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Weak and Strong Convergence of a Two-Step Iterations for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Mappings

Hossein Dehghan [1,*], Amir Gharajelo [2]

[1] *Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS), Zanjan, Iran*
[2] *Department of Mathematics, Roozbeh Institute of Higher Education, Zanjan, Iran*

Abstract. In this paper, we introduce and study a new two-step iterative scheme to approximate common fixed points for a finite family of generalized asymptotically quasi-nonexpansive mappings. We establish several strong and weak convergence results of the proposed algorithm in Banach spaces. These results generalize and refine many known results in the current literature.

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1. Introduction

In recent years, one-step and two-step iterative schemes(including Mann iteration and Ishikawa iteration processes as the most important cases) have been studied extensively by many authors to approximate fixed points of various classes of mappings (see for example [1, 4, 5, 8, 10]).

Approximating common fixed points of a finite family of nonlinear mappings plays an important role in solving systems of equations and inequalities that often arise in applied mathematics. For a finite family of mappings $\{T_i : i = 1, 2, \dots, m\}$, it is desirable to devise a iteration scheme which extends the modified Mann iteration and the modified Ishikawa

*Corresponding author.

Email addresses: `h_dehghan@iasbs.ac.ir`, `hossein.dehgan@gmail.com` (H. Dehghan), `amirgharajelo@yahoo.com` (A. Gharajelo)

iteration from one mapping to a finite family of mappings. Thereby, to achieve this goal, we introduce a new two-step iterative scheme for a finite family of mappings as follows:

Let C be a nonempty convex subset of a real Banach space X and $\{T_i : i = 1, 2, \ldots, m\}$ be a family of self-mappings of C. For a given $x_1 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes

$$\begin{aligned} y_n &= \sum_{i=1}^{m} a_{in} T_i^n x_n + b_n x_n \\ x_{n+1} &= \sum_{i=1}^{m} \left(\alpha_{in} T_i^n y_n + \beta_{in} T_i^n x_n \right) + \gamma_n x_n, \quad n \geq 1, \end{aligned} \tag{1}$$

where $\{a_{in}\}$, $\{b_n\}$, $\{\alpha_{in}\}$, $\{\beta_{in}\}$, and $\{\gamma_n\}$ are appropriate sequences in $[0, 1]$ for all $i \in \{1, 2, \ldots, m\}$ such that $b_n + \sum_{i=1}^{m} a_{in} = \gamma_n + \sum_{i=1}^{m} (\alpha_{in} + \beta_{in}) = 1$ for each $n \geq 1$.

If $T_1 = T_2 = \cdots = T_m$, then the iteration process (1) reduces to

$$\begin{aligned} y_n &= a_n T^n x_n + (1 - a_n) x_n \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n x_n + \gamma_n x_n, \quad n \geq 1, \end{aligned} \tag{2}$$

where $\{a_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are appropriate sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$.

Clearly, the iteration process (2) includes the modified Ishikawa iteration

$$\begin{aligned} y_n &= a_n T^n x_n + (1 - a_n) x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n) x_n, \quad n \geq 1, \end{aligned} \tag{3}$$

where $\{a_n\}$ and $\{\alpha_n\}$ are appropriate sequences in $[0, 1]$, and the modified Mann iteration

$$x_{n+1} = a_n T^n x_n + (1 - a_n) x_n, \quad n \geq 1, \tag{4}$$

where $\{a_n\}$ is appropriate sequences in $[0, 1]$. Therefore, (1) generalizes the modified Ishikawa iteration and the modified Mann iteration from one mapping to the finite family of mappings $\{T_j : j = 1, 2, \ldots, m\}$.

The aim of this paper is to obtain some strong and weak convergence results for the iterative process (1) of a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach spaces.

