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The Ehresmann–Schein–Nambooripad Theorem and its Successors

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Abstract. The Ehremann–Schein–Nambooripad Theorem expresses the fundamental connection between the notions of inverse semigroups and inductive groupoids, which exists because these concepts provide two distinct approaches to the study of one-one partial transformations. In the case of arbitrary partial transformations, the analogous two approaches are provided by restriction semigroups and inductive categories, the former being generalisations of inverse semigroups, and the latter of inductive groupoids. There is indeed also a generalisation of the Ehremann–Schein–Nambooripad Theorem which encapsulates the connection between these two more general objects. In this article, we will explore the origins of these theorems, and survey the basic theory surrounding them.

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1. Introduction

The Ehresmann–Schein–Nambooripad Theorem (hereafter, "ESN Theorem") was first formulated explicitly by Lawson [29, Theorem 4.1.8], bringing together the work of the three named authors. This theorem (presented below as our Theorem 1) expresses the important connection between the class of inverse semigroups* and that of inductive groupoids, where an inductive groupoid is a small ordered category, subject to certain conditions on its ordering, in which all arrows are invertible (note that a "small textquotedblright category is one which is based upon a set, rather than a class). It should in fact be no surprise that these two notions are so closely related, for they are, in essence, two distinct solutions to the same problem: that of axiomatising systems of one-one partial transformations. If we are content to work

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*Defined abstractly as a semigroup S in which every element s has a unique *generalised inverse* s': $ss's = s$ and $s'ss' = s'$. Equivalently (and often easier to prove), an inverse semigroup is a semigroup in which every element has *at least one* generalised inverse, and in which idempotents commute with each other. We will assume that the reader has a passing familiarity with the theory of inverse semigroups.

with a partially-defined composition of such partial transformations, then we obtain an inductive groupoid. On the other hand, if we insist upon an everywhere-defined composition, the notion of an inverse semigroup emerges. The ESN Theorem expresses the fact that, given any inverse semigroup, we may restrict its multiplication in such a way that we obtain an inductive groupoid on the same underlying set: the information that is seemingly lost in restricting multiplication is contained instead in the partial order structure of the inductive groupoid. Conversely, it is possible (using information encoded in the order structure) to extend the partial multiplication in an inductive groupoid to an everywhere-defined multiplication, and thereby construct an inverse semigroup. Indeed, the ESN Theorem goes further: it is expressed in category-theoretic terms, stating that certain categories of inverse semigroups are isomorphic to certain other categories of inductive groupoids. In this way, it tells us that there is also a fundamental connection between certain natural functions between inverse semigroups and other equally natural functions between inductive groupoids. The ESN Theorem has thus proved to be a useful tool in the theories of both inverse semigroups and inductive groupoids: it allows the study of one to inform that of the other.

The ESN Theorem has indeed been so beneficial that it has been highly desirable to obtain extensions of it to other situations. In particular, versions of the ESN Theorem have been proved in the cases of semigroups which are more general than inverse semigroups. For example, Nambooripad (see Section 2) arrived at a generalisation of the ESN Theorem which links regular semigroups (generalisations of inverse semigroups in which every element is assumed to have *at least one* generalised inverse) with so-called “regular groupoids”: generalisations of inductive groupoids. There has also been a great deal of activity in this area, in connection with the non-regular generalisations of inverse semigroups, such as ample semigroups and restriction semigroups (see Section 3). We will be particularly interested in the case of restriction semigroups, which may be derived from semigroups of arbitrary partial transformations. Restriction semigroups correspond, in an “ESN-like” manner, to inductive categories; the order structure on the latter is essentially the same as that defined upon an inductive groupoid, but we no longer insist upon the invertibility of arrows. This generalisation of the ESN Theorem may be expressed, by analogy with the original, in terms of isomorphisms of certain categories of restriction semigroups and certain other categories of inductive categories (see below for comments on our use of the word “category”). This theorem, together with its corollaries, is the subject of this survey article, which is intended as a sequel to a previous survey [20] on restriction semigroups, where the use of extensions of the ESN Theorem in the study of such semigroups was indicated only briefly. The present article is also intended as a companion piece to [22], in which certain extensions of the ESN Theorem were discussed, but in which very little historical context was given. Our approach will be to begin with the most general situation (namely, restriction semigroups and inductive categories) and use this to derive other cases of interest: in particular, that of inverse semigroups and inductive groupoids. We will not treat the case of regular semigroups, however, since these are a generalisation of inverse semigroups along different lines to restriction semigroups, and so the relevant results for regular semigroups do not follow from those for restriction semigroups.

