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Single-Input Control Systems on the Euclidean Group $\mathsf{SE}(2)$

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Abstract. We consider a general single-input left-invariant control affine system, evolving on the Euclidean group $\mathsf{SE}(2)$. Any such controllable control system is (detached feedback) equivalent to one of two typical cases. In each case, we consider an optimal control problem (with quadratic cost) which is then lifted, via the Pontryagin Maximum Principle, to a Hamiltonian system on the dual space $\mathfrak{se}(2)^*$. The reduced Hamilton equations are derived and the stability nature of all equilibrium states is then investigated. Finally, these equations are explicitly integrated by elliptic functions.

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1. Introduction

In recent decades, attention has been drawn to invariant control systems evolving on matrix Lie groups of low dimension. Such systems arise, for instance, in the airplane landing problem, the motion planning for wheeled robots, and the control of underactuated underwater vehicles (see, e.g., [31, 21, 20, 22, 15] and the references therein).

An arbitrary left-invariant control affine system on the Euclidean group $\mathsf{SE}(2)$ has the form $\dot{g} = g\left(A + u_1 B_1 + \cdots + u_\ell B_\ell\right)$, where $A, B_1, \ldots, B_\ell \in \mathfrak{se}(2)$, $1 \leq \ell \leq 3$. (The elements $B_1, \ldots, B_\ell$ are assumed to be linearly independent.) Specific (left-invariant) optimal control problems on the Euclidean group $\mathsf{SE}(2)$, associated with above mentioned control systems, have been studied by several authors (see, e.g., [11, 10, 29, 24, 23, 28]).

In this paper, we consider only *single-input* control systems (i.e., systems of the form $\dot{g} = g\,(A + uB)$). Such a system is controllable if and only if it has full rank. Moreover,

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any such controllable control system is (detached feedback) equivalent to exactly one of the following control systems: Σ_1 or $\Sigma_{2,\alpha}$ ($\alpha > 0$) with trace $\Gamma_1 = E_1 + \langle E_3 \rangle$ or $\Gamma_{2,\alpha} = \alpha E_3 + \langle E_1 \rangle$, respectively. (Here E_1 and E_3 denote elements of the standard basis for $\mathfrak{se}(2)$.) In each typical case, we consider an optimal control problem (with quadratic cost) of the form

$$\begin{gathered} \dot{g} = g\,(A + uB)\,, \qquad g \in \mathsf{SE}(2),\ u \in \mathbb{R} \\ g(0) = g_0, \quad g(T) = g_T \\ \mathscr{J} = \tfrac{1}{2}\int_0^T u^2(t)dt \to \min. \end{gathered}$$

Each problem is lifted, via the Pontryagin Maximum Principle, to a Hamiltonian system on the dual of the Lie algebra $\mathfrak{se}(2)$. Then the (minus) Lie-Poisson structure on $\mathfrak{se}(2)^*$ is used to derive the equations for extrema (cf. [11, 1, 13]; see also [25, 26] for similar computations on the rotation group $\mathsf{SO}(3)$). The (Lyapunov) stability nature of all equilibrium states is then investigated (by the energy-Casimir method). Finally, these equations are explicitly integrated by elliptic functions.

2. Preliminaries

2.1. Invariant Control Systems

Invariant control systems on Lie groups were first considered in 1972 by Brockett [8] and by Jurdjevic and Sussmann [12]. A *left-invariant control system* Σ is a (smooth) control system evolving on a (real, finite-dimensional) Lie group G, whose dynamics $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ is invariant under left translations. (The tangent bundle $T\mathsf{G}$ is identified with $\mathsf{G} \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G). For the sake of convenience, we shall assume that (the state space of the system) G is a *matrix* Lie group. For the purposes of this paper, we may also assume that (the input space) $U = \mathbb{R}^\ell$. Such a control system is described as follows (cf. [11, 1, 27])

$$\dot{g} = \Xi(g,u), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell \tag{1}$$

where $\Xi(g,u) = g\Xi(\mathbf{1},u) \in T_g\mathsf{G}$. Admissible controls are bounded and measurable maps $u(\cdot) : [0,T] \to \mathbb{R}^\ell$. We further assume that the parametrisation map $\Xi(\mathbf{1},\cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is an embedding. Hence, the *trace* $\Gamma = \operatorname{im}\Xi(\mathbf{1},\cdot)$ is a submanifold of $\mathfrak{g}$. We have that $\Gamma = \left\{\Xi_u = \Xi(\mathbf{1},u) : u \in \mathbb{R}^\ell\right\}$ (cf. [5, 6]). A *trajectory* for an admissible control $u(\cdot) : [0,T] \to \mathbb{R}^\ell$ is an absolutely continuous curve $g(\cdot) : [0,T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\Xi(\mathbf{1},u(t))$ for almost every $t \in [0,T]$.

A left-invariant control system Σ is said to be *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0,T] \to \mathbb{R}^\ell$ such that $g(0) = g_0$ and $g(T) = g_1$. Controllable systems on connected (matrix) Lie groups must have *full rank*; this means that the Lie algebra generated by the trace of the system, $\mathsf{Lie}(\Gamma)$, is $\mathfrak{g}$. The following result is well known (see, also, [30]).

