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Convolution And Rayleigh's Theorem For Generalized Fractional Hartley Transform

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Abstract. The fractional Hartley transform, which is a generalization of the Hartley transform, has many applications in several areas, including signal processing and optics.

In this paper we have introduced convolution theorem, modulation theorem and Parseval's identity (Rayleigh's Theorem) for the generalized fractional Hartley transform.

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Key Words: Convolution theorem, Modulation theorem, Parseval's identity, Hartley transform.

1. Introduction:

The fractional Fourier transform has become the focus of many research papers, because of its recent applications in many fields, including optics and signal processing. Hence fractional Hartley transform is also useful tool in many fields, due to it's close relation with fractional Fourier transforms.

Many properties of the fractional Fourier transform are well known, including its product and convolution theorem, which have been derived by Almeida [5] and Zayed [2].

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Using the eigen value function as used in fractional Fourier transform, different integral transform in Fourier class, including Hartley transform are generalized to fractional transform by Pei [4]. He had shown that for all non negative integer m, $e^{\frac{-t^2}{2}} H_m(t)$ is the eigen function of the Hartley transform and had given the formula for fractional Hartley transform as,

$$H^{\alpha}\{f(t)\}(s) = \int_{-\infty}^{\infty} f(t) K_{\alpha}(t,s)dt,$$

where

$$K_{\alpha}(t,s) = \sqrt{\frac{1-i\cot\phi}{2\pi}} e^{i\frac{s^2}{2}\cot\phi} . e^{i\frac{t^2}{2}\cot\phi} \frac{1}{2}\left[(1-ie^{i\phi})cas(\csc\phi.st) + (1+ie^{i\phi})cas(-\csc\phi.st)\right]. \quad (1.1)$$

Almedia [5] had defined convolution for the fractional Fourier transform as

$$H_{\alpha}(u) = \left|\sec\alpha\right| e^{-i\frac{u^2}{2}\tan\alpha} \int_{-\infty}^{\infty} F_{\alpha}(v) g\left[(u-v)\sec\alpha\right] . e^{i\left(\frac{v^2}{2}\right)\tan\alpha} dv \quad (1.2)$$

Since the convolution theorem for the Fourier transform, which states that the Fourier transform of the convolution of two functions is the product of their Fourier transform, the one for the fractional Fourier transform does not seem as nice or as practical. The reason is that the convolution operation defined by (1.2) is not the right sort of convolution for the fractional Fourier transform. Zayed had defined fractional Fourier type convolution as follows. For any function $f(t)$, if $\bar{f}(t) = f(t)e^{i\cot\phi\frac{t^2}{2}}$ then for any two function f and g the convolution operation $*$ is defined by Zayed [2] as,

$$h(t) = (f \star g)(t) = \sqrt{\frac{1-i\cot\phi}{2\pi}} . e^{-i\cot\phi\frac{t^2}{2}} (\bar{f} * \bar{g})(t),$$

where $*$ is the convolution operation for the Fourier transform as defined by (1.2).

In this paper first we have defined generalized fractional Hartley transform in section 2. We have proved convolution theorem for fractional Hartley transform in section 3. Also discussed the modulation theorem and Parseval's identity in section 4.

2. Generalized fractional Hartley transform

2.1 The test function space $E(R^n)$

An infinitely differentiable complex valued function ψ on R^n belongs to $E(R^n)$ if for each compact set $K \subset S_a$ where $S_a = \{t \in R^n, |t| \le a, a > 0\}$,

$$\gamma_{E,k}(\psi) = \sup_{t \in K} \left| D_t^k \psi(t) \right| < \infty, \qquad k = 1,2,3,.....$$

Note that the space E is complete and therefore a Frechet space.

2.2 The fractional Hartley transform on E'

It can be easily proved that function $K_\alpha(t,s)$ as a function of t, is a member of $E(R^n)$, where

$$K_\alpha(t,s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi}.e^{i\frac{t^2}{2}\cot\phi}\,\frac{1}{2}\left[(1 - ie^{i\phi})cas(\csc\phi.st) + (1 + ie^{i\phi})cas(-\csc\phi.st)\right],$$

and $\phi = \dfrac{\alpha\pi}{2}$.