Now, we recall the well-known concepts and results. For convenience, we use the notations $\lim_n \equiv \lim_{n\to\infty}$, $\liminf_n \equiv \liminf_{n\to\infty}$ and $\limsup_n \equiv \limsup_{n\to\infty}$. Let C be a nonempty subset of a real Banach space X and T be a self-mapping of C. The fixed point set of T is denoted by $F(T) = \{x \in C : Tx = x\}$. The mapping T is called

(i) *asymptotically nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_n r_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + r_n)\|x - y\|$ for all $x, y \in C$ and each $n \geq 1$;

(ii) *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_n r_n = 0$ such that $\|T^n x - p\| \leq (1 + r_n)\|x - p\|$ for all $x \in C$, $p \in F(T)$ and each $n \geq 1$;

(iii) *generalized asymptotically quasi-nonexpansive* [7] if $F(T) \neq \emptyset$ and there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0,\infty)$ with $\lim_n r_n = \lim_n s_n = 0$ such that $\|T^n x - p\| \leq (1+r_n)\|x-p\| + s_n$ for all $x \in C$, $p \in F(T)$ and each $n \geq 1$;

(iv) *uniformly L-Lipschitz* if there exists constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x-y\|$ for all $x, y \in C$ and each $n \geq 1$.

It is clear that a generalized asymptotically quasi-nonexpansive mapping is to unify various classes of mappings associated with the class of asymptotically quasi-nonexpansive mapping, asymptotically nonexpansive mappings and nonexpansive mappings. However, the converse of each of above statement may be not true. The example, shows that a generalized asymptotically quasi-nonexpansive mapping is not an asymptotically quasi-nonexpansive mapping, can be found in [7].

A family of self-mappings $\{T_i : i = 1,2,\ldots,m\}$ of C is said to satisfy Condition (A'') [3] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $f(d(x,F)) \leq \|x_n - T_i x_n\|$ for some $1 \leq i \leq m$ and for all $x \in C$ where $d(x,F) = \inf\{\|x-y\| : y \in F = \bigcap_{i=1}^m F(T_i)\}$. We recall that a Banach space X is said to satisfy Opial's condition [6] if x_n converging to x weakly and $x \neq y$ imply that

$$\limsup_n \left\|x_n - x\right\| < \limsup_n \left\|x_n - y\right\|.$$

In the sequel, the following lemmas are needed to prove our main results.

Lemma 1 ([9, Lemma 1]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^\infty \delta_n < \infty$ and $\sum_{n=1}^\infty b_n < \infty$, then $\lim_n a_n$ exists.

Lemma 2 ([2, Lemma 1.2]). *Let $k \geq 2$ and $\{y_n^{(1)}\}, \ldots, \{y_n^{(k)}\}$ be sequences in a uniformly convex Banach space X with $\limsup_n \|y_n^{(i)}\| \leq a$ for each $i = 1,2,\ldots,k$ and for some $a \geq 0$. Suppose $\{\alpha_n^{(1)}\}, \ldots, \{\alpha_n^{(k)}\}$ be sequences in $[0,1]$ such that $\sum_{i=1}^k \alpha_n^{(i)} = 1$ and $\lim_n \|\sum_{i=1}^k \alpha_n^{(i)} y_n^{(i)}\| = a$. If $\liminf_n \alpha_n^{(i)} > 0$ and $\liminf_n \alpha_n^{(j)} > 0$ for some $i, j \in \{1,2,\ldots,k\}$, then $\lim_n \|y_n^{(i)} - y_n^{(j)}\| = 0$.*

2. Convergence in Banach spaces

The aim of this section is to establish the strong convergence of the iterative process (1) to converge to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a Banach space. To proceed in this direction, the following lemma is needed.

Lemma 3. *Let X be a real Banach space, C be a nonempty closed convex subset of X and $\{T_i : i = 1,2,\ldots,m\}$ be a family of generalized asymptotically quasi-nonexpansive self-mappings of C with the sequences $\{r_n^{(1)}\},\ldots,\{r_n^{(m)}\}$ and $\{s_n^{(1)}\},\ldots,\{s_n^{(m)}\}$ such that $\sum_{n=1}^\infty r_n^{(i)} < \infty$ and $\sum_{n=1}^\infty s_n^{(i)} < \infty$ for each $i = 1,2,\ldots,m$. Let $\{x_n\}$ be the sequence defined by (1). If $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$, then we have the following conclusions.*

(i) $\lim_n \|x_n - p\|$ *exists for all* $p \in F$.

(ii) $\lim_n d(x_n, F)$ *exists.*

(iii) *If* $\liminf_n \alpha_{in} > 0$ *for some* $i \in \{1, 2, \ldots, m\}$, *then* $\lim_n \|y_n - p\| = \lim_n \|x_n - p\|$ *for all* $p \in F$.