Theorem 1 ([7]). *A left-invariant control system on the Euclidean group* $\mathsf{SE}(n)$ *is controllable if and only if it has full rank.*

We shall denote a (left-invariant control) system Σ by (G,Ξ) (see, e.g., [5, 6]). We say that a system $\Sigma = (\mathsf{G},\Xi)$ is *connected* if its state space G is connected. Let $\Sigma = (\mathsf{G},\Xi)$ and $\Sigma' = (\mathsf{G}',\Xi')$ be two connected full-rank systems with traces $\Gamma \subseteq \mathfrak{g}$ and $\Gamma' \subseteq \mathfrak{g}'$, respectively. We say that Σ and Σ' are (locally) *detached feedback equivalent* if there exist open neighbourhoods N and N' of (the unit elements) $\mathbf{1}$ and $\mathbf{1}'$, respectively, and a (local) diffeomorphism $\Phi = \phi \times \varphi : N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$ such that $\phi(\mathbf{1}) = \mathbf{1}'$ and $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),\varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. Two detached feedback equivalent systems have the same trajectories (up to a diffeomorphism in the state space), which are parametrised differently by admissible controls. We recall the following result.

Theorem 2 ([6]). *$\Sigma = (\mathsf{G},\Xi)$ and $\Sigma' = (\mathsf{G}',\Xi')$ are (locally) detached feedback equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

2.2. Invariant Optimal Control Problems

Consider a left-invariant control system (1) evolving on some matrix Lie group $\mathsf{G} \leq \mathsf{GL}(n,\mathbb{R})$ of dimension m. In addition, it is assumed that there is a prescribed (smooth) *cost function* $L : \mathbb{R}^\ell \to \mathbb{R}$ (which is also called a Lagrangian). Let g_0 and g_1 be arbitrary but fixed points of G. We shall be interested in finding a trajectory-control pair $(g(\cdot),u(\cdot))$ which satisfies

$$g(0) = g_0, \quad g(T) = g_1 \tag{2}$$

and in addition *minimizes* the total cost functional $\mathscr{J} = \int_0^T L(u(t))dt$ among all trajectories of (1) which satisfy the same boundary conditions (2). The terminal time $T > 0$ can be either fixed or it can be free.

The *Pontryagin Maximum Principle* is a necessary condition for optimality which is most naturally expressed in the language of the geometry of the cotangent bundle $T^*\mathsf{G}$ of G (cf. [1, 11]). The cotangent bundle $T^*\mathsf{G}$ can be trivialized (from the left) such that $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g}$. The dual space $\mathfrak{g}^*$ has a natural *Poisson structure*, called the "minus Lie-Poisson structure", given by

$$\{F,G\}_-(p) = -p\left([dF(p), dG(p)]\right)$$

for $p \in \mathfrak{g}^*$ and $F,G \in C^\infty(\mathfrak{g}^*)$. (Note that $dF(p)$ is a linear function on $\mathfrak{g}^*$ and so is an element of $\mathfrak{g}$.) The Poisson manifold $(\mathfrak{g}^*, \{\cdot,\cdot\})$ is denoted by $\mathfrak{g}^*_-$. Each left-invariant Hamiltonian on the cotangent bundle $T^*\mathsf{G}$ is identified with its reduction on the dual space $\mathfrak{g}^*_-$.

To an optimal control problem (with fixed terminal time)

$$\int_0^T L(u(t))dt \to \min \tag{3}$$

subject to (1) and (2), we associate, for each real number λ and each control parameter $u \in \mathbb{R}^\ell$, a Hamiltonian function on $T^*\mathsf{G} = \mathsf{G} \times \mathfrak{g}^*$:

$$H_u^\lambda(\xi) = \lambda L(u) + \xi\left(g\Xi(\mathbf{1},u)\right) = \lambda L(u) + p\left(\Xi(\mathbf{1},u)\right), \quad \xi = (g,p) \in T^*\mathsf{G}.$$

The Maximum Principle can be stated, in terms of the above Hamiltonians, as follows.

Maximum Principle. *Suppose the trajectory-control pair* $(\bar{g}(\cdot), \bar{u}(\cdot))$ *defined over the interval* $[0, T]$ *is a solution for the optimal control problem (1)-(2)-(3). Then, there exists a curve* $\xi(\cdot) : [0, T] \to T^*\mathsf{G}$ *with* $\xi(t) \in T^*_{\bar{g}(t)}\mathsf{G}, t \in [0, T]$, *and a real number* $\lambda \leq 0$, *such that the following conditions hold for almost every* $t \in [0, T]$ *:*

$$(\lambda, \xi(t)) \not\equiv (0, 0) \tag{4}$$

$$\dot{\xi}(t) = \vec{H}^{\lambda}_{\bar{u}(t)}(\xi(t)) \tag{5}$$

$$H^{\lambda}_{\bar{u}(t)}(\xi(t)) = \max_u H^{\lambda}_u(\xi(t)) = \textit{constant}. \tag{6}$$

An optimal trajectory $\bar{g}(\cdot) : [0, T] \to \mathsf{G}$ is the projection of an integral curve $\xi(\cdot)$ of the (time-varying) Hamiltonian vector field $\vec{H}^{\lambda}_{\bar{u}(t)}$ defined for all $t \in [0, T]$. A trajectory-control pair $(\xi(\cdot), u(\cdot))$ defined on $[0, T]$ is said to be an *extremal pair* if $\xi(\cdot)$ satisfies the conditions (4), (5) and (6). The projection $\xi(\cdot)$ of an extremal pair is called an extremal. An extremal curve is called normal if $\lambda = -1$ and abnormal if $\lambda = 0$. In this paper, we shall be concerned only with normal extremals. Suppose the maximum condition (6) eliminates the parameter u from the family of Hamiltonians (H_u), and as a result of this elimination, we obtain a smooth function H (without parameters) on $T^*\mathsf{G}$ (in fact, on $\mathfrak{g}^*_-$). Then the whole (left-invariant) optimal control problem reduces to the study of trajectories of a fixed Hamiltonian vector field $\vec{H}$. The following result holds.