The generalized fractional Hartley transform of $f(t) \in E'(R^n)$, where $E'(R^n)$ is the dual of the testing function space, can be defined as,

$$H^\alpha\{f(t)\}(s) = \langle f(t), K_\alpha(t,s)\rangle. \qquad (2.2.1)$$

Another simple form of fractional Hartley transform as in Sontakke [3] is

$$H^\alpha\{f(t)\}(s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} \int_{-\infty}^{\infty} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right] f(t)\,dt \qquad (2.2.2)$$

3. Convolution of fractional Hartley transform

3.1 Convolution theorem:

We define fractional Hartley type convolution as follows.

For any function $f(t)$, define the function $\bar{f}(t) = f(t)e^{i\cot\phi\frac{t^2}{2}}$. Then for any two function f and g we define the convolution operation '$*$' by

$$h(t) = (f \star g)(t) = \sqrt{\frac{1-i\cot\phi}{2\pi}}.e^{-i\cot\phi\frac{t^2}{2}}(\bar{f} * \bar{g})(t).$$

Now we state and prove our convolution theorem.

Theorem:

Let $h(t) = (f \star g)(t)$ and $H^{\alpha}\{h(t)\}$, $H^{\alpha}\{f(t)\}$ and $H^{\alpha}\{g(t)\}$ denote the fractional Hartley transform of $h(t)$, $f(t)$ and $g(t)$ respectively, then

$$2e^{i\frac{s^2}{2}\cot\phi}H^{\alpha}\{h(t)\}(s) = H^{\alpha}\{g(t)\}(\mathrm{s})\left[\left(H^{\alpha}\{f(t)\}(s) - H^{\alpha}\{f(-t)\}(s)\right)\right.$$
$$\left.+e^{-i\phi}.\cos\phi\left(H^{\alpha}\{f(t)\}(s) + H^{\alpha}\{f(-t)\}(s)\right)\right] + e^{-i\phi}.\sin\phi H^{\alpha}\{g(-t)\}(\mathrm{s})\left(H^{\alpha}\{f(-t)\}(s) - H^{\alpha}\{f(t)\}(s)\right)$$

Proof: From the definition of the fractional Hartley transform (2.2.2) and definition of convolution we have,

$$H^{\alpha}\{h(t)\}(s) = \sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]h(t)dt\,.$$

$$= \left(\sqrt{\frac{1-i\cot\phi}{2\pi}}\right)^2 e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]e^{-i\frac{t^2}{2}\cot\phi}\int_{-\infty}^{\infty}f(u)e^{i\frac{u^2}{2}\cot\phi}$$
$$g(t-u)e^{i\cot\phi\frac{(t-u)^2}{2}}dudt$$

$$= \left(\sqrt{\frac{1-i\cot\phi}{2\pi}}\right)^2 e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)g(t-u)\,e^{i\cot\phi\frac{u^2}{2}}e^{i\cot\phi\frac{(t-u)^2}{2}}\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]dtdu.$$

By making the change of variable $t - u = v$, we obtain

$$= \left(\sqrt{\frac{1-i\cot\phi}{2\pi}}\right)^2 e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)g(v)\,e^{i\cot\phi\frac{u^2}{2}}e^{i\cot\phi\frac{v^2}{2}}\left[\cos(\csc\phi.s(u+v)) - ie^{i\phi}\sin(\csc\phi.s(u+v)\right]dvdu$$

$$e^{i\frac{s^2}{2}\cot\phi}H^{\alpha}\{h(t)\}(\mathrm{s}) = C^{\alpha}\{f(u)\}H^{\alpha}\{g(v)\}(\mathrm{s}) - e^{-i\left(\phi-\frac{\pi}{2}\right)}S^{\alpha}\{f(u)\}\left[i\cos\phi H^{\alpha}\{g(v)\}(s)\right.$$
$$\left.-i\sin\phi H^{\alpha}\{g(-v)\}(s)\right],$$

where C^{α} and S^{α} denote fractional cosine and fractional sine transform.
Using the relation of fractional cosine transform and fractional sine transform to fractional Hartley transform, we get

$$2e^{i\frac{s^2}{2}\cot\phi}H^{\alpha}\{h(t)\}(s)=H^{\alpha}\{g(v)\}\left[\left(H^{\alpha}\{f(u)\}-H^{\alpha}\{f(-u)\}\right)+e^{-i\phi}.\cos\phi\left(H^{\alpha}\{f(u)\}+H^{\alpha}\{f(-u)\}\right)\right]$$
$$+e^{-i\phi}\sin\phi\,H^{\alpha}\{g(-v)\}\left(H^{\alpha}\{f(-u)\}-H^{\alpha}\{f(u)\}\right)$$

i.e.

