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On the Vanishing Properties of Local Cohomology Modules Defined by a Pair of Ideals

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Abstract. As a generalization of the ordinary local cohomology modules, recently some authors introduced the local cohomology modules with respect to a pair of ideals. In this paper, we get some results on Artinianness, vanishing, finiteness and other properties of these modules. Let R be a commutative Noetherian ring, I, J two ideals of R and M a finitely generated R-module such that $\dim_R M = n$. We prove that $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is I-cofinite Artinian and $H^n_{I,J}(M)/IH^n_{I,J}(M)$ has finite length. Also we show that, if R is local with $\dim R/I+J = 0$ and $\dim_R M/JM = d > 0$, then $H^d_{I,J}(M)$ is not finitely generated.

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1. Introduction

Throughout this paper, R is a commutative Noetherian ring with non-zero identity, I, J are two ideals of R and M is an R-module. For notations and terminologies not given in this paper, the reader is referred to [1] and [6], if necessary.

As a generalization of the ordinary local cohomology modules, Takahashi, Yoshino and Yoshizawa, in [6], introduced the local cohomology modules with respect to a pair of ideals (I,J). To be more precise, let $\mathrm{W}(I,J) = \{\mathfrak{p} \in \mathrm{Spec(R)} : I^t \subseteq \mathfrak{p} + J$ for some positive integer $t\}$. The set of elements x of M such that $\mathrm{Supp}_R Rx \subseteq \mathrm{W}(I,J)$, is said to be (I,J)-torsion submodule of M and is denoted by $\Gamma_{I,J}(M)$. It is easy to see that $\Gamma_{I,J}$ is a covariant, R-linear functor from the category of R-modules to itself. For an integer i, the local cohomology functor $H^i_{I,J}$ with respect to (I,J), is defined to be the i-th right derived functor of $\Gamma_{I,J}$. Also $H^i_{I,J}(M)$ is called

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the i-th local cohomology module of M with respect to (I,J). If $J=0$, then $H^i_{I,J}$ coincides with the ordinary local cohomology functor H^i_I.

Some authors studied the properties of these extended modules; see, for example, [2, 3, 5, 7]. In this direction, we study Artinianness, vanishing and finiteness of the local cohomology modules defined by a pair of ideals. Suppose that M is finitely generated with $\dim_R M = n$. It is well known that $H^n_I(M)$ is I-cofinite Artinian; [see 4, Proposition 5.1]. We generalize this result and prove that $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is I-cofinite Artinian.

Let R be local and M finitely generated with $\dim_R M = n > 0$. It follows by Grothendieck's Non-vanishing Theorem that $H^n_I(M)$ is not finitely generated, whenever $\dim R/I = 0$. As a generalization of this result, we show that if $\dim R/I+J = 0$ and $\dim_R M/JM = d > 0$, then $H^d_{I,J}(M)$ is not finitely generated.

2. Main Results

Recall that R is a Noetherian ring, I, J are two ideals of R and M is an R-module. The following result improves [5, Corollary 3.5].

Theorem 1. *Let M be finitely generated with $\dim_R M = n$. Then $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is I-cofinite Artinian.*

Proof. We use induction on n. If $n=0$, then M has finite length. Therefore $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M)$ has finite length and so $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M)$ is I-cofinite Artinian. Now suppose, inductively, that $n>0$, and the result has been proved for all R-modules of dimensions smaller than n satisfying the hypothesis. Since $H^n_{I,J}(M/\Gamma_{I,J}(M)) \cong H^n_{I,J}(M)$ by [6, Corollary 1.13(4)], we may assume in addition that M is an (I,J)-torsion free R-module. Thus I contains an element a which is non zero-divisor on M. Since $\dim M/aM \le n-1$, it follows by the inductive hypothesis that $H^{n-1}_{I,J}(M/aM)/JH^{n-1}_{I,J}(M/aM)$ is I-cofinite Artinian. The exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ induces an exact sequence

$$\cdots \to H^{n-1}_{I,J}(M/aM) \to H^n_{I,J}(M) \xrightarrow{a} H^n_{I,J}(M) \to 0$$

of local cohomology modules. Now the exact sequence

$$H^{n-1}_{I,J}(M/aM)/JH^{n-1}_{I,J}(M/aM) \to H^n_{I,J}(M)/JH^n_{I,J}(M) \xrightarrow{a} H^n_{I,J}(M)/JH^n_{I,J}(M) \to 0$$

implies that $0 :_{H^n_{I,J}(M)/JH^n_{I,J}(M)} a$ is I-cofinite Artinian. Therefore $H^n_{I,J}(M)/JH^n_{I,J}(M)$ is I-cofinite Artinian, by [4, Proposition 4.1]. This completes the inductive step. The result follows by induction.

Let $\tilde{W}(I,J)$ denote the set of ideals $\mathfrak{a}$ of R such that $I^t \subseteq \mathfrak{a}+J$ for some positive integer t. It is easy to see that, for any $\mathfrak{a} \in \tilde{W}(I,J)$, $\Gamma_{\mathfrak{a}}(M)$ is a subset of $\Gamma_{I,J}(M)$.