Theorem 3 ([13]). *For the left-invariant control problem*

$$\dot{g} = g\left(A + u_1 B_1 + \cdots + u_\ell B_\ell\right), \qquad g \in \mathsf{G},\ u \in \mathbb{R}^\ell$$
$$g(0) = g_0, \quad g(T) = g_T$$
$$\mathscr{J} = \frac{1}{2}\int_0^T \left(c_1 u_1^2(t) + \cdots + c_\ell u_\ell^2(t)\right) dt \to \min \quad (T \textit{ is fixed })$$

every (normal) extremal control is given by

$$u_i(t) = \tfrac{1}{c_i} p(t)(B_i), \quad i = 1, \ldots, \ell$$

where $p(\cdot) : [0, T] \to \mathfrak{g}^*$ *is an integral curve of the Hamiltonian vector field* $\vec{H}$ *on* $\mathfrak{g}^*_-$ *corresponding to the reduced Hamiltonian*

$$H(p) = p(A) + \tfrac{1}{2}\left(\tfrac{1}{c_1} p(B_1)^2 + \cdots + \tfrac{1}{c_\ell} p(B_\ell)^2\right).$$

Remark 1. *In coordinates on* $\mathfrak{g}^*_-$, *the (components of the) integral curves satisfy*

$$\dot{p}_i = -\sum_{j,k=1}^{m} c^k_{ij} p_k \frac{\partial H}{\partial p_j}, \quad i = 1, \ldots, m.$$

Here, c^k_{ij} *denote the structure constants of* $\mathfrak{g}$ *with respect to a basis* $\left(E_k\right)_{1 \leq k \leq m}$ *for* $\mathfrak{g}$ *(i.e.,* $[E_i, E_j] = \sum_{k=1}^m c^k_{ij} E_k$*) and* $p_i = p(E_i)$.

2.3. Jacobi Elliptic Functions

Given the modulus $k \in [0,1]$, the basic *Jacobi elliptic functions* $\mathrm{sn}(\cdot,k), \mathrm{cn}(\cdot,k)$ and $\mathrm{dn}(\cdot,k)$ can be defined as

$$\begin{aligned}\mathrm{sn}(x,k) &= \sin \mathrm{am}(x,k)\\ \mathrm{cn}(x,k) &= \cos \mathrm{am}(x,k)\\ \mathrm{dn}(x,k) &= \sqrt{1-k^2\sin^2\mathrm{am}(x,k)}\end{aligned}$$

where $\mathrm{am}(\cdot,k) = F(\cdot,k)^{-1}$ is the amplitude and $F(\varphi,k) = \int_0^{\varphi} \frac{dt}{\sqrt{1-k^2\sin^2 t}}\cdot$ (For the degenerate cases $k=0$ and $k=1$, we recover the circular functions and the hyperbolic functions, respectively.) Nine other elliptic functions are defined by taking reciprocals and quotients; in particular, we get $\mathrm{ns}(\cdot,k) = \frac{1}{\mathrm{sn}(\cdot,k)}\cdot$ Simple elliptic integrals can be expressed in terms of appropriate inverse (elliptic) functions. The following formulas hold true for $b < a \le x$ and $b \le x \le a$, respectively (see [2] or [14]):

$$\int_x^{\infty} \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}} = \frac{1}{a}\mathrm{ns}^{-1}\left(\tfrac{1}{a}x, \tfrac{b}{a}\right) \tag{7}$$

$$\int_x^{a} \frac{dt}{\sqrt{(a^2-t^2)(t^2-b^2)}} = \frac{1}{a}\mathrm{dn}^{-1}\left(\tfrac{1}{a}x, \tfrac{\sqrt{a^2-b^2}}{a}\right)\cdot \tag{8}$$

2.4. The Energy-Casimir Method

The *energy-Casimir method* [9] gives sufficient conditions for Lyapunov stability of equilibrium states for certain types of Hamilton-Poisson dynamical systems (cf. [16, 19]). The method is restricted to certain types of systems, since its implementation relies on an abundant supply of Casimir functions.

The standard energy-Casimir method states that if z_e is an equilibrium point of a Hamiltonian vector field $\vec{H}$ (associated with an energy function H) and if there exists a Casimir function C such that z_e is a critical point of $H+C$ (on the whole state space) and $d^2(H+C)(z_e)$ is (positive or negative) definite, then z_e is Lyapunov stable.

Ortega and Ratiu have obtained a generalisation of the standard energy-Casimir method (cf. [18, 17]). This extended version states that if $C = \lambda_1 C_1 + \cdots + \lambda_k C_k$, where $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $C_1, \ldots, C_k$ are conserved quantities (i.e., they Poisson commute with the energy function H), then definiteness of $d^2(\lambda_0 H + C)(z_e), \lambda_0 \in \mathbb{R}$ is only required on the intersection (subspace) $W = \ker dH(z_e) \cap \ker dC_1(z_e) \cap \cdots \cap dC_k(z_e)$.

