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On Statistical Boundedness of Metric Valued Sequences

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Abstract. In this work, statistical boundedness is defined in a metric space and, statistical boundedness of metric valued sequences and their subsequences are studied. The interplay between the statistical boundedness and boundedness in a metric spaces are also studied, and it is shown that boundedness imply statistical boundedness and if the number of elements of the metric space is finite then these two concepts coincide. Moreover, here is given analogy of Balzano-Weierstrass Theorem.

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1. Introduction and Definitions

The statistical convergence of real or complex valued sequences was first introduced by Fast [7], but the idea of statistical convergence goes back to Zygmund [16]. In recent years, statistical convergence has become popular research area for many mathematicians [2, 3, 8, 9, 10, 12, 13, 14, 15] etc.

On the other hand, analysis on metric spaces has rapidly developed in present time [see 11]. This development is usually based on some generalizations of the differentiability.

Some approaches which based on the convergence of the metric valued sequences have been studied in [6] [see also 1, 4, 5]. But in these studies it is not deductive clear that the usual convergence is the best possible way to obtain the smooth structure for arbitrary metric space.

A lot of different convergence methods were defined (Cesaro, Nörlund, Weighted Mean, Abel etc.) and applied to many branches of mathematics. Almost all convergence methods depend on the algebraic structure of the space. It is clear that metric space does not have the algebraic structure in general. So, the generalization of boundedness by using these methods for metric valued sequences is impossible. However, the notion of statistical convergence is easy to extend for arbitrary metric spaces.

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The concept d-statistical convergence for metric valued sequences was given first time in [12]. In this work, d- statistical boundedness of metric valued sequences is defined, and the relations between usual boundedness, convergence, d- statistical convergence and d-statistical boundedness are investigated.

Let (X, d) be a metric space. For convenience denote by $\tilde{X}$ the set of all sequences of points from X. That is, $\tilde{X} = \{\tilde{x} = (x_n) : x_n \in X\}$.

Let us remember the usual definition of convergence and boundedness in any arbitrary metric space:

Definition 1 (In usual case). *Let $\tilde{x} = (x_n) \in \tilde{X}$ be a sequence.*

(i) $\tilde{x}$ is called convergent to a point $a \in X$ if for every $\epsilon > 0$ there exist an $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that

$$d(x_n, a) < \epsilon \tag{1}$$

for every $n \geq n_0$.

(ii) $\tilde{x}$ is called bounded if for every $x \in X$ there is a $M > 0$ such that

$$d(x_n, x) < M \tag{2}$$

for all $n \in \mathbb{N}$.

The set of convergent and bounded sequences in $\tilde{X}$ will be denoted by $C(\tilde{X})$ and $B(\tilde{X})$, respectively. It is clear that $C(\tilde{X}) \subset B(\tilde{X})$. But, in general the inverse is not true: To see this, take into consider the sequence $\tilde{x} = (x_n)$ with

$$x_n = \begin{cases} x, & n \text{ even} \\ y, & n \text{ odd} \end{cases}$$

for an arbitrary $x, y \in X$. It is clear that $\tilde{x}$ is bounded, but it is not convergent.

Definition 2 (Statistical case). *Let $\tilde{x} = (x_n) \in \tilde{X}$ be a sequence.*

(i) $\tilde{x}$ is called $d-$statistical convergent to a point $a \in X$ if, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k : k \leq n \text{ and } d(x_k, a) \geq \epsilon\}\right| = 0 \tag{3}$$

is satisfied.

(ii) $\tilde{x}$ is called d-statistical bounded if, for an arbitrary $x \in X$ there is a $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k : k \leq n \text{ and } d(x_k, x) \geq M\}\right| = 0 \tag{4}$$

is satisfied.

In (3) and later $|B|$ denotes the number of elements of the set B.

The set of d-statistical convergent sequences and d-statistical bounded sequences in $\tilde{X}$ will be denoted by $C^d_{st}(\tilde{X})$ and $B^d_{st}(\tilde{X})$, respectively.

Definition 3 (Asymptotic Density)**.**

(i) Let $K \subseteq \mathbb{N}$ and

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n} |K(n)|,$$

where $K(n) := \{k \in K : k \leq n\}$. If the limit exists and finite, then the number $\delta(K)$ is called asymptotic density of the set K [13].

