the following corollary.

Corollary 3. *If* $f(z) \in \sum_{p,2-p}$ *satisfies the following inequality*

$$\text{Re}\{-z^{p+1}[(p+2)\left(P^{\alpha}_{\beta,p}f(z)\right)' + z\left(P^{\alpha}_{\beta,p}f(z)\right)'']\} > -\frac{p(\pi-2)}{4-\pi}\ (z \in U),$$

then

$$\text{Re}\{-z^{p+1}\left(P^{\alpha}_{\beta,p}f(z)\right)'\} > 0\ (z \in U).$$

The result is the best possible.

Remark 3. *For $\alpha = 0$, Corollary 3 reduces to the result of Pap [8].*

Taking $A = 1 - \frac{2\delta}{p}$ $(0 \leq \delta < p)$, $B = -1, m = 1 - p$ and$\lambda = \beta$ $(\beta > 0)$ in Theorem 1 and using (19), we have the following corollary.

Corollary 4. *If $f(z) \in \sum_p$ satisfies the following inequality*

$$\mathrm{Re}\{-z^{p+1}[(p+2)\left(P^{\alpha}_{\beta,p}f(z)\right)' + z\left(P^{\alpha}_{\beta,p}f(z)\right)'']\} > \delta \ (0 \leq \delta < p;\ z \in U),$$

then

$$\mathrm{Re}\{-z^{p+1}\left(P^{\alpha}_{\beta,p}f(z)\right)'\} > p + 2(p-\delta)(\ln 2 - 1) \ (z \in U).$$

The result is the best possible.

Theorem 2. *If $f \in \sum^{Q}_{p,m}(\beta,\alpha,\lambda,A,B)$ $(\beta > -1)$, then*

$$-\frac{z^{p+1}\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{p} \prec q_3(z) \prec \frac{1+Az}{1+Bz} \ (\beta > -1; z \in U), \tag{25}$$

where the function $q_3(z)$ given by

$$q_3(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}\ {}_2F_1(1,1;\frac{\beta+\alpha-1}{\lambda(p+m)}+1;\frac{Bz}{Bz+1}) & (B \neq 0) \\ 1 + \frac{\beta+\alpha-1}{\beta+\alpha+\lambda(p+m)-1}Az & (B = 0), \end{cases}$$

is the best dominant of (25). Furthermore,

$$\mathrm{Re}\left(-\frac{z^{p+1}\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{p}\right) > \eta \ (z \in U), \tag{26}$$

where

$$\eta(\beta,\alpha,p,\lambda,A,B) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}\ {}_2F_1(1,1;\frac{\beta+\alpha-1}{\lambda(p+m)}+1;\frac{B}{B-1}) & (B \neq 0) \\ 1 - \frac{\beta+\alpha-1}{\beta+\alpha+\lambda(p+m)-1}A & (B = 0). \end{cases}$$

The result is the best possible.

Proof. Setting

$$\phi(z) = -\frac{z^{p+1}\left(Q^{\alpha}_{\beta,p}f(z)\right)'}{p} \ (z \in U). \tag{27}$$

Then the function $\phi(z)$ is of the form (12) and is analytic in U. Differentiating (27), and with the aid of the identity (9) we get

$$\begin{aligned}\phi(z)+\frac{\lambda z\phi^{'}(z)}{\beta+\alpha-1} &= -\frac{z^{p+1}}{p}\left\{(1-\lambda)\left(Q^{\alpha}_{\beta,p}f(z)\right)^{'}+\lambda\left(Q^{\alpha-1}_{\beta,p}f(z)\right)^{'}\right\}\\ &\prec \frac{1+Az}{1+Bz}\ (z\in U). \end{aligned} \tag{28}$$

Now, by using Lemma 1 for $\gamma=\frac{\beta+\alpha-1}{\lambda}$, we deduce that

$$\phi(z)\prec q(z)=\frac{\beta+\alpha-1}{\lambda(p+m)}z^{-\frac{\beta+\alpha-1}{\lambda(p+m)}}\int_0^z t^{\frac{\beta+\alpha-1}{\lambda(p+m)}-1}(\frac{1+At}{1+Bt})dt,$$

and the proof is completed similarly to Theorem 1.