3. The Euclidean Group $\mathsf{SE}(2)$

The Euclidean group

$$\mathsf{SE}(2) = \left\{ \begin{bmatrix} 1 & 0 \\ \mathbf{v} & R \end{bmatrix} : \mathbf{v} \in \mathbb{R}^{2\times 1},\ R \in \mathsf{SO}(2) \right\}$$

is a (real) three-dimensional connected matrix Lie group. The group is solvable and unimodular. The associated Lie algebra is given by

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

Let

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{se}(2)$ with the following table for the bracket operation

$[\cdot,\cdot]$	E_1	E_2	E_3
E_1	0	0	$-E_2$
E_2	0	0	E_1
E_3	E_2	$-E_1$	0

With respect to this basis, the group $\mathsf{Aut}(\mathfrak{se}(2))$ of Lie algebra automorphisms of $\mathfrak{se}(2)$ is given by

$$\left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} : x, y, v, w \in \mathbb{R}, x^2 + y^2 \neq 0, \varsigma = \pm 1 \right\}. \tag{9}$$

As $\mathfrak{se}(2)$ is not semisimple, the Killing form is degenerate. Moreover, it can be shown that there does not exist any non-degenerate invariant scalar product on $\mathfrak{se}(2)$. Therefore, we use the non-degenerate bilinear form

$$\left\langle \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & -x_3 \\ x_2 & x_3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ y_1 & 0 & -y_3 \\ y_2 & y_3 & 0 \end{bmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

(on $\mathfrak{se}(2)$) to identify $\mathfrak{se}(2)$ with $\mathfrak{se}(2)^*$ (cf. [11]). That is, we identify $P \in \mathfrak{se}(2)$ with $\langle P, \cdot \rangle \in \mathfrak{se}(2)^*$. Then each extremal curve $p(\cdot)$ in $\mathfrak{se}(2)^*$ is identified with a curve $P(\cdot)$ in $\mathfrak{se}(2)$ via the formula $\langle P(t), X \rangle = p(t)(X)$ for all $X \in \mathfrak{se}(2)$. Thus

$$P(t) = \begin{bmatrix} 0 & 0 & 0 \\ P_1(t) & 0 & -P_3(t) \\ P_2(t) & P_3(t) & 0 \end{bmatrix}$$

where $P_i(t) = \langle P(t), E_i \rangle = p(t)(E_i) = p_i(t), \quad i = 1, 2, 3.$

Now consider a Hamiltonian H on the (minus) Lie-Poison structure for $\mathfrak{se}(2)^*$. The equations of motion take the following form

$$\dot{p}_i = -p([E_i, dH(p)]), \quad i = 1, 2, 3$$

or, explicitly,

$$\begin{cases} \dot{p}_1 = \dfrac{\partial H}{\partial p_3} p_2 \\ \dot{p}_2 = -\dfrac{\partial H}{\partial p_3} p_1 \\ \dot{p}_3 = \dfrac{\partial H}{\partial p_2} p_1 - \dfrac{\partial H}{\partial p_1} p_2. \end{cases} \tag{10}$$

We note that $C : \mathfrak{se}(2)^* \to \mathbb{R}$, $C(p) = p_1^2 + p_2^2$ is a Casimir function.

4. Classification of Systems

Consider a general single-input left-invariant control affine system Σ with trace $\Gamma = A + \langle B \rangle \subset \mathfrak{se}(2)$. We shall assume that Σ has full rank (i.e., $\mathsf{Lie}\{A, B\} = \mathfrak{se}(2)$). This means (by proposition 1) that Σ is precisely a controllable system. Note that the Lie algebra rank condition is equivalent to the conditions (i) A and B are linearly independent and (ii) $\{A, B\} \not\subset \langle E_1, E_2 \rangle$ (cf. [30]). The following result gives a classification of all such control systems under the detached feedback equivalence (see, also, [6, 4, 3]).

Theorem 4. *Any controllable single-input (left-invariant control affine) system Σ is (locally) detached feedback equivalent to exactly one of the following systems: Σ_1 or $\Sigma_{2,\alpha}$ $(\alpha > 0)$ with respective parametrisations*

$$\Xi_1(\mathbf{1}, u) = E_1 + uE_3, \qquad \Xi_{2,\alpha}(\mathbf{1}, u) = \alpha E_3 + uE_1.$$

Proof. Throughout, we use the algebraic characterisation from proposition 2. Let the trace of the system Σ be given by $\Gamma = \sum_{i=1}^3 a_i E_i + \left\langle \sum_{i=1}^3 b_i E_i \right\rangle$.

First, consider the case $b_3 \neq 0$. Then

$$\begin{aligned} \Gamma &= (a_1 - \tfrac{b_1 a_3}{b_3})E_1 + (a_2 - \tfrac{b_2 a_3}{b_3})E_2 + \left\langle \tfrac{b_1}{b_3} E_1 + \tfrac{b_2}{b_3} E_2 + E_3 \right\rangle \\ &= a_1' E_1 + a_2' E_2 + \left\langle b_1' E_1 + b_2' E_2 + E_3 \right\rangle \end{aligned}$$

for some corresponding constants $a_i', b_i' \in \mathbb{R}$, $i = 1, 2$. Hence,

$$\psi = \begin{bmatrix} a_1' & -a_2' & b_1' \\ a_2' & a_1' & b_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is a Lie algebra automorphism mapping Γ_1 to Γ. (Note that $\det \psi = 0$ if and only if $a_1' = a_2' = 0$, a contradiction.)