$$2e^{i\frac{s^2}{2}\cot\phi}H^{\alpha}\{h(t)\}(\mathrm{s})=H^{\alpha}\{g(t)\}(\mathrm{s})\left[\left(H^{\alpha}\{f(t)\}(s)-H^{\alpha}\{f(-t)\}(s)\right)\right.$$
$$\left.+e^{-i\phi}.\cos\phi\left(H^{\alpha}\{f(t)\}(s)+H^{\alpha}\{f(-t)\}(s)\right)\right]$$
$$+e^{-i\phi}\sin\phi\,H^{\alpha}\{g(-t)\}(s)\left(H^{\alpha}\{f(-t)\}(s)-H^{\alpha}\{f(t)\}(s)\right) \qquad (3.1.1)$$

3.2 Convolution of various combinations of even and odd functions:

Next we consider different cases of convolution for even and odd functions.

Case I: If the function *f* and *g* both are odd functions, i.e. $f(-t)=-f(t)$ and $g(-t)=-g(t)$ then

$$H^{\alpha}\{h(t)\}(s)=H^{\alpha}\{f*g\}(s)=\left(1+e^{i\phi}\sin\phi\right)e^{-i\frac{s^2}{2}\cot\phi}H^{\alpha}\{f(t)\}(s)H^{\alpha}\{g(t)\}(s).$$

Case II: If the function *f* and *g* both are even function i.e. $f(-t)=f(t)$ and $g(-t)=g(t)$ then

$$H^{\alpha}\{h(t)\}(s)=H^{\alpha}\{f*g\}(s)=e^{-i\phi}\cos\phi.e^{-i\frac{s^2}{2}\cot\phi}.H^{\alpha}\{f(t)\}(s).H^{\alpha}\{g(t)\}(s)$$
$$=e^{-i\left(\frac{s^2}{2}\cot\phi+\phi\right)}\cos\phi.H^{\alpha}\{f(t)\}(s).H^{\alpha}\{g(t)\}(s).$$

Case III: If *f* is even, *g* is odd then

$$H^{\alpha}\{f*g\}(s)=H^{\alpha}\{h(t)\}(s)=e^{-i\left(\frac{s^2}{2}\cot\phi+\phi\right)}\cos\phi.H^{\alpha}\{f(t)\}(s).H^{\alpha}\{g(t)\}(s).$$

Case IV: If *f* is odd, *g* is even then

$$H^{\alpha}\{h(t)\}(s)=H^{\alpha}\{f*g\}(s)=\left(1-e^{-i\phi}\sin\phi\right)e^{-i\frac{s^2}{2}\cot\phi}H^{\alpha}\{g(t)\}(s)H^{\alpha}\{f(t)\}(s).$$

Case V: If *f* is even function and *g* any function then,

$$H^{\alpha}\{h(t)\}(s) = H^{\alpha}\{f * g\}(s) = e^{-i\left(\frac{s^2}{2}\cot\phi+\phi\right)}\cos\phi . H^{\alpha}\{f(t)\}(s) . H^{\alpha}\{g(t)\}(s).$$

Case VI: If *f* is odd function and *g* any function then,

$$H^{\alpha}\{h(t)\}(s) = H^{\alpha}\{f * g\}(s) = e^{-i\frac{s^2}{2}\cot\phi} H^{\alpha}\{f(t)\}(s) \left(H^{\alpha}\{g(t)\}(s) - e^{-i\phi}\sin\phi\, H^{\alpha}\{g(-t)\}(s)\right).$$

Case VII: If *f* is any function and *g* even function then,

$$2e^{i\frac{s^2}{2}\cot\phi} H^{\alpha}\{h(t)\}(s) = H^{\alpha}\{f * g\}(s)$$
$$= H^{\alpha}\{g(t)\}(s)\left[H^{\alpha}\{f(t)\}(s)\left(1+e^{-i\phi}(\cos\phi-\sin\phi)\right) + H^{\alpha}\{f(-t)\}(s)\left(e^{-i\phi}(\cos\phi+\sin\phi)-1\right)\right].$$