Proof. Let $p \in F$, $r_n = \max_{1\le i\le m} r_{in}$ and $s_n = \max_{1\le i\le m} s_{in}$. For each $n \ge 1$, we note that

$$\begin{aligned}
\|y_n - p\| &= \left\|\sum_{i=1}^m a_{in} T_i^n x_n + b_n x_n - p\right\| \\
&\le \sum_{i=1}^m a_{in} \left\|T_i^n x_n - p\right\| + b_n \|x_n - p\| \\
&\le \sum_{i=1}^m a_{in} \left((1 + r_{in})\|x_n - p\| + s_{in}\right) + b_n(1 + r_n)\|x_n - p\| \\
&\le (1 + r_n)\|x_n - p\| + s_n. \qquad (5)
\end{aligned}$$

It follows from (5) that

$$\begin{aligned}
\|x_{n+1} - p\| &\le \sum_{i=1}^m \Big(\alpha_{in}\|T_i^n y_n - p\| + \beta_{in}\|T_i^n x_n - p\|\Big) + \gamma_n \|x_n - p\| \\
&\le \sum_{i=1}^m \big(\alpha_{in}[(1 + r_{in})\|y_n - p\| + s_{in}] + \beta_{in}[(1 + r_{in})\|x_n - p\| + s_{in}]\big) \\
&\quad + \gamma_n (1 + r_n)^2 \|x_n - p\| \\
&\le \sum_{i=1}^m \big(\alpha_{in}[(1 + r_n)[(1 + r_n)\|x_n - p\| + s_n] + s_n] \\
&\quad + \beta_{in}[(1 + r_n)^2\|x_n - p\| + s_n]\big) + \gamma_n (1 + r_n)^2 \|x_n - p\| \\
&\le (1 + r_n)^2\|x_n - p\| + (2 + r_n)s_n. \qquad (6)
\end{aligned}$$

Since $\{r_n\}$ and $\{s_n\}$ are independent of p, by taking infimum over all $p \in F$ in both sides of (6), we obtain

$$d(x_{n+1}, F) \le (1 + r_n)^2 d(x_n, F) + (2 + r_n)s_n. \qquad (7)$$

Using Lemma 1, the conclusions (i) and (ii) of lemma follow from (6) and (7), respectively.

(iii) Since $\lim_n \|x_n - p\|$ exists, it follows from (5) that $\limsup_n \|y_n - p\| \le \lim_n \|x_n - p\|$. Also, by (6)

$$\begin{aligned}
\|x_{n+1} - p\| &\le \sum_{i=1}^m \big(\alpha_{in}[(1 + r_n)\|y_n - p\| + s_n] + \beta_{in}[(1 + r_n)\|x_n - p\| + s_n]\big) \\
&\quad + \gamma_n (1 + r_n)^2 \|x_n - p\|
\end{aligned}$$

$$\begin{aligned}&\leq (1+r_n)^2\left(\sum_{i=1}^{m}\alpha_{in}\|y_n-p\|+\left(1-\sum_{i=1}^{m}\alpha_{in}\right)\|x_n-p\|\right)\\&\quad +(2+r_n)s_n.\end{aligned}$$

for all $n \geq 1$. Since $\liminf_n \sum_{i=1}^{m} \alpha_{in} > 0$, we have

$$\frac{\|x_{n+1}-p\|-(1+r_n)^2\|x_n-p\|}{(1+r_n)^2\sum_{i=1}^{m}\alpha_{in}}+\|x_n-p\|\leq\|y_n-p\|+\frac{(2+r_n)s_n}{(1+r_n)^2\sum_{i=1}^{m}\alpha_{in}}$$

for sufficiently large numbers n. By taking $\liminf_n$ in both sides, we obtain

$$\lim_n\|x_n-p\|\leq\liminf_n\|y_n-p\|.$$

This completes the proof.