Next, consider the case $b_3 = 0$. Since $a_3 \neq 0$ (as Σ is of full rank), and either $b_1 \neq 0$ or $b_2 \neq 0$, we get that

$$\psi = \begin{bmatrix} b_1 & -\mathrm{sgn}(a_3) b_2 & \frac{a_1}{\alpha} \\ b_2 & \mathrm{sgn}(a_3) b_1 & \frac{a_2}{\alpha} \\ 0 & 0 & \mathrm{sgn}(a_3) \end{bmatrix}$$

is a Lie algebra automorphism. Let $\alpha = |a_3|$, then $\psi \cdot \Gamma_{2,\alpha} = \Gamma$.

A simple argument shows that Σ_1 is not equivalent to any system $\Sigma_{2,\alpha}$ and that $\Sigma_{2,\alpha}$ is not equivalent to $\Sigma_{2,\beta}$ for any $\alpha \neq \beta$, $\alpha, \beta > 0$.

Left-Invariant Control Problems

Henceforth, we consider only the systems Σ_1 and $\Sigma_{2,\alpha}$. In each of these typical cases, we investigate an optimal control problem (with quadratic cost):

$$\left.\begin{array}{c} \dot{g} = g\left(E_1 + uE_3\right) \\ g(0) = g_0, \quad g(T) = g_T \\ \mathscr{J} = \dfrac{1}{2}\displaystyle\int_0^T u^2(t)dt \to \min \end{array}\right\} \qquad \mathsf{LiCP}(1)$$

and

$$\left.\begin{array}{c} \dot{g} = g\left(\alpha E_3 + uE_1\right) \\ g(0) = g_0, \quad g(T) = g_T \\ \mathscr{J} = \dfrac{1}{2}\displaystyle\int_0^T u^2(t)dt \to \min \end{array}\right\} \qquad \mathsf{LiCP}(2)$$

The following two results follow easily from proposition 3.

Theorem 5 ([31]). *For the* LiCP(1)*, the extremal control is given by $u = p_3$, where $H(p) = p_1 + \frac{1}{2}p_3^2$ and*

$$\begin{cases} \dot{p}_1 = p_2 p_3 \\ \dot{p}_2 = -p_1 p_3 \\ \dot{p}_3 = -p_2. \end{cases} \tag{11}$$

Theorem 6. *For the* LiCP(2)*, the extremal control is given by $u = p_1$, where $H(p) = \frac{1}{2}p_1^2 + \alpha p_3$ and*

$$\begin{cases} \dot{p}_1 = \alpha p_2 \\ \dot{p}_2 = -\alpha p_1 \\ \dot{p}_3 = -p_1 p_2. \end{cases} \tag{12}$$

5. Stability

The equilibrium states for (11) are

$$e_1^{\mu} = (\mu, 0, 0) \quad \text{and} \quad e_2^{\nu} = (0, 0, \nu)$$

where $\mu, \nu \in \mathbb{R}, \nu \neq 0$.

Theorem 7. *The equilibrium states have the following behaviour.*

(i) The equilibrium state e_1^μ is stable if $\mu < 0$ and unstable if $\mu \geq 0$.

(ii) Each equilibrium state e_2^ν is stable.

Proof. The linearization of the system is given by

$$\begin{bmatrix} 0 & p_3 & p_2 \\ -p_3 & 0 & -p_1 \\ 0 & -1 & 0 \end{bmatrix}.$$

(i) Assume $\mu > 0$. The linearization of the system (at e_1^μ) has eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{\mu}$. Thus e_1^μ is unstable. Now, assume $\mu = 0$. Then the linearization of the system has eigenvalues $\lambda_{1,2,3} = 0$. Thus, as the geometric multiplicity is strictly less than the algebraic multiplicity, e_1^μ is unstable.

Assume $\mu < 0$. Let $H_\chi = H + \chi(C)$ be an energy-Casimir function, i.e., $H_\chi(p_1, p_2, p_3) = \frac{1}{2}p_3^2 + p_1 + \chi\left(p_1^2 + p_2^2\right)$, where $\chi \in C^\infty(\mathbb{R})$. The derivative

$$dH_\chi = \begin{bmatrix} 1 + 2p_1\dot{\chi}\left(p_1^2 + p_2^2\right) & 2p_2\dot{\chi}\left(p_1^2 + p_2^2\right) & p_3 \end{bmatrix}$$

vanishes at e_1^μ if and only if $\dot{\chi}\left(\mu^2\right) = -\frac{1}{2\mu}\cdot$ Then, the Hessian (at e_1^μ) $d^2H_\chi(\mu, 0, 0) = \operatorname{diag}\left(4\mu^2\ddot{\chi}\left(\mu^2\right) - \frac{1}{\mu},\ -\frac{1}{\mu},\ 1\right)$ is positive definite if and only if $\ddot{\chi}\left(\mu^2\right) > \frac{1}{4\mu^3}\cdot$ The function $\chi(x) = -\frac{1}{4\mu^3}x^2$ satisfies these requirements. Hence, by the standard energy-Casimir method, e_1^μ is stable.