As we will see in Section 2, the original construction of an inverse semigroup from an inductive groupoid, and vice versa, was provided by Schein [44, 45]. His method was direct and entirely algebraic, as was that of subsequent authors in connection with generalisations of Schein's results — for example, Armstrong [1] in the case of ample semigroups. Though of course perfectly valid, this approach entails a somewhat lengthy proof, with the verification of associativity (in an inverse semigroup constructed from a given inductive groupoid) being particularly long, tedious and fiddly. A new approach was subsequently promoted by Lawson [29], using the order-theoretic techniques of Ehresmann [11]. By allowing order-theoretic considerations to play a much more prominent role, considerably shorter and more elegant proofs may be obtained.

The structure of the article is as follows. After the example of [20], we begin the article proper in Section 2 with a historical survey of the development of the theorems of interest, beginning with the origins of the theory of inverse semigroups, and arriving eventually at the generalised ESN Theorem for restriction semigroups. In Section 3, we give a brief introduction to restriction semigroups, but we only include those details which are pertinent to the subsequent considerations — for a more rounded account of restriction semigroups, the reader is referred to [20] or [15]. In particular, these articles motivate the study of such semigroups.

Inductive categories are defined in Section 4. We outline the basics of their theory, before demonstrating their connection with restriction semigroups in Section 5. By the end of Section 5, we will have shown that every restriction semigroup gives rise to an inductive category, and vice versa. In this way, we will have taken care of the "objects" part of the desired correspondence of categories. We therefore turn to the "arrows" part in the next two sections. In Section 6, we introduce functions called "$\vee$-premorphisms" between restriction semigroups and show that these correspond to order-preserving functors between inductive categories. In Section 7, we consider certain morphisms between restriction semigroups and show that these correspond to so-called "inductive functors" between inductive categories. By combining the results of Sections 5, 6 and 7, we obtain two isomorphisms of categories in Theorems 5 and 6. Together, these two theorems give the restriction semigroup version of the ESN Theorem. In the final section (8), we use the results on restriction semigroups and inductive categories to deduce the original version of the ESN Theorem.

Our approach to inductive categories is based very heavily upon Lawson's approach to inductive groupoids [29], but one important difference to note is the fact that we will be composing functions from left to right. Thus, for example, our domain and range in a category (see Section 4) are the opposite way round to those in [29]: for Lawson, $\exists \mathbf{r}(x) \cdot x$ with $\mathbf{r}(x) \cdot x = x$, and $\exists x \cdot \mathbf{d}(x)$ with $x \cdot \mathbf{d}(x) = x$ (cf. our Definition 8).

Our imitation of Lawson [29] extends also to the adoption of the order-theoretic approach indicated above, rather than the original, direct, algebraic approach. Thus, for example, our auxiliary results, Propositions 1 and 2, which are analogues of results of Lawson in the inverse case, allow us to sidestep the lengthy associativity proof mentioned above. As far as we are aware, our proofs of Theorems 2 and 3 (the mutual correspondence between restriction semigroups and inductive categories) are the first direct proofs — these theorems first appeared in [28], but were deduced there from other results. However, these direct proofs differ very little, on the whole, from those given by Armstrong [1] in the ample case, and Lawson [26] in the full restriction case (and, indeed, that of Schein [44, 45] in the inverse case). The only significant difference is the shorter proof of associativity.

It should be noted that, as indicated in [20], there are three types of restriction semigroup: left, right and two-sided. We will only be concerned with the two-sided version here, which we will refer to throughout simply as "restriction semigroups". An "ESN-type" theorem may be obtained for left restriction semigroups, but the objects to which such semigroups correspond are not categories, but "one-sided generalisations" of categories, termed *inductive constellations* — for further details, see [17].

Some comments should be made here on the terminology to be used throughout this article. First of all, it is worth emphasising that we will be using the term "groupoid" only to mean a small category in which all arrows are invertible. "Groupoid" is also occasionally used (for example, in [23, p. 1]) to refer to a set with an everywhere-defined binary operation (according to which definition, a semigroup is then an "associative groupoid") — the word "groupoid" will never be used in this sense here.

It is also important to point out that we will be using the word "category" in two slightly different, though equivalent, manners, according to the two different ways in which it is possible to view a category:

(♠) the "traditional" objects and morphisms version of a category (see, for example, [24, Definition 1.1]), in which the objects are mathematical entities such as sets, semigroups, groups, rings, etc., and the morphisms are functions between these, such as (homo)morphisms;

(♣) the "generalised monoid" version of a category, which sees a category as a set equipped with a partially-defined binary operation, subject to certain conditions (see Definition 8). All categories viewed in this way will be small categories. From this point of view, a category may be regarded as a directed graph, in which the objects (here termed "identities") are the vertices and the morphisms (or "arrows") are the edges. Here, two arrows may only be composed if the terminal vertex of the first coincides with the initial vertex of the second. Wherever it is defined, the composition is assumed, amongst other things, to be associative. A monoid is therefore such a category with precisely one identity.