Theorem 1. *Let X, C, $T_1,\dots,T_m$ and $\{x_n\}$ be as in Lemma 3 with the restriction that $F=\bigcap_{i=1}^{m}F(T_i)$ be nonempty and closed. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_n d(x_n,F)=0$.*

Proof. The necessity is obvious and then we prove only the sufficiency. Let $p \in F$. From Lemma 3(i), we know that $\lim_n \|x_n - p\|$ exists and hence $\{\|x_n - p\|\}$ is bounded. We put $M = \sup_{n\geq 1}\|x_n - p\|$. It follows from (6) that

$$\|x_{n+1}-p\|\leq(1+\delta_n)\|x_n-p\|+d_n, \tag{8}$$

where $\delta_n=(1+r_n)^2-1$ and $d_n=(2+r_n)s_n$ so that $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}d_n<\infty$. Thus, for positive integers k and n, we have

$$\begin{aligned}\|x_{n+k}-p\| &\leq \|x_{n+k-1}-p\|+M\delta_{n+k-1}+d_{n+k-1}\\&\leq \|x_{n+k-2}-p\|+M(\delta_{n+k-2}+\delta_{n+k-1})+d_{n+k-2}+d_{n+k-1}\\&\ \ \vdots\\&\leq \|x_n-p\|+M\sum_{i=n}^{n+k-1}\delta_i+\sum_{i=n}^{n+k-1}d_i.\end{aligned}\tag{9}$$

It follows from Lemma 3(ii) that $\lim_n d(x_n,F)$ exists. Thus $\lim_n d(x_n,F)=0$. Now, we show that $\{x_n\}$ is a Cauchy sequence. By $\lim_n d(x_n,F)=0$, $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}d_n<\infty$, we get that for any $\epsilon>0$, there exists a positive integer n_0 such that

$$d(x_{n_0},F)<\frac{\epsilon}{6},\quad \sum_{i=n_0}^{\infty}\delta_i<\frac{\epsilon}{3M}\text{ and }\sum_{i=n_0}^{\infty}d_i<\frac{\epsilon}{3}.$$

Therefore, there exists $p_0 \in F$ such that $\|x_{n_0}-p_0\|<\epsilon/6$. It follows from (9) that

$$\|x_{n_0+k}-x_{n_0}\| \leq \|x_{n_0+k}-p_0\|+\|x_{n_0}-p_0\|$$

$$\begin{aligned}
&\leq \quad 2\|x_{n_0} - p_0\| + M \sum_{i=n_0}^{n_0+k-1} \delta_i + \sum_{i=n_0}^{n_0+k-1} d_i \\
&< \quad 2\left(\frac{\epsilon}{6}\right) + M\left(\frac{\epsilon}{3M}\right) + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

for all $k \geq 1$. Thus $\{x_n\}$ is a Cauchy sequence and hence $x_n \to q$ for some $q \in C$. Moreover,

$$d(q, F) \leq \|x_n - q\| + d(x_n, F) \to 0 \text{ as } n \to \infty.$$

Since F is closed, then $q \in F$. This completes the proof.

3. Convergence in Uniformly Convex Banach Spaces

In this section, some weak and strong convergence results are established for iterative scheme (1) in uniformly convex Banach spaces without using the condition $\liminf_n d(x_n, F) = 0$ appearing in the preceding section. For this we have to consider Condition (A'') and Opial property.

The following lemma has the important ingredients for proving our main results.

Lemma 4. *Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X and $\{T_i : i = 1, 2, \ldots, m\}$ be a family of uniformly L-Lipschitz and generalized asymptotically quasi-nonexpansive self-mappings of C with the sequences $\{r_n^{(1)}\}, \ldots, \{r_n^{(m)}\}$ and $\{s_n^{(1)}\}, \ldots, \{s_n^{(m)}\}$ such that $\sum_{n=1}^{\infty} r_n^{(i)} < \infty$ and $\sum_{n=1}^{\infty} s_n^{(i)} < \infty$ for each $i = 1, 2, \ldots, m$. Let $\{x_n\}$ be the sequence defined by (1). Then we have the following conclusions.*

(i) If $0 < \liminf_n \alpha_{in} \leq \limsup_n (1 - \gamma_n) < 1$, then $\lim_n \|T_i^n y_n - x_n\| = 0$.

(ii) If $0 < \liminf_n \beta_{in} \leq \limsup_n (1 - \gamma_n) < 1$, then $\lim_n \|T_i^n x_n - x_n\| = 0$.

(iii) If $\liminf_n \alpha_{jn} > 0$ for some $j = 1, 2, \ldots, m$ and $0 < \liminf_n a_{in} \leq \limsup_n (1 - b_n) < 1$, then $\lim_n \|T_i^n x_n - x_n\| = 0$.