(ii) Let $H_\lambda = \lambda_0 H + \lambda_1 C$, where $\lambda_0 = 0$, $\lambda_1 = 1$. Then we get $dH_\lambda(0, 0, \nu) = \begin{bmatrix} 2p_1 & 2p_2 & 0 \end{bmatrix}|_{(0,0,\nu)} = 0$ and $d^2H_\lambda(0, 0, \nu) = \operatorname{diag}(2,\ 2,\ 0)$. Also,

$$\ker dH(e_2^\nu) \cap \ker dC(e_2^\nu) = \operatorname{span}\{(-\nu, 0, 1), (0, 1, 0)\}$$

and so $d^2H_\lambda(0, 0, \nu)\big|_{W\times W} = \operatorname{diag}\left(2\nu^2,\ 2\right)$ is positive definite. Hence, by the extended energy-Casimir method, e_2^ν is stable.

The equilibrium states for (12) are

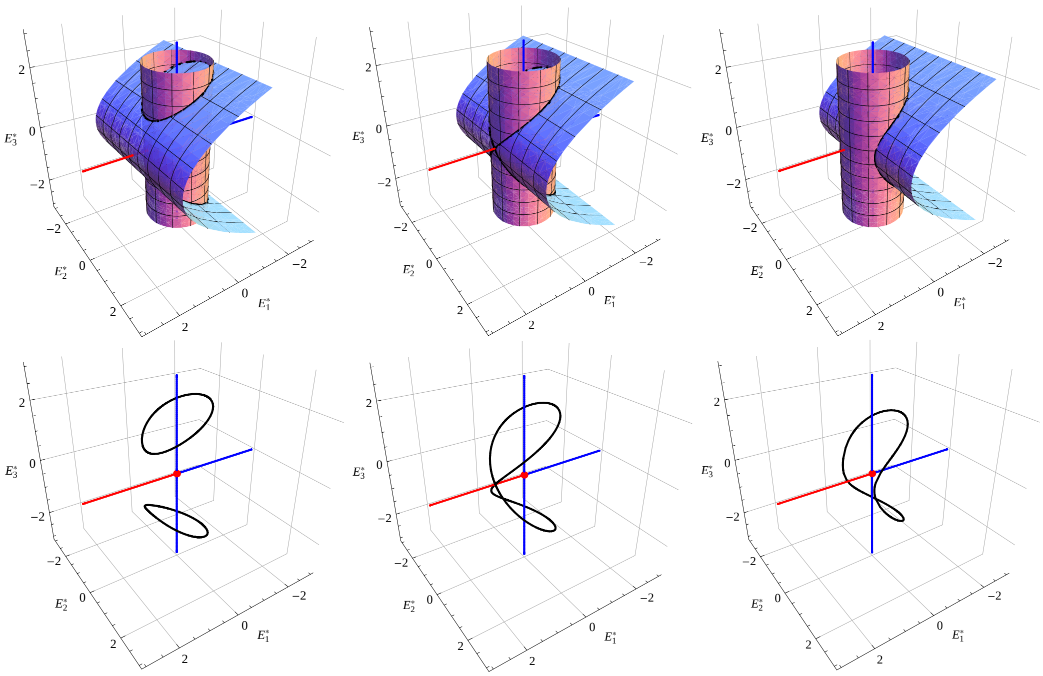
$$e_3^\mu = (0, 0, \mu), \quad \mu \in \mathbb{R}.$$

Again, using the extended energy-Casimir method (as in theorem 7), we obtain the following result.

Theorem 8. *Each equilibrium state e_3^μ is stable.*

6. Explicit Integration

First, let us consider the invariant control problem $\mathsf{LiCP}(1)$. There are three typical cases for the reduced extremal equations (11), corresponding to $H > \sqrt{C}$, $H = \sqrt{C}$ and $-\sqrt{C} < H < \sqrt{C}$. (Note that $H = -\sqrt{C}$ and $C = 0$ correspond to constant solutions, whereas the situation $H < \sqrt{C}$ is impossible.) In figure 1, we graph the level sets of H and C and their intersection. We also graph the stable equilibrium points (illustrated in blue) and unstable equilibrium points (illustrated in red), as presented in theorem 7. The reduced Hamilton



(a) $H > \sqrt{C}$ (b) $H = \sqrt{C}$ (c) $-\sqrt{C} < H < \sqrt{C}$

Figure 1: Typical cases of reduced extremals of $\mathsf{LiCP}(1)$.

equations (11) can be integrated by Jacobi elliptic functions. In each of the typical cases, we obtain explicit expressions for the integral curves of $\vec{H}$. We start by considering the case $H > \sqrt{C}$.

Theorem 9. *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and $h_0^2 - c_0 > 0$. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that*

$p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where

$$\begin{cases} \bar{p}_1(t) = \sqrt{c_0}\,\dfrac{k - \mathrm{sn}\,(\Omega t, k)}{1 - k\,\mathrm{sn}\,(\Omega t, k)} \\ \bar{p}_2(t) = \sigma\sqrt{c_0}\,\dfrac{k'\mathrm{cn}\,(\Omega t, k)}{1 - k\,\mathrm{sn}\,(\Omega t, k)} \\ \bar{p}_3(t) = -\sigma\sqrt{2\delta}\,\dfrac{\mathrm{dn}\,(\Omega t, k)}{1 - k\,\mathrm{sn}\,(\Omega t, k)}. \end{cases}$$

Here $\delta = \sqrt{h_0^2 - c_0}$, $\Omega = \sqrt{h_0 + \delta}$, $k = \sqrt{\frac{h_0 - \delta}{h_0 + \delta}}$ and $k' = \sqrt{\frac{2\delta}{h_0 + \delta}}$.