(ii) If $\delta(K) = 1$ then the set $K \subseteq \mathbb{N}$ is called a statistical dense subset of $\mathbb{N}$ [7].

2. Main Theorems and Their Proofs

The relation between bounded sequences and convergent sequences in an arbitrary metric space is known. How will be it for $d-$ statistical boundedness and $d-$ statistical convergence?

In this section, we will answer this question and give some relations between $d-$ statistical boundedness and $d-$ statistical convergence and with usual boundedness and convergence.

Theorem 1. *Let (X,d) be a metric space and $\tilde{x} = (x_n) \in \tilde{X}$. The following statements hold:*

(i) If $\tilde{x}$ is bounded, then $\tilde{x}$ is d- statistical bounded.

(ii) If $\tilde{x}$ is d- statistical convergent to $a \in X$, then $\tilde{x}$ is d- statistical bounded.

Proof.

(i) If $\tilde{x}$ is bounded, then for an arbitrary $x \in X$ there is $M > 0$ such that for all $n \in \mathbb{N}$

$$d(x_n, x) < M.$$

That is,

$$\{k : k \leq n, d(x_k, x) \geq M\} = \emptyset.$$

Hence, we have

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k : k \leq n \text{ and } d(x_k, x) \geq M\}\right| = 0.$$

(ii) For any arbitrary $\epsilon > 0$ and large $M > 0$ we have

$$\{k : k \leq n \text{ and } d(x_k, a) \geq M\} \subset \{k : k \leq n \text{ and } d(x_k, a) \geq \epsilon\}.$$

This inclusion gives

$$|\{k : k \leq n \text{ and } d(x_k, a) \geq M\}| \leq |\{k : k \leq n \text{ and } d(x_k, a) \geq \epsilon\}|.$$

From this inequality, we have

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k : k \le n \text{ and } d(x_k, a) \ge M\}\right| = 0$$

for every $\epsilon > 0$. This gives the proof.

Remark 1. *The inverse of (i) in Theorem 1 is not true in general.*

Example 1. *Let us take $X = \mathbb{R}$ with usual metric $d(x, y) := |x - y|$ and consider the sequence $\tilde{x} = (x_n) = (1, 1, -1, 2, -1, 1, -1, 1, 3, \dots)$ with*

$$x_n := \begin{cases} k, & n = k^2, \\ (-1)^n, & n \ne k^2. \end{cases}$$

It is clear that $\tilde{x}$ is not bounded in usual case. Let's show that $\tilde{x}$ is d-statistical bounded. For this aim, choose $x = 0$ and a sufficiently large $M > 0$. Then,

$$\frac{1}{n}|\{k : k \le n, |x_k| \ge M\}| = \frac{1}{n}|\{k : k = m^2 \le n, |m| \ge M\}| \le \frac{[|\sqrt{n}|] - [|M|] + 2}{n}$$

This calculation shows $\tilde{x} = (x_n)$ is d-statistical bounded.

Remark 2. *The inverse of (ii) in Theorem 1 is not true in general.*

Example 2. *Assume that $x, y \in X$ are distinct points and let us define the sequence $\tilde{x} = (x_n)$ with*

$$x_n := \begin{cases} x, & \text{if } n = 2k + 1, k \in \mathbb{N} \\ y, & \text{if } n = 2k \end{cases}$$

that is,

$$\tilde{x} = (\overset{1}{\widehat{x}}, y, \overset{3}{\widehat{x}}, y, \overset{5}{\widehat{x}}, y, \overset{7}{\widehat{x}}, \dots).$$

Let us choose $M > 0$ satisfying

$$M > 2\, max\{d(x, z), d(y, z)\}$$

for an arbitrary fixed $z \in X$. Then we have

$$\{k : k \le n, d(x_k, z) \ge M\} = \emptyset.$$

So, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k : k \le n \text{ and } d(x_k, z) \ge M\}\right| = 0.$$