(iv) If $\lim_n \|T_i^n x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, m$, then $\lim_n \|T_i x_n - x_n\| = 0$ for all $i = 1, 2, \ldots, m$.

Proof. (i) Let $p \in F$. By Lemma 3(i), $\lim_n \|x_n - p\|$ exists. Let $\lim_n \|x_n - p\| = a$ for some $a \geq 0$. Then,

$$\limsup_n \|T_i^n x_n - p\| \leq \limsup_n ((1 + r_{in})\|x_n - p\| + s_{in}) \leq a \tag{10}$$

for all $i = 1, 2, \ldots, m$. Also, by taking $\limsup_n$ in both sides of (5), we obtain that

$$\limsup_n \|y_n - p\| \leq \limsup_n \|x_n - p\| = a$$

and so

$$\limsup_n \|T_i^n y_n - p\| \leq \limsup_n ((1 + r_{in})\|y_n - p\| + s_{in}) \leq a \tag{11}$$

for all $i = 1, 2, \ldots, m$. Moreover, we note that

$$a = \lim_n \|x_{n+1} - p\| = \lim_n \left\| \sum_{i=1}^{m} \left(\alpha_{in} T_i^n y_n + \beta_{in} T_i^n x_n \right) + \gamma_n x_n - p \right\|$$
$$= \lim_n \left\| \sum_{i=1}^{m} \left(\alpha_{in}(T_i^n y_n - p) + \beta_{in}(T_i^n x_n - p) \right) + \gamma_n (x_n - p) \right\|.$$

This together with (10), (11) and Lemma 2 implies that the conclusions (i) and (ii) of lemma are satisfied.
Next, we shall prove (iii). Since $\liminf_n \alpha_{jn} > 0$, it follows from Lemma 3(iii) that $\lim_n \|y_n - p\| = a$. Therefore,

$$a = \lim_n \left\| y_n - p \right\| = \lim_n \left\| \sum_{i=1}^{m} a_{in} T_i^n x_n + b_n x_n - p \right\|$$
$$= \lim_n \left\| \sum_{i=1}^{m} a_{in}(T_i^n x_n - p) + b_n (x_n - p) \right\|.$$

This together with (10) and Lemma 2 implies that $\lim_n \|T_i^n x_n - x_n\| = 0$.

(v) Using (1), we have

$$\|y_n - x_n\| \le \sum_{i=1}^{m} a_{in} \|T_i^n x_n - x_n\| \to 0,$$
$$\|T_i^n y_n - x_n\| \le \|T_i^n y_n - T_i^n x_n\| + \|T_i^n x_n - x_n\|$$
$$\le L\|y_n - x_n\| + \|T_i^n x_n - x_n\| \to 0.$$

Then,

$$\|x_{n+1} - x_n\| \le \sum_{i=1}^{m} \left(\alpha_{in} \|T_i^n y_n - x_n\| + \beta_{in} \|T_i^n x_n - x_n\|) \right) \to 0. \tag{12}$$

For each $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} \|T_i x_n - x_n\| &\le \|T_i x_n - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\le L\|x_n - T_i^n x_n\| + L\|x_n - x_{n+1}\| + \|T_i^{n+1} x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \end{aligned}$$

which together with (12) implies that

$$\lim_n \|T_1 x_n - x_n\| = \lim_n \|T_2 x_n - x_n\| = \cdots = \lim_n \|T_m x_n - x_n\| = 0.$$

This completes the proof of lemma.

Lemma 5. *Let X, C and $T_1, T_2, \dots, T_m$ be as in Lemma 4 and $\{x_n\}$ be the sequence defined by (1) such that the parameters satisfy one of the following control conditions:*

(C1) $0 < \liminf_n \alpha_{in} \le \limsup_n (1-\gamma_n) < 1$ *for all* $i = 1, 2, \dots, m$ *and* $\limsup_n (1-b_n)L < 1$;

(C2) $0 < \liminf_n \beta_{in} \le \limsup_n (1-\gamma_n) < 1$ *for all* $i = 1, 2, \dots, m$;

(C3) $\liminf_n \alpha_{jn} > 0$ *for some* $j \in \{1, 2, \dots, m\}$ *and* $0 < \liminf_n a_{in} \le \limsup_n (1-b_n) < 1$ *for all* $i = 1, 2, \dots, m$.