Proof. We start by explaining how the expression for $\bar{p}(\cdot)$ can be found. Assume $\bar{p}(\cdot)$ is an integral curve of $\vec{H}$ satisfying $H(\bar{p}(0)) = h_0$, $C(\bar{p}(0)) = c_0 > 0$ and $\delta^2 = h_0^2 - c_0 > 0$. Then, as $\bar{p}(\cdot)$ solves (11), we get that

$$\frac{d}{dt}\bar{p}_1(t) = \pm\sqrt{2\left(c_0 - \bar{p}_1(t)^2\right)\left(h_0 - \bar{p}_1(t)\right)}. \tag{13}$$

This (separable) differential equation is transformed into standard form (see [2]) and formula (7) is then applied. After further simplification, this yields $\bar{p}_1(t)$ as specified. Then, as $C(\bar{p}(t)) = c_0$, we get that

$$\bar{p}_2(t)^2 = c_0 - \bar{p}_1(t)^2 = \frac{2c_0\delta\left(1 - \mathrm{sn}\,(\Omega t, k)^2\right)}{\left(h_0 + \delta\right)(1 - k\mathrm{sn}\,(\Omega t, k))^2} = \frac{c_0(k')^2\mathrm{cn}\,(\Omega t, k)^2}{(1 - k\mathrm{sn}\,(\Omega t, k))^2}$$

yielding $\bar{p}_2(t)$ as specified, for some $\sigma \in \{-1, 1\}$. Finally, as $\frac{d}{dt}\bar{p}_3(t) = -\bar{p}_2$ and

$$-\sigma\sqrt{c_0}\int \frac{k'\mathrm{cn}\,(\Omega t, k)}{1 - k\,\mathrm{sn}\,(\Omega t, k)}\,dt = -\sigma\sqrt{c_0}\frac{k'}{k\Omega}\,\frac{\mathrm{dn}\,(\Omega t, k)}{1 - k\,\mathrm{sn}\,(\Omega t, k)}$$

we get $\bar{p}_3(t)$ as prescribed.

This motivates $\bar{p}(\cdot)$ as a prospective integral curve of $\vec{H}$. Now notice, as $\delta^2 = h_0^2 - c_0 > 0$, that $0 < k < 1$ and so $1 - k\,\mathrm{sn}\,(\Omega t, k) > 0$. Hence $\bar{p}(t)$ is well defined and smooth for all $t \in \mathbb{R}$ (and all h_0, c_0 such that $h_0^2 - c_0 > 0$, $c_0 > 0$). We now verify that $\bar{p}(\cdot)$ is a solution to (11). We get that

$$\frac{d}{dt}\bar{p}_1(t) - \bar{p}_2(t)\bar{p}_3(t) = \left(-\sqrt{c_0}(k')^2\Omega + \sqrt{2c_0\delta}\right)\,\frac{\mathrm{cn}\,(\Omega t, k)\,\mathrm{dn}\,(\Omega t, k)}{(1 - k\mathrm{sn}\,(\Omega t, k))^2}.$$

Substitution and simplification then yields $\frac{d}{dt}\bar{p}_1(t) = \bar{p}_2(t)\bar{p}_3(t)$. Likewise, we get $\frac{d}{dt}\bar{p}_2(t) = -\bar{p}_1(t)\bar{p}_3(t)$ and $\frac{d}{dt}\bar{p}_3(t) = -\bar{p}_2(t)$. Hence $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(2)^*$ is a periodic integral curve of $\vec{H}$.

Any integral curve $p(\cdot)$ developing on $H^{-1}(h_0) \cap C^{-1}(c_0)$ must be of the form $p(t) = \bar{p}(t + t_0)$ for some $\sigma \in \{-1, 1\}$ and $t_0 \in \mathbb{R}$ (see figure 1a). We now prove this fact.

Let $\sigma = \operatorname{sgn}(p_3(0))$. We may assume $\sigma \neq 0$. Next we note that $(\bar{p}_1(t), \bar{p}_2(t))$ parametrises the circle $S = \{(x, y) : x^2 + y^2 = c_0\}$. But $p_1(0)^2 + p_2(0)^2 = c_0$, i.e., $(p_1(0), p_2(0)) \in S$. Therefore, there exists $t_0 \in \mathbb{R}$ such that $\bar{p}_1(t_0) = p_1(0)$ and $\bar{p}_2(t_0) = p_2(0)$. Then we have that

$$p_3(0)^2 = 2(h_0 - p_1(0)) = 2(h_0 - \bar{p}_1(t_0)) = \bar{p}_3(t_0)^2.$$

Hence, as $\operatorname{sgn}(p_3(t_0)) = \sigma = \operatorname{sgn}(p_3(0))$, we get that $p_3(0) = \bar{p}_3(t_0)$. Thus the integral curves $t \mapsto p(t)$ and $t \mapsto \bar{p}(t + t_0)$ solve the same Cauchy problem, and therefore are identical. (Throughout this proof we used Mathematica to facilitate calculations.)