Replacing $\phi(z)$ by $z^pP^{\alpha}_{\beta,p}f(z)$ in (22) and following the lines of the proof of Theorem 1, we can prove the following result.

Theorem 3. *If $f\in\sum_{p,m}$ satisfies*

$$z^p\left\{(1-\lambda)P^{\alpha}_{\beta,p}f(z)+\lambda P^{\alpha-1}_{\beta,p}f(z)\right\}\prec\frac{1+Az}{1+Bz}\ (\beta>0;z\in U),$$

then

$$z^pP^{\alpha}_{\beta,p}f(z)\prec q_1(z)\prec\frac{1+Az}{1+Bz}\ (\beta>0;z\in U),$$

and

$$\text{Re}\left(z^pP^{\alpha}_{\beta,p}f(z)\right)>\rho\ (\beta>0;z\in U),$$

where q_1 and ρ are given as in Theorem 1. The result is the best possible.

Replacing $\phi(z)$ by $z^pQ^{\alpha}_{\beta,p}f(z)$ in (27) and following the lines of the proof of Theorem 2, we can prove the following result.

Theorem 4. *If $f\in\sum_{p,m}$ satisfies*

$$z^p\left\{(1-\lambda)Q^{\alpha}_{\beta,p}f(z)+\lambda Q^{\alpha-1}_{\beta,p}f(z)\right\}\prec\frac{1+Az}{1+Bz}\ (\beta>-1;z\in U),$$

then

$$z^pQ^{\alpha}_{\beta,p}f(z)\prec q_3(z)\prec\frac{1+Az}{1+Bz}\ (\beta>-1;z\in U),$$

and

$$\text{Re}\left(z^pQ^{\alpha}_{\beta,p}f(z)\right)>\eta\ (\beta>-1;z\in U),$$

where q_3 and η are given as in Theorem 2. The result is the best possible.

Theorem 5. *Let* $-1 \leq B_j < A_j \leq 1$ $(j = 1,2)$ *and* $\beta > 0$. *If each of the functions* $f_j(z) \in \sum_p$ *satisfies the following subordination condition*

$$z^p \left\{(1-\lambda)P^{\alpha}_{\beta,p}f_j(z) + \lambda P^{\alpha-1}_{\beta,p}f_j(z)\right\} \prec \frac{1+A_jz}{1+B_jz} \; (j = 1,2; z \in U). \tag{29}$$

then

$$z^p \left\{(1-\lambda)P^{\alpha}_{\beta,p}H(z) + \lambda P^{\alpha-1}_{\beta,p}H(z)\right\} \prec \frac{1+(1-\frac{2\gamma}{p})z}{1-z} \; (z \in U), \tag{30}$$

where

$$H(z) = P^{\alpha}_{\beta,p}(f_1 * f_2)(z) \tag{31}$$

and

$$\gamma = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}[1 - \frac{1}{2}\,{}_2F_1(1,1;\frac{\beta}{\lambda}+1;\frac{1}{2})].$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Suppose that each of the functions $f_j(z) \in \sum_p$ $(j = 1,2)$ satisfies the condition (29). Then, by letting

$$\varphi_j(z) = z^p\{(1-\lambda)P^{\alpha}_{\beta,p}\left(f_j(z)\right) + \lambda P^{\alpha-1}_{\beta,p}\left(f_j(z)\right)\} \; (j = 1,2), \tag{32}$$

we have

$$\varphi_j(z) \in P(\gamma_j) \; (\gamma_j = \frac{1-A_j}{1-B_j};\; j = 1,2).$$

Making use of the identity (8) in (32), we have

$$P^{\alpha}_{\beta,p}\left(f_j(z)\right) = \frac{\beta}{\lambda}z^{-\frac{\beta}{\lambda}-p}\int_0^z t^{\frac{\beta}{\lambda}-1}\varphi_j(t)dt \; (j = 1,2). \tag{33}$$