Thus, whenever we refer to an inductive category (or, indeed, an inductive groupoid), we are thinking of it as a category in sense (♣). On the other hand, when we speak of a category *of* inductive categories (or of inverse semigroups, etc.), the underlined usage of the word category is thought of as being in sense (♠). We will continue to emphasise the distinction between (♠) and (♣) whenever we feel that it aids clarity.

In the interests of keeping the length of the article down, we have included as few proofs as possible. In particular, we have omitted the proofs of most results which we consider to be elementary, or which may easily be found elsewhere. Nevertheless, some such proofs have been included where they are particularly instructive. For example, Lemma 5 provides a good introduction to the properties of the restriction and corestriction in an ordered category (see Definition 11), and their interplay with the (partial) multiplicative structure.

It should be noted that some of the citations given here for certain results are slightly imprecise. For example, different parts of the above-mentioned Lemma 5 have been attributed to Armstrong [1] and Lawson [29]. The imprecision here stems from the fact that although our Lemma 5 concerns arbitrary ordered categories, Armstrong's version deals with ordered *cancellative* categories (see Definition 10), and Lawson's with ordered *groupoids*. Nevertheless, the proof in the case of arbitrary ordered categories is substantially the same as those in the more specialised cases — it is therefore appropriate to cite both Armstrong and Lawson here. Any such imprecise citations are marked with an asterisk *.

An "[F]" given as a citation for a lemma or theorem indicates that the result is of the nature of "folklore": it is both fundamental and reasonably easy to prove — the proofs of such results will therefore be omitted in most cases, unless the proof is particularly instructive. In such cases, I have made little effort to track down the first appearance of these results in the literature.

As the reader has probably concluded from this Introduction, this article has been written very much from the "semigroup point-of-view". We therefore assume a basic knowledge of semigroup theory on the part of the reader. For any undefined terminology or notation, the reader is referred to [23] or [29]. As one final note, we mention that, as per the oft-followed convention in semigroup theory, homomorphisms will be referred to throughout simply as "morphisms".

2. Historical Background

Following Lawson [29], we start by placing these theories in the context of Klein's *Erlanger Programm*. This was the point of view famously advocated by Felix Klein at the end of the nineteenth century that every geometry (Euclidean, hyperbolic, projective, etc.) should be regarded as the theory of invariants of a particular group of transformations.† To put this another way, not only can a group of structure-preserving bijections be associated with a given geometry, but also such a group can be used to define the geometry in the first place. This group-theoretic approach to geometry placed the burgeoning theory of groups at the centre-stage of late-nineteenth-century mathematics and thus ensured its future development (see [53]). The concept of a group became inextricably linked to the geometrical notion of symmetry. However, despite the initial success of the Erlanger Programm, it was quickly realised that there exist geometries which cannot be slotted into this rough scheme, that is, there exist geometries whose symmetries do not form groups. A prime example of this is differential geometry. In the early twentieth century, efforts were made to bring such "rogue geometries" into the fold by generalising the group concept, thereby devising an algebraic structure which would serve to describe the symmetries of the geometry. The advent of the General Theory of Relativity, with its reliance on differential geometry, ensured that this problem received a great deal of attention. The question of how to describe symmetries in differential geometry was eventually answered by Veblen and Whitehead with the introduction of the notion of a *pseudogroup*. This was a generalisation of Sophus Lie's "(infinite) continuous transformation group" [31], now termed a *Lie pseudogroup*.

Definition 1 ([49, p. 38])**.** *A* pseudogroup Γ *is a collection of partial homeomorphisms between*

†See [25] for Klein's text, [19, 2] for comments thereupon, and [18] for an English translation.

open subsets of a topological space such that Γ *is closed under composition and inverses, where we compose* $\alpha, \beta \in \Gamma$ *only if* $\operatorname{im} \alpha = \operatorname{dom} \beta$.