This shows that $\tilde{x}$ is d–statistical bounded.
Now let us show that $\tilde{x}$ is not d–statistical convergent to x(or y). For this aim, set $\epsilon < d(x, y)$. Then we have

$$|\{k : k \le n, d(x_k, x(\text{or } y)) \ge \epsilon\}| = \begin{cases} = \frac{n}{2}, & n \text{ even}; \\ < \frac{n}{2}, & n \text{ odd}. \end{cases}$$

According to this,

$$\lim_{n\to\infty}\frac{1}{n}\left|\{k : k \le n \text{ and } d(x_k, x(or\ y)) \ge \epsilon\}\right| = \frac{1}{2} \ne 0.$$

So, $\tilde{x}$ is not d–statistical convergent to x(or y).

Examples 1 and 2 show that the inclusions

$$B(\tilde{X}) \subset B^d_{st}(\tilde{X}) \text{ and } C^d_{st}(\tilde{X}) \subset B^d_{st}(\tilde{X})$$

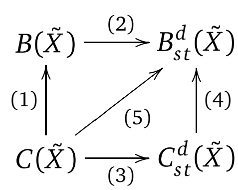
are sharp.

Corollary 1. *If $\tilde{x} = (x_n)$ is convergent to a point $a \in X$, then $\tilde{x}$ is d- statistical bounded.*

Proof. From [12] we know that convergence implies d- st convergence in an arbitrary metric space. Therefore, taking into account (ii) in Theorem 1, we get the proof.

Another way, if $\tilde{x}$ is convergent to $a \in X$, $\tilde{x}$ is bounded. By using (i) in Theorem 1, we have the result.

Corollary 2. *Let (X, d) be an arbitrary metric space. Then the following diagram holds:*



where $A \to B$ means that $A \subset B$.

Proof. The inclusion (5) is obtained directly from Corollary 1. The inclusions (2) and (4) are obtained from Theorem 1-(i) and (ii), respectively. The inclusion (3) is obtained from Proposition 2.1 in [12].

The inverse of all inclusions in the diagram are not true. Take into consider Example 1 in [12] for the inverse of (3), and in this work Example 1 and Example 2 for the inverse of (2) and (4). The following theorem gives necessary and sufficient condition on X for which the inverse of the inclusion (2) in the Corollary 2 is true:

Theorem 2. *Let (X, d) be a metric space with $X \ne \emptyset$. The following two statements are equivalent:*

(i) The set of all bounded sequences $\tilde{x} = (x_k) \in \tilde{X}$ is the same as the set of all d – statistical bounded sequences $\tilde{x} \in \tilde{X}$.

(ii) The cardinality of X is finite.

Proof. $(ii) \Rightarrow (i)$ The cardinality of X is finite. That is, there exist a number $n_0 \in \mathbb{N}$ such that $X = \{x_0, x_1, \ldots, x_{n_0}\}$, since the cardinality of X is finite. From the hypothesis, the diameter of X,

$$d(X) := sup\{d(x_k, x_l) : x_k, x_l \in X; k, l = \overline{0, n_0}\}$$

is finite.

If $\tilde{x}$ is bounded, then from Theorem 1-(i), $\tilde{x}$ is also d–statistical bounded.

So, we must only show the inverse is true under the hypothesis. Let's take $d-$ statistical bounded sequence $\tilde{x} = (x_k) \in \tilde{X}$, i.e., $\exists M > 0$ and $x \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n, d(x_k, x) \geq M\}| = 0. \tag{5}$$

Especially, if we take $M := 2d(X)$, we have, from (5),

$$\{k : k \leq n, d(x_k, x) \geq 2d(X)\} = \emptyset.$$

In other words,

$$\{k : k \leq n, d(x_k, x) < 2d(X)\} = n.$$

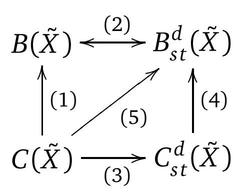
Therefore,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n, d(x_k, x) < 2d(X)\}| = 1.$$

This shows that $\tilde{x}$ is a bounded sequence.

$(i) \Rightarrow (ii)$ Suppose that $|X| = \infty$. Then, there exist at least one sequence that $d-$ statistical bounded but not bounded (see Example 1). That's why, our assumption is not true.