From (31) and (33), we get

$$\begin{aligned} P^{\alpha}_{\beta,p}H(z) &= \left(\frac{\beta}{\lambda}z^{-\frac{\beta}{\lambda}-p}\int_0^z t^{\frac{\beta}{\lambda}-1}\varphi_1(t)dt\right) * \left(\frac{\beta}{\lambda}z^{-\frac{\beta}{\lambda}-p}\int_0^z t^{\frac{\beta}{\lambda}-1}\varphi_2(t)dt\right) \\ &= \frac{\beta}{\lambda}z^{-\frac{\beta}{\lambda}-p}\int_0^z t^{\frac{\beta}{\lambda}-1}\varphi_0(t)dt \end{aligned} \tag{34}$$

where

$$\begin{aligned} \varphi_0(z) &= z^p\left\{(1-\lambda)P^{\alpha}_{\beta,p}H(z) + \lambda P^{\alpha-1}_{\beta,p}H(z)\right\} \\ &= \frac{\beta}{\lambda}z^{-\frac{\beta}{\lambda}}\int_0^z t^{\frac{\beta}{\lambda}-1}\left(\varphi_1 * \varphi_2\right)(t)dt. \end{aligned} \tag{35}$$

Since $\varphi_1(z) \in P(\gamma_1)$ and $\varphi_2(z) \in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(z) \in P(\gamma_3) \; (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)). \tag{36}$$

According to Lemma 2, we have

$$\mathrm{Re}\{(\varphi_1 * \varphi_2)(z) \geq 2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+|z|}. \tag{37}$$

Now by using (37) in (35) and then appealing to Lemma 4, we get

$$\begin{aligned}
\mathrm{Re}\{\varphi_o(z)\} &= \frac{\beta}{\lambda}\int_0^1 u^{\frac{\beta}{\lambda}-1}\,\mathrm{Re}\{(\varphi_1 * \varphi_2)(uz)\}du \\
&\geq \frac{\beta}{\lambda}\int_0^1 u^{\frac{\beta}{\lambda}-1}(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|})du \\
&> \frac{\beta}{\lambda}\int_0^1 u^{\frac{\beta}{\lambda}-1}(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u})du \\
&= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}[1 - \frac{\beta}{\lambda}\int_0^1 u^{\frac{\beta}{\lambda}-1}(1+u)^{-1}du] \\
&= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}[1 - \frac{1}{2}{}_2F_1(1,1;\frac{\beta}{\lambda}+1;\frac{1}{2})] \\
&= \gamma \ (z \in U).
\end{aligned}$$

which completes the proof of the assertion (30).

When $B_1 = B_2 = -1$, we consider the functions $f_j(z) \in \sum_p (j = 1,2)$, which satisfy (29) and

$$P^{\alpha}_{\beta,p}\left(f_j(z)\right) = \frac{\beta}{\lambda} z^{-\frac{\beta}{\lambda}-p}\int_0^z t^{\frac{\beta}{\lambda}-1}\left(\frac{1+A_j t}{1-t}\right)dt \ (j = 1,2),$$

for which we have

$$\varphi_j(z) = \frac{1+A_j t}{1-t} \ (j = 1,2)$$

and

$$(\varphi_1 * \varphi_2)(z) = 1 + \frac{(1+A_1)(1+A_2)z}{1-z}.$$

Thus it follows from (35) and Lemma 4 that

$$\begin{aligned}
\varphi_o(z) &= \frac{\beta}{\lambda}\int_0^1 u^{\frac{\beta}{\lambda}-1}\left(1-(1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right)du \\
&= 1-(1+A_1)(1+A_2) + (1+A_1)(1+A_2)(1-z)^{-1}\ {}_2F_1(1,1;\frac{\beta}{\lambda}+1;\frac{z}{z-1}) \\
&\to 1-(1+A_1)(1+A_2) + \frac{1}{2}(1+A_1)(1+A_2){}_2F_1(1,1;\frac{\beta}{\lambda}+1;\frac{1}{2}) \\
&\quad \text{as } z \to -1,
\end{aligned}$$

which evidently completes the proof of Theorem 5.

Letting $A_j = 1 - \frac{2\eta_j}{p}$ $(0 \leq \eta_j < p)$, $B_j = -1 (j = 1,2)$, $\alpha = 0$ and $\frac{\lambda}{\beta} = \tau$, in Theorem 5, we get the following result.