In the traditional case of groups of transformations, the move to an abstract setting had yielded the notion of an abstract group; researchers now asked the question: what is the corresponding abstract structure for a pseudogroup? It turned out there are two closely-linked solutions to this problem, both of which are connected with the question of how to compose partial mappings. We have seen that Veblen and Whitehead chose to compose two partial mappings α and β only if $\operatorname{im} \alpha = \operatorname{dom} \beta$, thereby giving their pseudogroups a *partial* composition. Attempts were subsequently made to "complete" this composition to give a pseudogroup an everywhere-defined operation. One such attempt was made by J. A. Schouten and J. Haantjes, for example, in [48, p. 361], where two partial mappings α and β on a set X were composed whenever $\operatorname{im} \alpha \cap \operatorname{dom} \beta \neq \emptyset$. However, this operation was still partial, for it did not take into account the possibility of the "empty transformation": the partial transformation whose domain is $\emptyset \subseteq X$, which will result whenever $\operatorname{im} \alpha \cap \operatorname{dom} \beta = \emptyset$ (see [20, §3] for a brief introduction to partial transformations).

The final step came in the early 1950s with the observation by Viktor Vladimirovich Wagner[‡] that the composition of partial mappings is a special case of the composition of binary relations. In the case of binary relations, however, it is much clearer that the composition may be empty. This simple observation enabled Wagner to overcome the psychological difficulties which had so far barred the admission of an empty transformation into the study of partial mappings. The introduction of the empty transformation meant that an everywhere-defined composition of partial mappings could finally be utilised, namely, the familiar (left-to-right) composition usually employed for such mappings in modern semigroup theory [23, p. 148]:

$$\operatorname{dom} \alpha\beta = \left[\operatorname{im} \alpha \cap \operatorname{dom} \beta\right] \alpha^{-1}, \ x(\alpha\beta) = (x\alpha)\beta, \text{ for any } x \in \operatorname{dom} \alpha\beta. \tag{1}$$

Wagner now turned his attention to the study of systems of one-one partial transformations with this everywhere-defined composition [50]. Though a differential geometer by training, Wagner recognised in such systems the structure of a semigroup; given a set X, he defined what we now term the *symmetric inverse monoid* $\mathscr{I}_X$ on X. Further, upon moving to an abstract setting, Wagner observed that these were semigroups with an involution which generalised the group-theoretic notion of inversion. In [51], he gave the modern definition of an inverse semigroup, though under the name of *generalised group* (обобщенная группа). He subsequently developed the theory of generalised groups further in a much longer paper [52], where they were intimately connected with the notion of a so-called *generalised heap* or *generalised groud* (обобщенная груда); loosely speaking, whereas an inverse semigroup arises from the study of systems of partial one-one transformations of a single set into itself, generalised heaps come from the study of systems of partial one-one transformations from one set to another, and must therefore be equipped with a ternary operation, rather than a binary operation (given partial one-one mappings α, β, γ from subsets of a set A to subsets of a set B, the ternary operation, denoted $[\cdot\ \cdot\ \cdot]$, is defined to be the composition $[\alpha\ \beta\ \gamma] = \alpha\beta^{-1}\gamma$).

[‡]Виктор Владимирович Вагнер. I have chosen to transliterate the "В" of "Вагнер" as "W", as this was apparently Wagner's own preference — see [47, p. 152]

Inverse semigroups were introduced and studied independently by Gordon Preston, first in his 1953 DPhil thesis [38], under the name "mapping semigroups", and then in a much more polished form in three papers of 1954 [39, 40, 41]. It was in these latter three papers that the term "inverse semigroup" appeared for the very first time. Preston was influenced by earlier work on systems of one-one partial transformations: his initial motivation was the axiomatisation of certain semigroups of one-one partial transformations that had been studied by David Rees [43]. For more details on Preston's development of inverse semigroups, see [42]; on the history of inverse semigroups more generally, see [46, 47].

With the definition of an inverse semigroup established, it was clear that the desired abstract model for a pseudogroup had been obtained. If a pseudogroup is given the composition of (1), then it is simply an inverse semigroup of homeomorphisms between open sets of a topological space.

The second solution to the problem of finding an abstract model for a pseudogroup is due to Charles Ehresmann and goes back to the composition of partial mappings. Rather than trying to "complete" the operation on a pseudogroup, Ehresmann realised that if the partial composition defined by Veblen and Whitehead is retained, then a pseudogroup has the structure of a *groupoid*, that is, a small category in which all arrows are invertible.[§] In fact, as Ehresmann also observed, pseudogroups form *ordered* groupoids, with the obvious partial order of restriction of mappings. By imposing extra conditions on ordered groupoids, Ehresmann went even further and studied the special case of so-called *inductive* groupoids, although his notion of "inductivity" was a little more stringent than in the modern definition. The ordering in the groupoid played a much more prominent role in Ehresmann's work than it had in the work of previous authors. In essence, whereas both Wagner and Preston axiomatised $(\mathscr{I}_X, \circ)$ to obtain an inverse semigroup (where $\circ$ is the composition of (1)), Ehresmann axiomatised $(\mathscr{I}_X, \cdot, \subseteq)$ to obtain an inductive groupoid (where $\cdot$ is Veblen and Whitehead's partial composition, and $\subseteq$ denotes the ordering of partial transformations) — see [29, p. 9].