Corollary 3. *Let (X, d) be a metric space with finite number elements. Then, the following diagram holds:*



Corollary 4. *Let (X, d) be a metric space with $X \neq \emptyset$. Then, the following two statements are equivalent:*

(i) $B(\tilde{X}) = C(\tilde{X}) = B^d_{st}(\tilde{X}) = C^d_{st}(\tilde{X})$

(ii) The set X is a singleton.

Proof. The proof is open from above the Theorem 2 and Theorem 2.2 in [12].

Theorem 3. *Let X be a set which has at least two elements and endowed ρ discrete metric. Then, we have*

$$\tilde{X} = B(\tilde{X}) = B^{\rho}_{st}(\tilde{X}) \tag{6}$$

and

$$C(\tilde{X}) \subset C^{\rho}_{st}(\tilde{X}). \tag{7}$$

Proof. We know that any space with discrete metric has finite diameter. Therefore, any sequence in this space is bounded. That is, there is a $M > 0$ and arbitrary $x \in X$ such that $\rho(x_n, x) < M$ for every $\tilde{x} = (x_n) \in \tilde{X}$. So, we have

$$\{k : k \leq n, \rho(x_k, x) \geq M\} = \emptyset.$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n, \rho(x_k, x) \geq M\}| = 0.$$

This shows that $\tilde{x} \in B_{st}^{\rho}(\tilde{X})$ for all $\tilde{x} \in \tilde{X}$.

Let us take an arbitrary sequence $\tilde{x} \in B_{st}^{\rho}$. If we choose $M > 1$, then $\{k : k \leq n, \rho(x_k, x) \geq M\} = \emptyset$. It means that $\rho(x_k, x) < M$ for all $k \in \mathbb{N}$. Thus, we see that $\tilde{x} \in B(\tilde{X})$.

Now let us see that $C(\tilde{X}) \subset C_{st}^{\rho}(\tilde{X})$. Let $\tilde{x} \in C(\tilde{X})$. Then for an arbitrary $\varepsilon > 0$ there exist $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $x_n = x_{n_0}$ for all $n \geq n_0$. Therefore, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k : k \leq n, \rho(x_k, x_{n_0}) \geq \varepsilon\}| \leq \lim_{n\to\infty} \frac{n_0}{n} = 0.$$

This shows that $\tilde{x} \in C_{st}^{\rho}(\tilde{X})$.

If we consider the sequence $\tilde{x} = (x_n)$ with

$$x_n := \begin{cases} x & \text{if } n = k^2, k \in \mathbb{N} \\ y & \text{if } n \neq k^2 \end{cases}$$

then $\tilde{x}$ is ρ- statistical convergent to x but not convergent. So, the inclusion in (7) is sharp.

Corollary 5. *Let X be a set which has at least two elements and endowed bounded metric. Then, (6) and (7) hold.*

Theorem 4. *Let (X, d) be a metric space, $\tilde{x} = (x_n) \in \tilde{X}$ and let $\tilde{x}' = (x_{n_k})$ be a subsequence of $\tilde{x}$. If $\tilde{x}$ is d-statistical bounded then $\tilde{x}'$ is also d-statistical bounded.*

Proof. Suppose that $\tilde{x}$ is d – statistical bounded. It is clear that there is a number $M > 0$ and $x \in X$ such that

$$\{n_k : n_k \leq n, d(x_{n_k}, x) \geq M\} \subset \{k : k \leq n, d(x_k, x) \geq M\}.$$

Then, since

$$|\{n_k : n_k \leq n, d(x_{n_k}, x) \geq M\}| \leq |\{k : k \leq n, d(x_k, x) \geq M\}|,$$

we have

$$0 \leq \lim_{n\to\infty} \frac{1}{n} \left| \left\{ n_k : n_k \leq n \text{ and } d(x_{n_k}, x) \geq M \right\} \right| \leq 0.$$

The last inequality shows that $\tilde{x}'$ is d – statistical bounded.

Lemma 1. *Let (X,d) be a metric space and $\widetilde{x}=(x_n)\in\widetilde{X}$. Then, the sequence $\tilde{x}$ is $d-st$ bounded if and only if the real sequence $\left(d(x_n,x)\right)$ is statistical bounded for an arbitrary $x\in X$.*

It can be obtained directly from the Definition 1.2-(ii). So, the proof is omitted here.