Case VIII: If *f* is any function and *g* is odd function then,

$$2e^{i\frac{s^2}{2}\cot\phi} H^{\alpha}\{h(t)\}(s) = H^{\alpha}\{g(t)\}(s)\left[H^{\alpha}\{f(t)\}(s)\left(1+e^{-i\phi}(\sin\phi+\cos\phi)\right) - H^{\alpha}\{f(-t)\}(s)\left(1+e^{-i\phi}(\sin\phi-\cos\phi)\right)\right].$$

4. Modulation Theorem for fractional Hartley transform

4.1 If $H^{\alpha}\{f(t)\}(s)$ is fractional Hartley transform of $f(t)$ then

$$H^{\alpha}\{f(t)\cos ut\}(s) = \frac{1}{2}e^{-i\frac{u^2}{4}\sin 2\phi}\left\{e^{-isu\cos\phi}H^{\alpha}\{f(t)\}(s+u\sin\phi) + e^{isu\cos\phi}H^{\alpha}\{f(t)\}(s-u\sin\phi)\right\}.$$

Proof: Using the definition of fractional Hartley transform

$$H^{\alpha}\{f(t)\cos ut\}(s) = \sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi . st) - ie^{i\phi}\sin(\csc\phi . st)\right] . f(t)\cos ut\,dt,$$

by solving, we get

$$H^{\alpha}\left\{f(t)\cos ut\right\}(s)=\frac{1}{2}e^{-i\frac{u^2}{4}\sin 2\phi}\left\{e^{-isu\cos\phi}H^{\alpha}\left\{f(t)\right\}(s+u\sin\phi)\right.$$
$$\left.+e^{isu\cos\phi}H^{\alpha}\left\{f(t)\right\}(s-u\sin\phi)\right\}. \qquad (4.1.1)$$

4.2 If $H^{\alpha}\left\{f(t)\right\}(s)$ is fractional Hartley transform of $f(t)$, then

$$H^{\alpha}\left\{f(t)\sin ut\right\}(s)=\frac{1}{2}e^{-i\frac{u^2}{4}\sin 2\phi}\left\{\left[e^{-isu\cos\phi}\left(i\cot\phi H^{\alpha}\left\{f(t)\right\}(s+u\sin\phi)-\sin\phi H^{\alpha}\left\{f(-t)\right\}\right.\right.\right.$$
$$\left.\left.\left.(s+u\sin\phi)\right)\right]-\left[e^{isu\cos\phi}\left(i\cot\phi H^{\alpha}\left\{f(t)\right\}(s-u\sin\phi)-\sin\phi H^{\alpha}\left\{f(-t)\right\}(s-u\sin\phi)\right)\right]\right\}$$

Proof: Using the definition of fractional Hartley transform

$$H^{\alpha}\left\{f(t)\sin ut\right\}(\mathrm{s})=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]f(t)\sin ut\,dt$$

by solving, we get

$$H^{\alpha}\left\{f(t)\sin ut\right\}(\mathrm{s})=\frac{1}{2}e^{-i\frac{u^2}{4}\sin 2\phi}\left\{\left[e^{-isu\cos\phi}\left(i\cot\phi H^{\alpha}\left\{f(t)\right\}(s+u\sin\phi)-\sin\phi H^{\alpha}\left\{f(-t)\right\}(s+u\sin\phi)\right)\right]\right.$$
$$\left.-\left[e^{isu\cos\phi}\left(i\cot\phi H^{\alpha}\left\{f(t)\right\}(s-u\sin\phi)-\sin\phi H^{\alpha}\left\{f(-t)\right\}(s-u\sin\phi)\right)\right]\right\}. \qquad (4.2.1)$$

4.3 If $H^{\alpha}\left\{f(t)\right\}(s)$ is fractional Hartley transform of $f(t)$ then

$$H^{\alpha}\left\{f(t)e^{iut}\right\}(\mathrm{s})=\frac{1}{2}e^{-i\frac{u^2}{4}\sin 2\phi}\left\{(1-\cot\phi)e^{-isu\cot\phi}.H^{\alpha}\left\{f(t)\right\}(s+u\sin\phi)+(1+\cot\phi)e^{isu\cot\phi}\right.$$
$$\left.H^{\alpha}\left\{f(t)\right\}(s-u\sin\phi)-i\sin\phi\left(H^{\alpha}\left\{f(-t)\right\}(s+u\sin\phi)-H^{\alpha}\left\{f(-t)\right\}(s-u\sin\phi)\right)\right\}.$$