The motivation for Ehresmann's work came from the study of *local structures*: structures defined on topological spaces by using pseudogroups in a manner analogous to the way in which groups are used to define geometries (see [29, §1.2]). Ehresmann's category-theoretic work began in [9] and continued through a number of further papers, which may all be found in his collected works [11] — see [29, §§1.6 and 4.4] for more details on Ehresmann's publications. See also [4, 5, 6] on the history of groupoids.

The theories of inverse semigroups and ordered groupoids developed along their separate paths for some time after their inceptions. It seems that Ehresmann was aware of the connection between his work and that of Wagner (see [29, p. 131]); indeed, it was Ehresmann who first defined the *pseudoproduct* which is necessary for the construction of an inverse semi-

[§]The notion of a groupoid seems to have originated with Heinrich Brandt [3], although Brandt's groupoids had a slightly stronger definition than the modern one. In modern terminology [29, p. 105], Brandt groupoids are *connected* groupoids: for any identities x, y in the groupoid, there is an arrow s with $\mathbf{d}(s) = x$ and $\mathbf{r}(s) = y$ (see Section 4 for the definition of this notation). An observation later made by Schein [44, 45] is thus reasonably clear: an arbitrary groupoid is a union of Brandt groupoids, since each "connected component" is a Brandt groupoid. On the origins of Brandt groupoids, see [21, §4]. We note also that the theory of groupoids predates that of categories, which was initiated by a 1945 paper of Eilenberg and Mac Lane [12] — see [7, Chapter 8] for further details on the development of category theory.

group from an inductive groupoid [9, 10] (viz. our equation (6)). However, it was left to Boris Schein [44, 45] to make the connection explicit. By relaxing Ehresmann's original conditions for "inductivity", Schein arrived at the modern notion of an inductive groupoid: an ordered groupoid in which every pair of identities has a greatest lower bound, or *meet*. Furthermore, he observed that, without the order structure, such objects had in fact already been studied (under the name "partial group": textquotedblleft groupe partiel") by Robert Croisot in 1948 [8]. For this reason, the modern notion of a groupoid was termed by Schein a *Croisot groupoid*. A Croisot groupoid was said further to be *replenishable* if its multiplication could be extended to an everywhere-defined operation; Schein showed that a Croisot groupoid is replenishable if and only if it can be ordered in such a way as to make it inductive. Any inductive Croisot groupoid may be replenished in such a way that an inverse semigroup is obtained [45, Theorem 3.4]. Conversely, Schein demonstrated that if we take any inverse semigroup S and define in it a partial operation, which he termed the *abutting multiplication* (defined in our equation (9)), then the semigroup becomes a Croisot groupoid $\mathrm{tr}(S)$ (the *trace* of S) with respect to this new multiplication; furthermore, the natural partial order of S makes $\mathrm{tr}(S)$ inductive cite[p. 109]schein1979. Thus, given any inverse semigroup, we may construct an inductive (Croisot) groupoid, and vice versa. We see then that inverse semigroups and inductive groupoids share a very close connection which reflects their common origins in the theory of pseudogroups.

The results of Schein were subsequently generalised to the case of regular semigroups by K. S. S. Nambooripad. Beginning in his 1973 PhD thesis [34], and developing his ideas through a series of papers based thereupon [35, 36], Nambooripad showed that every regular semigroup gives rise to a generalisation of an inductive groupoid, which he termed a *regular groupoid*, and, conversely, that a regular semigroup may be obtained from any such regular groupoid. A regular groupoid, together with a certain collection of mappings between $\mathscr{R}$-classes of the idempotents of the corresponding regular semigroup, and another collection of mappings between $\mathscr{L}$-classes of the idempotents, was termed by Nambooripad a *regular system*. He showed that there is a one-one correspondence between regular systems and regular semigroups [35, Part II, Theorems 1 and 2]. Indeed, going further, he also demonstrated that there is a similar correspondence between morphisms of regular systems (which are defined in an appropriate manner) and morphisms of the associated regular semigroups, and vice versa. In this way, without ever using the word "category", Nambooripad showed that there is an isomorphism between the category of regular systems and morphisms and the category of regular semigroups and morphisms. We may then deduce the specialisation of this result to the inverse case: that the category of inductive groupoids and inductive functors is isomorphic to the category of inverse semigroups and morphisms, where we have switched to the terminology to be used throughout the present article: an *inductive functor* is an order-preserving functor (or *ordered functor*) which preserves meet. It was shown further [37, pp. 286–7] that if we require our functors between inductive groupoids to be merely ordered, and not necessarily inductive, then these correspond to functions between inverse semigroups, termed $\vee$-*premorphisms*; a $\vee$-premorphism, as introduced by McAlister [32], is a function $\theta : S \to T$ between inverse semigroups S and T such that $(st)\theta \leq (s\theta)(t\theta)$. We may thus establish an isomorphism between the category of inductive groupoids and ordered functors, and the category of inverse semigroups and $\vee$-premorphisms. As noted in the Introduction, these category-theoretic results, which very nicely sum up the connection between inverse semigroups and inductive groupoids, were gathered together into a single theorem (Theorem 4.1.8) in [29]; this was named the *Ehresmann-Schein-Nambooripad Theorem* to reflect its disparate origins:

Theorem 1. *The category of inverse semigroups and $\vee$-premorphisms is isomorphic to the category of inductive groupoids and ordered functors; the category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors.*[¶]

With this connection between inverse semigroups and inductive groupoids established, it was natural to seek other generalisations. We have seen that there is an extension of this result to the regular case, due to Nambooripad, but what of the non-regular generalisations of inverse semigroups, such as the ample (a.k.a. type-A) semigroups developed by John Fountain [13, 14], and the related semigroups surveyed in [20]? Furthermore, since a groupoid is a very specialised type of category, we might ask what type of semigroup may be associated with a more general inductive category, or even with an arbitrary inductive category (in sense (♣)). This question has indeed been answered, via a succession of generalisations of the ESN Theorem, which we now discuss.

The first of these generalisations is due to Sheena Armstrong [1] and is rooted in the work of John Meakin [33]. We saw above that, at one point, Nambooripad used mappings between $\mathscr{R}$- and $\mathscr{L}$-classes as part of his description of the structure of regular semigroups. Inspired by some observations of Schein [44], Meakin had, entirely independently of Nambooripad, embarked upon a similar approach to the structure of an inverse semigroup S by means of so-called "structure mappings", that is, mappings between $\mathscr{R}$-classes of S. In essence, given an inverse semigroup S, Meakin's structure mappings permit the location of those products which do not belong to $\mathrm{tr}(S)$. Indeed, these structure mappings encode the same information as the "restriction" and "corestriction" that we will introduce in Definition 11. Armstrong generalised this approach to the study of ample semigroups by considering mappings between classes of the generalised Green's relations, $\mathscr{R}^*$ and $\mathscr{L}^*$, in terms of which ample semigroups are defined (see Section 3). In her Theorem 3.9 (our Corollaries 4 and 6), Armstrong extended the ESN Theorem to the case of ample semigroups and inductive cancellative categories (to be defined in Section 4), although, adapting Schein's terminology, she referred to these as *inductive weak Croisot groupoids*. Like Schein, Armstrong did not give her result a category-theoretic formulation such as Theorem 1.

In the presentation of [20], ample semigroups have two successive generalisations: full restriction semigroups (formerly termed weakly ample semigroups) and restriction semigroups (formerly, weakly E-ample semigroups). Each of the two further generalisations of the ESN Theorem to these cases is due to Mark Lawson. The case of full restriction semigroups and inductive unipotent categories (see Section 4) appears in his DPhil thesis [26, Theorem 3.16] (our Corollary 9) as a generalisation of Armstrong's result, whilst that of restriction semigroups and arbitrary inductive categories may be found in a later paper [28, Theorem 5.7] (our Theorem 6). This second generalisation is carried out in the order-theoretic style of

[¶]Every occurrence of the word "category" in this theorem is used in sense (♠).

Ehresmann, whereas the work of Lawson's thesis, and that of Armstrong, simply adapts the direct approach of Schein [44, 45], complete with the consequent lengthy associativity proof noted in the Introduction. Indeed, in a parallel paper [27], Lawson also gave an order-theoretic treatment of the inverse case; this last paper appears to contains the seeds of [29], for which book the ESN Theorem provides the main focus.

We conclude this historical introduction by noting that, although Lawson [28] did phrase his results in category-theoretic terms, he only considered the case where the arrows between (full) restriction semigroups are morphisms; he did not consider a "$\vee$-premorphisms" version. Such a treatment may be found instead in [22]. We note also that [22] contains "ESN-type" theorems (not only for restriction semigroups, but also in the special case of inverse semigroups) involving the functions dual to $\vee$-premorphisms, so-called "$\wedge$-premorphisms", which have an important role to play in the theory of partial actions (see, for example, [16, 30]). However, we will not consider these functions in the present article.