Lemma 2. *Let (X,d) be a metric space and $\widetilde{x}=(x_n)\in\widetilde{X}$ be a $d-$statistical bounded sequence. Then, the sequence $\widetilde{x}$ has at least one bounded subsequence.*

Proof. From the Lemma 1 the real sequence $\left(d(x_n,x)\right)$ is statistical bounded in real numbers. That is, there is a positive number M such that $\delta(A)=1,\delta(B)=0$ where

$$A:=\left\{k:d(x_k,x)<M\right\},B:=\left\{k:d(x_k,x)\geq M\right\}.$$

Let $k_1\in\mathbb{N}$ be the minimal element of A and $d(x_{k_1},x)<M$. Since $\delta(A)=1$, it can be choosen $k_2\geq k_1$ such that the minimal element of the set

$$\left\{k:k>k_1,\ k\in A\right\}$$

satisfying $d(x_{k_2},x)<M$.

In the n-th step we can choose $k_n\geq k_{n-1}$ which is the minimal element of the set

$$\left\{k:k\geq k_{n-1},\ k\in A\right\}$$

such that $d(x_{k_n},x)<M$.

So, we obtain non-decreasing sequence $\left(k_n\right)_{n\in\mathbb{N}}$ such that $\widetilde{x}'=(x_{k_n})$ is the subsequence of $\tilde{x}$ satisfying $d(x_{k_n},x)<M$ for all $k_n\in\mathbb{N}$. This shows that the subsequence $\widetilde{x}'$ is bounded.

The following theorem gives the analogy of Balzano-Weierstrass Theorem:

Theorem 5. *Let (X,d) be an arbitrary metric space. Every $d-$statistical bounded sequence has at least $d-$statistical convergent subsequence.*

Proof. Let us take $d-$statistical bounded sequence $\widetilde{x}=(x_n)\in\widetilde{X}$ and denote the real sequence $(d(x_n,x))$ for arbitrary $x\in X$ by (y_n). From Lemma 1, the real sequence (y_n) is statistical bounded with usual metric on $\mathbb{R}$. Take into account the following sets for sufficiently large $M>0$,

$$A([0,M],n):=\left\{k:k\leq n,\ y_k\in[0,M)\right\},$$
$$B([0,M],n):=\left\{k:k\leq n,\ y_k\in[M,\infty)\right\}$$

such that $A\cup B=\{1,2,3,\ldots,n\}$ and $\delta(A)=1,\ \delta(B)=0$.

Also, denote $I_0:=[0,M]$ with the length $l(I_0)=M$ and divide it into two parts as

$$I_0^1:=\left[0,\frac{M}{2}\right]\text{ and }I_0^2:=\left[\frac{M}{2},M\right].$$

It is clear that asymptotic density of these sets satisfy

$$0\leq\delta(I_0^1)\leq 1,\ 0\leq\delta(I_0^2)\ \leq 1.$$

If $\delta(I_0^1)=0$ (or $\delta(I_0^2)=0$), consider I_0^2 (or consider I_0^1) otherwise consider the interval having big asymptotic density and denote it by I_1. Note that the length of I_1 is $l(I_1)=\frac{M}{2}$ and $I_1\subset I_0$.

If we divide the new closed interval into two parts as I_1^1 and I_2^1, we can choose a new one and denote this interval by I_2 from the above explanation such that the length of this interval $l(I_2)=\frac{M}{2^2}$ and $I_2\subset I_1$. After continuing this procedure we obtain closed nested intervals which length tends to zero. From the nested Theorem, we get

$$\bigcap_{n=1}^{\infty} I_n=\{y^*\}. \tag{8}$$

Now our aim to construct a subsequence of (y_n) such that it is convergent to y^*. For this turn to begin of the proof and choose k_1 which is the minimal element of I_0, choose $k_2\geq k_1$ which is the minimal element of I_1 and so on. If continue this process we obtain $k_n\geq k_{n-1}$ which is the minimal element of I_n. Also, we can choose $k_{n+1}\geq k_n$ that is minimal element of I_{n+1}. Otherwise the number of elements of I_n^1 (or I_n^2) is at most k_n. This is contradiction to assumption on I_n^1 (or I_n^2). So, non-decreasing sequence (k_n) gives the subsequence (y_{k_n}) of the sequence (y_n) satisfying