Proof: Using (4.1.1) and (4.2.1)

4.4 Parseval's identity for fractional Hartley transform:

Statement: If the fractional Hartley transform of $f(t)$ and $g(t)$ be $H^{\alpha}\left\{f(t)\right\}(s)$ and $H^{\alpha}\left\{g(t)\right\}(s)$ respectively then

$$1)\int_{-\infty}^{\infty} f(t).g^*(t)\,dt = \cos^2\frac{\phi}{2}\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(s).H^{\alpha}\{g(t)\}(s)\,ds + \sin^2\frac{\phi}{2}\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(-s).H^{\alpha}\{g(t)\}(-s)\,ds$$

$$-2i\sin\phi\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(s).H^{\alpha}\{g(t)\}(-s)\,ds + 2i\sin\phi\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(-s).H^{\alpha}\{g(t)\}(s)\,ds$$

and

$$2)\int_{-\infty}^{\infty} |f(t)|^2\,dt = \cos^2\frac{\phi}{2}\int_{-\infty}^{\infty}\left(H^{\alpha}\{f(t)\}(s)\right)^2 ds + \sin^2\frac{\phi}{2}\int_{-\infty}^{\infty}\left(H^{\alpha}\{f(t)\}(-s)\right)^2 ds,$$

where $g^*(t)$ is a complex conjugate of $g(t)$.

Proof: The Parseval's relation for the fractional Fourier transform is as follows,

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F_{\alpha}(s)G_{\alpha}^*(s)ds. \tag{4.4.1}$$

Now using the relation between fractional Fourier transform and fractional Hartley transform is as follows.

$$F_{\alpha}\{f(t)\}(\mathrm{s}) = \frac{1}{2}\left[(1+e^{-i\phi})H^{\alpha}\{f(t)\}(s) + (1-e^{-i\phi})H^{\alpha}\{f(t)\}(-s)\right]$$

ie

$$F_{\alpha}\{f(t)\}(\mathrm{s}) = \frac{1}{2}\begin{bmatrix} H^{\alpha}\{f(t)\}(s) + H^{\alpha}\{f(t)\}(-s) + \cos\phi\left(H^{\alpha}\{f(t)\}(s) - H^{\alpha}\{f(t)\}(-s)\right) \\ -i\sin\phi\left(H^{\alpha}\{f(t)\}(s) - H^{\alpha}\{f(t)\}(-s)\right) \end{bmatrix}$$

and

$$G_{\alpha}^{\bullet}\{g(t)\}(\mathrm{s}) = \frac{1}{2}\begin{bmatrix} H^{\alpha}\{g(t)\}(s) + H^{\alpha}\{g(t)\}(-s) + \cos\phi\left(H^{\alpha}\{g(t)\}(s) - H^{\alpha}\{g(t)\}(-s)\right) \\ +i\sin\phi\left(H^{\alpha}\{g(t)\}(s) - H^{\alpha}\{g(t)\}(-s)\right) \end{bmatrix}.$$

Where $G_{\alpha}^{\bullet}(s)$ is a complex conjugate of $G_{\alpha}(s)$, hence equation (4.4.1) becomes

$$\int_{-\infty}^{\infty} f(t).g^*(t)\,dt = \cos^2\frac{\phi}{2}\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(s).H^{\alpha}\{g(t)\}(s)\,ds + \sin^2\frac{\phi}{2}\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(-s).H^{\alpha}\{g(t)\}(-s)\,ds$$
$$-2i\sin\phi\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(s).H^{\alpha}\{g(t)\}(-s)\,ds + 2i\sin\phi\int_{-\infty}^{\infty} H^{\alpha}\{f(t)\}(-s).H^{\alpha}\{g(t)\}(s)\,ds.$$

In particular if $f = g$ then,

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \cos^2 \frac{\phi}{2} \int_{-\infty}^{\infty} \left(H^{\alpha}\{f(t)\}(s)\right)^2 ds + \sin^2 \frac{\phi}{2} \int_{-\infty}^{\infty} \left(H^{\alpha}\{f(t)\}(-s)\right)^2 ds.$$

5. Conclusion

We have proved convolution theorem, modulation theorem and Parseval's identity for fractional Hartley transform.

Convolution of two functions for fractional Hartley transform may be used in filter design.

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