Remark 2. *Note that, for any $h_0, c_0 \in \mathbb{R}$, $h_0^2 - c_0 > 0$, $c_0 > 0$ and $\sigma \in \{-1, 1\}$, we have that $\bar{p}(\cdot) : \mathbb{R} \to \mathfrak{se}(2)^*$ is a periodic integral curve of $\vec{H}$. Consequently, any integral curve $p(\cdot)$ of $\vec{H}$ (satisfying the conditions of theorem 9) has maximal domain $\mathbb{R}$ and is periodic (on $\mathbb{R}$).*

We now proceed to the case $-\sqrt{C} < H < \sqrt{C}$.

Theorem 10. *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and $h_0^2 - c_0 < 0$. Then there exists $t_0 \in \mathbb{R}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = -\dfrac{\sqrt{c_0} - 2\delta + \left(\sqrt{c_0} + 2\delta\right) \operatorname{dn}(\Omega t, k)}{1 + \operatorname{dn}(\Omega t, k)} \\[2ex] \bar{p}_2(t) = \dfrac{4\sqrt[4]{c_0}\,\delta}{\sqrt{\sqrt{c_0} + \delta}} \; \dfrac{\sqrt{k' + \operatorname{dn}(\Omega t, k)}\operatorname{sn}(\Omega t, k)}{\sqrt{(1 + \operatorname{dn}(\Omega t, k))^3}} \\[2ex] \bar{p}_3(t) = \dfrac{4\delta}{\sqrt{\sqrt{c_0} + \delta}} \; \dfrac{\operatorname{cn}(\Omega t, k)}{\sqrt{k' + \operatorname{dn}(\Omega t, k)}\sqrt{1 + \operatorname{dn}(\Omega t, k)}}. \end{cases}$$

Here $\delta = \frac{1}{\sqrt{2}}\sqrt[4]{c_0}\sqrt{h_0 + \sqrt{c_0}}$, $\Omega = \frac{\sqrt{c_0}+\delta}{\sqrt[4]{c_0}}$, $k = \frac{2\sqrt{\sqrt{c_0}\delta}}{\sqrt{c_0}+\delta}$ and $k' = \frac{\sqrt{c_0}-\delta}{\sqrt{c_0}+\delta}$.

Remark 3. *The proof of this theorem is similar to that of theorem 9. The crucial difference is that before solving equation* (13)*, one needs to deinterlace the roots of the two quadratics involved. (This leads to a somewhat more involved computation which relies on formula* (8)*.) Again, we note that $\bar{p}(\cdot)$ is always a periodic integral curve and that any integral curve $p(\cdot)$ has maximal domain $\mathbb{R}$, on which it is periodic.*

We finish with the case $H = \sqrt{C}$.

Theorem 11. *Suppose $p(\cdot) : (-\varepsilon, \varepsilon) \to \mathfrak{se}(2)^*$ is an integral curve of $\vec{H}$ such that $H(p(0)) = h_0$, $C(p(0)) = c_0 > 0$ and $h_0 = \sqrt{c_0}$. Then there exists $t_0 \in \mathbb{R}$ and $\sigma \in \{-1, 1\}$ such that $p(t) = \bar{p}(t + t_0)$ for $t \in (-\varepsilon, \varepsilon)$, where*

$$\begin{cases} \bar{p}_1(t) = \frac{1}{2}h_0(\cosh(2\sqrt{h_0}t) - 3)\operatorname{sech}^2(\sqrt{h_0}t) \\ \bar{p}_2(t) = \sigma h_0 \sinh(2\sqrt{h_0}t)\operatorname{sech}^3(\sqrt{h_0}t) \\ \bar{p}_3(t) = 2\sigma\sqrt{h_0}\operatorname{sech}(\sqrt{h_0}t). \end{cases}$$

Remark 4. *This result can be obtained by limiting $h_0 \to \sqrt{c_0}$ from the left (i.e., using theorem 10) and adding possible changes in sign. Note however that this cannot be done from the right (i.e., using theorem 9).*

Finally, for the invariant control problem $\mathsf{LiCP}(2)$, there is only one typical case. As before, we graph the level sets of H and C and their intersection in figure 2. A simple computation then gives the solutions in this case.

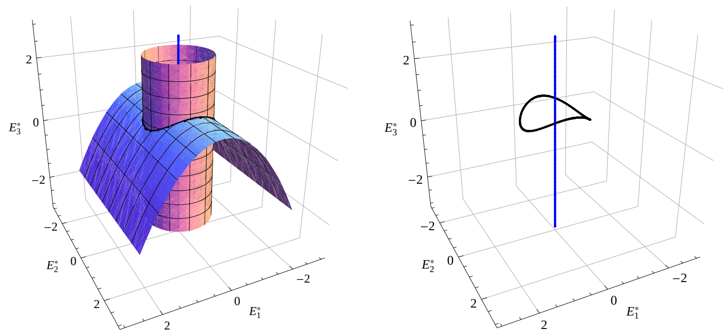


Figure 2: Typical reduced extremal of $\mathsf{LiCP}(2)$.

Theorem 12. *The reduced Hamilton equations* (12) *have the solutions*

$$\begin{cases} p_1(t) = \sqrt{c_0}\sin\left(\alpha t + t_0\right) \\ p_2(t) = \sqrt{c_0}\cos\left(\alpha t + t_0\right) \\ p_3(t) = \frac{h_0}{\alpha} - \frac{c_0}{2\alpha}\sin^2\left(\alpha t + t_0\right) \end{cases}$$

where $c_0 = C(p(0))$, $h_0 = H(p(0))$ *and* $t_0 \in \mathbb{R}$.

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