$$\left|y_{k_n}-y^*\right|<l(I_k)=\frac{M}{2^n}. \tag{9}$$

For every $\varepsilon>0$, there is a $k_{n_0}=k_{n_0}(\varepsilon)\in\mathbb{N}$ such that $\frac{1}{2^{k_n}}<\varepsilon$ for every $k_n\geq k_{n_0}$. So, from (9) we have

$$\left|y_{k_n}-y^*\right|<\varepsilon. \tag{10}$$

The equation (10) shows (y_{k_n}) is convergent to y^* in usual case. It means that there is an element $x^*\in X$ such that the sequence $y_n:=d(x_{k_n},x)$ convergent to $y^*=d(x^*,x)$.

In [12], we know that this implies the sequence $d(x_{k_n},x)$ is $d-$statistical convergent to $d(x^*,x)$. Hence, we have

$$\left\{k_n : k_n\leq n,\ d(x_{k_n},x^*)\geq\varepsilon\right\}\subset\left\{k_n : k_n\leq n,\ \left|d(x_{k_n},x)-d(x^*,x)\right|\geq\varepsilon\right\}.$$

From this inclusion and the Definition 2-(i) we have

$$0\leq\lim_{n\to\infty}\frac{1}{n}\left|\left\{k_n : k_n\leq n, d(x_{k_n},x^*)\geq\varepsilon\right\}\right|\leq\lim_{n\to\infty}\frac{1}{n}\left|\left\{k_n : k_n\leq n,\left|d(x_{k_n},x)-d(x^*,x)\right|\geq\varepsilon\right\}\right|=0.$$

The last inequality says the sequence $\tilde{x}$ has a d-statistical convergent subsequence.

The following theorem gives the relation between statistical equivalence and statistical boundedness for sequences.

Theorem 6. *Let (X,d) be a metric space, $\tilde{x}$ and $\tilde{y}$ belong to $\tilde{X}$ and assume that $\tilde{x}$ be a $d-$ statistical bounded sequence. If $\tilde{x}\asymp\tilde{y}$, then $\tilde{y}$ is also $d-$ statistical bounded.*

Proof. We can follow the method in the proof of Lemma 3.2 given in [12]. Let us define a subset A of $\mathbb{N}$ as "$n\in A\Leftrightarrow x_n\neq y_n$". Then, subset $\mathbb{N}\backslash A$ is statistical dense from the Definition 2-(ii). This implies that

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m\in A : m\leq n\}\right|=0. \tag{11}$$

Let $M > 0$ be a sufficiently large number. According to the definition of A we have

$$\begin{aligned}&\{m \in A : m \leq n \text{ } and \text{ } d(y_m, a) \geq M\}\\ &\quad \subseteq \{m \in A : m \leq n\} \cup \{m \in A : m \leq n \text{ } and \text{ } d(x_m, a) \geq M\},\end{aligned}$$

for all $n \in \mathbb{N}$ and arbitrary point $a \in X$. By using this inclusion and equality (11), we get

$$\begin{aligned}&\limsup_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(y_m, a) \geq M\}\right|}{n}\\ &\leq \limsup_{n\to\infty} \frac{|\{m \in A : m \leq n\}|}{n} + \limsup_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(x_m, a) \geq M\}\right|}{n}\\ &= \limsup_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(x_m, a) \geq M\}\right|}{n}.\end{aligned}$$

Since $\tilde{x}$ is $d-$ statistical bounded, we have

$$\limsup_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(x_m, a) \geq M\}\right|}{n} = 0.$$

Consequently the inequality

$$\limsup_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(y_m, a) \geq M\}\right|}{n} \leq 0 \tag{12}$$

holds. By using (12) we obtain

$$\begin{aligned}0 &\leq \liminf_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(y_m, a) \geq M\}\right|}{n}\\ &\leq \limsup_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(y_m, a) \geq M\}\right|}{n} \leq 0.\end{aligned}$$

Hence the limit relation

$$\lim_{n\to\infty} \frac{\left|\{m \in \mathbb{N} : m \leq n \text{ and } d(y_m, a) \geq M\}\right|}{n} = 0$$

holds. Therefore, this gives the desired result.