Then $\lim_n \|T_i^n x_n - x_n\| = 0$ *for all* $i = 1, 2, \dots, m$*, and so by Lemma 4(iv),* $\lim_n \|T_i x_n - x_n\| = 0$ *for all* $i = 1, 2, \dots, m$.

Proof. (C1) It follows from Lemma 4(i) that $\lim_n \|T_i^n y_n - x_n\| = 0$ for all $i = 1, 2, \dots, m$. Using (1) we have

$$\begin{aligned}\|y_n - x_n\| &\le \sum_{i=1}^m a_{in}\|T_i^n x_n - x_n\| \\ &\le \sum_{i=1}^m a_{in}(\|T_i^n x_n - T_i^n y_n\| + \|T_i^n y_n - x_n\|) \\ &\le \sum_{i=1}^m a_{in}(L\|y_n - x_n\| + \|T_i^n y_n - x_n\|) \\ &= (1-b_n)L\|y_n - x_n\| + \sum_{i=1}^m a_{in}\|T_i^n y_n - x_n\|.\end{aligned}$$

Thus, $\lim_n (1-(1-b_n)L)\|y_n - x_n\| = 0$. Since $\limsup_n (1-b_n)L < 1$, then

$$\lim_n \|y_n - x_n\| = 0. \tag{13}$$

Next, we observe that

$$\begin{aligned}\|T_i^n x_n - x_n\| &\le \|T_i^n x_n - T_i^n y_n\| + \|T_i^n y_n - x_n\| \\ &\le L\|y_n - x_n\| + \|T_i^n y_n - x_n\|.\end{aligned}$$

This together with (13) implies that $\lim_n \|T_i^n x_n - x_n\| = 0$ for all $i = 1, 2, \dots, m$. This completes the proof of (C1). (C2) and (C3) follow from (ii) and (iii) of Lemma 4, respectively.

Now, we state and prove the weak and strong convergence theorems of (1).

Theorem 2. *Let X, C, $T_1, \dots, T_m$ and $\{x_n\}$ be as in Lemma 5. Then we have the followings.*

(i) If $\{T_i : i = 1, 2, \dots, m\}$ satisfies Condition (A''), then $\{x_n\}$ converges strongly to a common fixed point of the family.

(ii) If X satisfies Opial's condition and $I - T_i$ is demiclosed at 0 for all $i = 1, 2, \dots, m$, then $\{x_n\}$ converges weakly to a common fixed point of the family.

Proof. (i) It follows from Lemma 5 that $\lim_n \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, m$. Therefore, by using Condition (A''), there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\lim_n f(d(x_n, F)) \le \lim_n \|x_n - T_i x_n\| = 0$$

for some $i = 1, 2, \ldots, m$. That is $\lim_n d(x_n, F) = 0$. By Theorem 1, we conclude that $\{x_n\}$ converges strongly to a point $p \in F$.

(ii) Let $p \in F$. It follows from Lemma 3 that $\lim_n \|x_n - p\|$ exists and hence $\{x_n\}$ is bounded. Since X is uniformly convex, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging weakly to some $u \in C$. By Lemma 4, $\lim_n \|x_n - T_i x_n\| = 0$ for all $i = 1, 2, \ldots, m$. Since $I - T_i$ is demiclosed for all $i = 1, 2, \ldots, m$, we obtain $u \in F$. Suppose that subsequences $\{x_{n_k}\}$ and $\{x_{n_l}\}$ of $\{x_n\}$ converge weakly to u and v, respectively. As proved above $u, v \in F$. Again by Lemma 3, $\lim_n \|x_n - u\|$ and $\lim_n \|x_n - v\|$ exist. Assume that $u \neq v$. Then by the Opial property

$$\begin{aligned} \lim_n \|x_n - u\| &= \lim_k \|x_{n_k} - u\| < \lim_k \|x_{n_k} - v\| = \lim_n \|x_n - v\| \\ &= \lim_l \|x_{n_l} - v\| < \lim_l \|x_{n_l} - u\| = \lim_n \|x_n - u\| \end{aligned}$$

This contradiction proves that $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, m\}$ and the proof is completed.

Remark 1. *If $T_1 = T_2 = \cdots = T_m$ in Theorem 2 we obtain weak and strong convergence of the modified Mann iteration (4) and the modified Ishikawa iteration.*

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