Theorem 2. *Let M be finitely generated with $\dim_R M = n$ and t a positive integer. If $H^i_{I,J}(M) = 0$, for all $i > t$, then $H^t_{I,J}(M)/\mathfrak{a}H^t_{I,J}(M) = 0$, for any $\mathfrak{a} \in \tilde{W}(I,J)$.*

Proof. Let $\mathfrak{a} \in \tilde{W}(I,J)$ be fixed. We prove the claim by using induction on n. If $n = 0$, then the claim is clear. Assume, inductively, that $n > 0$ and the result has been proved for any R-module of dimension less than n satisfying the hypothesis. Since $H^i_{I,J}(M/\Gamma_{I,J}(M)) \cong H^i_{I,J}(M)$ for all $i > 0$, by [6, Corollary 1.13(4)], we may assume in addition that $\Gamma_{I,J}(M) = 0$. We have $\Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{I,J}(M)$, thus $\Gamma_{\mathfrak{a}}(M) = 0$, and therefore $\mathfrak{a}$ contains an element a which is non zero-divisor on M. The exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ induces the following exact sequence

$$\cdots \to H^i_{I,J}(M) \xrightarrow{a} H^i_{I,J}(M) \to H^i_{I,J}(M/aM) \to H^{i+1}_{I,J}(M) \to \cdots$$

of local cohomology modules. In view of the hypothesis and the above exact sequence, $H^i_{I,J}(M/aM) = 0$ for all $i > t$. Since a is non zero-divisor on M, we have $\dim M/aM \le n-1$, and therefore the inductive hypothesis implies that $H^t_{I,J}(M/aM)/\mathfrak{a}H^t_{I,J}(M/aM) = 0$. The above exact sequence implies that $H^t_{I,J}(M)/aH^t_{I,J}(M) \cong H^t_{I,J}(M/aM)$. Since $a \in \mathfrak{a}$, therefore

$$H^t_{I,J}(M)/\mathfrak{a}H^t_{I,J}(M) \cong H^t_{I,J}(M/aM)/\mathfrak{a}H^t_{I,J}(M/aM).$$

The inductive step is complete. The result follows by induction.

Corollary 1. *Let M be a finitely generated module such that $\dim_R M = n$. Then $H^n_{I,J}(M)/\mathfrak{a}H^n_{I,J}(M)$ has finite length, for any $\mathfrak{a} \in \tilde{W}(I,J)$. Specially, $H^n_{I,J}(M)/IH^n_{I,J}(M)$ has finite length.*

Proof. Let $\mathfrak{a} \in \tilde{W}(I,J)$ be fixed. If $n = 0$, then M has finite length and so $\Gamma_{I,J}(M)/\mathfrak{a}\Gamma_{I,J}(M)$ has finite length. Now assume that $n > 0$. It follows by [6, Theorem 4.7(1)] and Theorem 2, that $H^n_{I,J}(M)/\mathfrak{a}H^n_{I,J}(M) = 0$.

Corollary 2. *Let M be finitely generated of finite dimension such that $\dim_R M/JM = d$. Then $H^{d+1}_{I,J}(M)/\mathfrak{a}H^{d+1}_{I,J}(M)$ is finitely generated, for any $\mathfrak{a} \in \tilde{W}(I,J)$. Specially, $H^{d+1}_{I,J}(M)/IH^{d+1}_{I,J}(M)$ is finitely generated.*

Proof. Let $\mathfrak{a} \in \tilde{W}(I,J)$ be fixed. If $d = -1$, then the claim is trivial. Now assume that $d \ge 0$. It follows by [6, Theorem 4.7(2)] and Theorem 2, that $H^{d+1}_{I,J}(M)/\mathfrak{a}H^{d+1}_{I,J}(M) = 0$.

Corollary 3. *Let R be local and M a finitely generated module such that $\dim_R M/JM = d$. Then $H^d_{I,J}(M)/\mathfrak{a}H^d_{I,J}(M)$ is finitely generated, for any $\mathfrak{a} \in \tilde{W}(I,J)$. In particular, $H^d_{I,J}(M)/IH^d_{I,J}(M)$ is finitely generated.*

Proof. Let $\mathfrak{a} \in \tilde{W}(I,J)$ be fixed. If $d = 0$, then the claim is trivial. Now assume that $d > 0$. It follows by [6, Theorem 4.3] and Theorem 2, that $H^d_{I,J}(M)/\mathfrak{a}H^d_{I,J}(M) = 0$.

Proposition 1. *Let R be local, M finitely generated and t a non-negative integer. If $H^i_{I,J}(M)$ is finitely generated, for all $i > t$, then $H^i_{I,J}(M) = 0$, for all $i > t$.*

Proof. We may assume that $I \neq R$, otherwise $\Gamma_{I,J}$ is identity functor. Proposition 4.10, in [6], says that $H^i_{I,J}(M) = 0$, for all $i > \mathrm{ara}(I\overline{R})$, where $\overline{R} = R/\sqrt{J + \mathrm{Ann}_R(M)}$. Let $s = \mathrm{ara}(I\overline{R})$. When $t \geq s$, there is nothing to prove. Now, assume that $t < s$. In view of Theorem 2, we have $H^s_{I,J}(M)/IH^s_{I,J}(M) = 0$, so Nakayama's Lemma shows that $H^s_{I,J}(M) = 0$. By keeping this process, we deduce that $H^i_{I,J}(M) = 0$, for all $i > t$.

Corollary 4. *Let R be local with* $\dim R/I + J = 0$ *and M finitely generated. Then $H^d_{I,J}(M)$ is not finitely generated, where* $\dim_R M/JM = d > 0$.

Proof. Note that $\sup\{i : H^i_{I,J}(M) \neq 0\} = d$, by [6, Theorem 4.5]. Now the claim follows by Proposition 1.

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