3. Restriction Semigroups

In this section, we provide a brief introduction to the notion of a *two-sided restriction semigroup*, which we will refer to here simply as a *restriction semigroup*. These are semigroups which arise from semigroups of arbitrary partial transformations in much the same way that inverse semigroups arise from semigroups of one-one partial transformations. A sketch of the history of these semigroups is given in [20], and it to this article (as well as to [15]) that we direct the interested reader for further details (including proofs) and references. Indeed, in the very brief account of these semigroups given here, we do not include any justification for their study, usually provided by means of partial transformations, and move straight to the abstract definition; said justification may be found in [20].

Let S be a semigroup and suppose that S has some distinguished subsemilattice of idempotents $E \subseteq E(S)$, where, as usual, $E(S)$ denotes the subset of idempotents of a semigroup S. We define two (equivalence) relations in S with respect to E:

$$a \,\widetilde{\mathscr{R}}_E\, b \Longleftrightarrow \forall e \in E\,[ea = a \Leftrightarrow eb = b]\,;$$
$$a \,\widetilde{\mathscr{L}}_E\, b \Longleftrightarrow \forall e \in E\,[ae = a \Leftrightarrow be = b]\,.$$

Thus, two elements of S are $\widetilde{\mathscr{R}}_E$- ($\widetilde{\mathscr{L}}_E$-)related if and only if they have the same left (right) identities in E. In the case where $E = E(S)$, we omit the subscripts from the relations and write $\widetilde{\mathscr{R}}$ for $\widetilde{\mathscr{R}}_{E(S)}$ and $\widetilde{\mathscr{L}}$ for $\widetilde{\mathscr{L}}_{E(S)}$. As the notation suggests, $\widetilde{\mathscr{R}}_E$ and $\widetilde{\mathscr{L}}_E$ are generalisations of Green's relations $\mathscr{R}$ and $\mathscr{L}$ in the sense that, for any $E \subseteq E(S)$, $\mathscr{R} \subseteq \widetilde{\mathscr{R}} \subseteq \widetilde{\mathscr{R}}_E$ and $\mathscr{L} \subseteq \widetilde{\mathscr{L}} \subseteq \widetilde{\mathscr{L}}_E$. It is useful to note the following conditions, derived from the above, for an element $a \in S$ to be $\widetilde{\mathscr{R}}_E$- or $\widetilde{\mathscr{L}}_E$-related to an idempotent $e \in E$:

$$a \,\widetilde{\mathscr{R}}_E\, e \Longleftrightarrow ea = a \text{ and } \forall f \in E\,\left[fa = a \Rightarrow fe = e\right]\,;$$
$$a \,\widetilde{\mathscr{L}}_E\, e \Longleftrightarrow ae = a \text{ and } \forall f \in E\,\left[af = a \Rightarrow ef = e\right]\,.$$

Without giving any justification, we define *left* and *right* restriction semigroups in terms of $\widetilde{\mathscr{R}}_E$ and $\widetilde{\mathscr{L}}_E$, respectively:

Definition 2. *Let S be a semigroup with distinguished subsemilattice of idempotents $E \subseteq E(S)$. We call S a* left restriction semigroup (with respect to E) *if*

(1) every element $a \in S$ is $\widetilde{\mathscr{R}}_E$-related to a (necessarily unique) element of E, which we denote by a^+;

(2) $\widetilde{\mathscr{R}}_E$ is a left congruence;

(3) for all $a \in S$ and all $e \in E$, $ae = (ae)^+a$.

If $E = E(S)$, we term S a full *left restriction semigroup.*

Definition 3. *Let S be a semigroup with distinguished subsemilattice of idempotents $E \subseteq E(S)$. We call S a* right restriction semigroup (with respect to E) *if*

(1) every element $a \in S$ is $\widetilde{\mathscr{L}_E}$-related to a (necessarily unique) element of E, which we denote by a^*;

(2) $\widetilde{\mathscr{L}_E}$ is a left congruence;

(3) for all $a \in S$ and all $e \in E$, $ea = a(ea)^*$.

If $E = E(S)$, we term S a full *right restriction semigroup.*

As might be expected, we obtain a (two-sided) restriction semigroup by combining the preceding two definitions:

Definition 4. *Let S be a semigroup with distinguished subsemilattice of idempotents $E \subseteq E(S)$. We call S a* (two-sided) restriction semigroup (with respect to E) *if it is both a left restriction semigroup with respect to E and a right restriction semigroup with respect to E.*

Note that $e^+ = e^* = e$, for any $e \in E$. We observe also that any inverse semigroup is a left/right/two-sided restriction semigroup with respect to $E(S)$, with $a^+ = aa^{-1}$ and $a^* = a^{-1}a$; thus, in an inverse semigroup, $\widetilde{\mathscr{R}} = \mathscr{R}$ and $\widetilde{\mathscr{L}} = \mathscr{L}$.