Suppose (X, d) and (Y, d') are two metric spaces. We say that Y is a metric subspace of X and X is a metric superspace of Y if, and only if Y is a subset of X and d' is a restriction of d, i.e.

$$d'(x, y) := d_Y(x, y).$$

Theorem 7. *Let (X, d) be a metric space and (Y, d') be a metric subspace. Then, the following statements hold:*

(i) If $(y_n) \in \widetilde{Y}$ is $d'-$statistical bounded then (y_n) is $d-$statistical bounded.

(ii) If $(x_n) \in \widetilde{X}$ is $d-$statistical bounded, then (x_n) $d'-$statistical bounded in subspace that contains all terms of (x_n).

Proof.

(i) From the definition of subspace metric and statistical boundednes and for arbitrary $y^* \in Y$ and sufficiently large positive $M > 0$ we have

$$\{k : k \leq n,\ d'(y_k, y^*) \geq M\} = \{k : k \leq n,\ d(y_k, y^*) \geq M\}.$$

Therefore,

$$0 = \lim_{n\to\infty} \frac{1}{n} \left|\{k : k \leq n,\ d'(y_k, y^*) \geq M\}\right| = \lim_{n\to\infty} \frac{1}{n} \left|\{k : k \leq n,\ d(y_k, y^*) \geq M\}\right|.$$

(ii) It can be prove by using the same arguments in (i).

Let $X \neq \emptyset$ be a metric space with equivalent metrics d_1 and d_2. That is, there are positive constant m_1 and m_2 such that

$$m_1 d_1(x,y) \leq d_2(x,y) \leq m_2 d_1(x,y) \tag{13}$$

for every $x, y \in X$.

Theorem 8. *Let X be a metric space with equivalent d_1 and d_2 metrics. Then, the following statements hold:*

(i) $B_{st}^{d_1}(\widetilde{X}) = B_{st}^{d_2}(\widetilde{X})$,

(ii) $C_{st}^{d_1}(\widetilde{X}) = C_{st}^{d_2}(\widetilde{X})$.

Proof.

(i) Assume that $\tilde{x} = (x_n) \in \widetilde{X}$ is an arbitrary d_1- statistical bounded sequence, i.e. $\tilde{x} \in B_{st}^{d_1}(\widetilde{X})$. Then, there is a positive $M > 0$ and arbitrary $x^* \in X$ such that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k : k \leq n, d_1(x_k, x^*) \geq M\}\right| = 0.$$

Also, from (13) we have

$$\begin{aligned} \{k : k \leq n, d_1(x_k, x^*) \geq M\} &\subset \{k : k \leq n, d_2(x_k, x^*) \geq m_1 M\} \\ &\subset \left\{k : k \leq n, d_1(x_k, x^*) \geq \frac{m_1}{m_2} M\right\} \end{aligned}$$

and

$$\left|\{k : k \leq n, d_1(x_k, x^*) \geq M\}\right| \leq \left|\{k : k \leq n, d_2(x_k, x^*) \geq m_1 M\}\right|$$

$$\leq \quad \left|\left\{k : k \leq n,\, d_1(x_k, x^*) \geq \frac{m_1}{m_2} M\right\}\right|.$$

By using this inequality, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k : k \leq n,\, d_2(x_k, x^*) \geq m_1 M\}\right| = 0.$$

This shows that $\tilde{x} = (x_n) \in \widetilde{X}$ is d_2- statistical bounded sequence, i.e. $\tilde{x} \in B_{st}^{d_2}(\widetilde{X})$. To obtain inverse, it is enough to consider (13) as follows:

$$\frac{1}{m_2} d_2(x, y) \leq d_1(x, y) \leq \frac{1}{m_1} d_2(x, y)$$

for every $x, y \in X$.

(ii) The proof is clear from Theorem 2.3 in [12].

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