Corollary 5. *If* $f \in \sum_p$ *satisfies*

$$\operatorname{Re} z^p \left\{(1+p\tau)f_j(z) + \tau z f_j^{'}(z)\right\} > \eta_j \ (j=1,2;\ z \in U),$$

then

$$\operatorname{Re} z^p \left\{(1+p\tau)(f_1 * f_2)(z) + \tau z \left((f_1 * f_2)(z)\right)^{'}\right\} > \gamma,$$

where

$$\gamma = 1 - 4(1-\frac{\eta_1}{p})(1-\frac{\eta_2}{p})[1-\frac{1}{2}{}_2F_1(1,1;\frac{1}{\tau}+1;\frac{1}{2})].$$

Remark 4. *For* $p=1$*, the result (asserted by Corollary 5 above) was also obtained by Yang [15].*

Theorem 6. *Let* $-1 \le B_j < A_j \le 1$ $(j=1,2)$ *and* $\beta > -1$*. If each of the functions* $f_j(z) \in \sum_p$ *satisfies the following subordination condition*

$$z^p \left\{(1-\lambda)Q^{\alpha}_{\beta,p} f_j(z) + \lambda Q^{\alpha-1}_{\beta,p} f_j(z)\right\} \prec \frac{1+A_j z}{1+B_j z} \ (j=1,2; z \in U).$$

then

$$z^p \left\{(1-\lambda)Q^{\alpha}_{\beta,p} E(z) + \lambda Q^{\alpha-1}_{\beta,p} E(z)\right\} \prec \frac{1+(1-\frac{2\xi}{p})z}{1-z} \ (z \in U),$$

where

$$E(z) = Q^{\alpha}_{\beta,p}(f_1 * f_2)(z)$$

and

$$\xi = 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}[1-\frac{1}{2}{}_2F_1(1,1;\frac{\beta+\alpha-1}{\lambda}+1;\frac{1}{2})].$$

The result is the best possible when $B_1 = B_2 = -1$.

The proof is similar to Theorem 5.

References

[1] M. Aouf, New criteria for multivalent meromorphic starlike functions of order alpha, Proc. Japan Acad. Ser. A, 69, 65-70. 1993.

[2] E. Aqlan, J. Jahangiri and S. Kulkarni, Certain integral operators applied to meromorphic p-valent functions, J. Nat. Geom., 24, 111-120. 2003.

[3] T. Bulboaca, Differential Subordinations and Superordinations, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[4] D. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc., 52, 191-195. 1975.

[5] V. Kumar and S. Shukla, Certain integrals for classes of p-valent meromorphic functions, Bull. Austral. Math. Soc. 25, 85–97. 1982.

[6] A. Lashin, On certain subclasses of meromorphic functions associated with certain integral operators, Comput. Math. Appl. (To appear).

[7] S. Miller and P. Mocanu, Differential subordinations: Theory and Applications, Series on Monographs and Textbooks in pure and Applied mathematics (No. 225),Marcel Dekker, New York and Basel, 2000.

[8] M. Pap, On certain subclasses of meromorphic m-valent close-to-convex functions, Pure Math. Appl., 9, 155-163. 1998.

[9] D. Pashkouleva, The starlikeness and spiral-convexity of analytic functions, in: H.M. Srivastava and S.Owa (Editors), Current Topics in Analytic Function Theory, pp. 266-273, World Scientific Puplishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[10] J. Patel and P. Sahoo, On certain subclass of meromorphically univalent functions, Kyungpook Math. J. 42, 121-132. 2002.

[11] H. Srivastava and J. Patel, Applications of differential subordination to certain subclasses of meromorphically multivalent functions, J. Inequal. Pure Appl. Math. 6, no. 3, Art. 88, 1-15. 2005.

[12] J. Stankiewicz and Z. Stankiewicz, Some applications of the Hadamard convolution in the theory of functions, Ann. Univ. Mariae Curie-Sklodowska Sect. A, 40, 251-265. 1986.

[13] B. Uralegaddi and C. Somanatha, Certain classes of meromorphic multivalent functions, Tamkang J. Math. 23, 223-231. 1992.

[14] E. Whittaker and G. Watson, A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, Fourth Edition (Reprinted), Cambridge University Press, Cambridge, 1927.

[15] D. Yang, Certain convolution operators for meromorphic functions, South. Asian Bull. Math. 25, 175-186. 2001.

[16] D. Yang, On a class of meromorphic starlike multivalent functions, Bull. Inst. Math. Acad. Sinica, 24, 151-157. 1996.