It should be noted that left restriction semigroups form a variety of algebras of type (2,1), as do right restriction semigroups; two-sided restriction semigroups form a variety of algebras of type (2,1,1). In all three cases, the "full" versions form only quasi-varieties. The "varieties" standpoint has become an extremely useful way of viewing these semigroups, but we will have no occasion to adopt this view here — the interested reader is directed to [15].

For the rest of this article, the distinguished subsemilattice of a given restriction semigroup S will be denoted by E, unless stated otherwise. We will therefore suppress mention of E, except where clarity demands it, and refer simply to "the restriction semigroup S".

We record here some very useful properties of restriction semigroups which will be used many times in the course of this article; these properties follow immediately the left (right) congruence properties of $\widetilde{\mathscr{R}}_E$ (respectively, $\widetilde{\mathscr{L}_E}$):

Lemma 1 ([F]). *Let S be a restriction semigroup. For any $s, t \in S$, $(st)^+ = (st^+)^+$ and $(st)^* = (s^*t)^*$.*

Just like an inverse semigroup, any restriction semigroup possesses a partial order which is natural in the sense that it is compatible with the semigroup multiplication, and that it restricts to the usual partial order on idempotents from E (namely, $e \leq f$ if and only if $e = ef$). The partial order in a restriction semigroup may be given by

$$a \leq b \Longleftrightarrow a = eb, \text{ for some } e \in E, \tag{2}$$

or, equivalently:

$$a \leq b \Longleftrightarrow a = bf, \text{ for some } f \in E. \tag{3}$$

In fact, the idempotents e and f in (2) and (3) can be taken to be a^+ and a^*, respectively:

$$a \leq b \Longleftrightarrow a = a^+ b \Longleftrightarrow a = ba^*. \tag{4}$$

Given the above comments on inverse semigroups, it is easy to see that if the restriction semigroup in question is in fact inverse, then the above ordering coincides with the usual partial order on an inverse semigroup.

We have already observed that any inverse semigroup is a restriction semigroup. In fact, there is another special type of restriction semigroup which we will have occasion to consider: so-called *ample* semigroups. These form a class of semigroups intermediate between restriction semigroups and inverse semigroups, and are defined in terms of the following specialisations of $\widetilde{\mathscr{R}}_E$ and $\widetilde{\mathscr{L}_E}$:

$$\begin{aligned} a \,\mathscr{R}^*\, b &\Longleftrightarrow \forall x, y \in S^1 [xa = ya \Leftrightarrow xb = yb]; \\ a \,\mathscr{L}^*\, b &\Longleftrightarrow \forall x, y \in S^1 [ax = ay \Leftrightarrow bx = by]. \end{aligned}$$

These equivalence relations are again generalisations of Green's relations $\mathscr{R}$ and $\mathscr{L}$, and, indeed, we have $\mathscr{R} \subseteq \mathscr{R}^* \subseteq \widetilde{\mathscr{R}} \subseteq \widetilde{\mathscr{R}}_E$ and $\mathscr{L} \subseteq \mathscr{L}^* \subseteq \widetilde{\mathscr{L}} \subseteq \widetilde{\mathscr{L}_E}$ on any semigroup S, for any subsemilattice E. As with $\widetilde{\mathscr{R}}_E$ and $\widetilde{\mathscr{L}_E}$, we have simpler conditions for an element $a \in S$ to be $\mathscr{R}^*$- or $\mathscr{L}^*$-related to an idempotent $e \in E(S)$:

$$\begin{aligned} a \,\mathscr{R}^*\, e &\Longleftrightarrow ea = a \text{ and } \forall x, y \in S^1 [xa = ya \Rightarrow xe = ye]; \\ a \,\mathscr{L}^*\, e &\Longleftrightarrow ae = a \text{ and } \forall x, y \in S^1 [ax = ay \Rightarrow ex = ey]. \end{aligned} \tag{5}$$

We may now define *left* and *right* ample semigroups:

Definition 5. *We call a semigroup S a* left ample semigroup *if*

(1) every element $a \in S$ is $\mathscr{R}^*$-related to a (necessarily unique) element of $E(S)$, which we denote by a^+;

(2) for all $a \in S$ and all $e \in E(S)$, $ae = (ae)^+ a$.

Definition 6. *We call a semigroup S a* right ample semigroup *if*

(1) every element $a \in S$ is $\mathscr{L}^*$-related to a (necessarily unique) element of $E(S)$, which we denote by a^*;

(2) for all $a \in S$ and all $e \in E(S)$, $ea = a(ea)^*$.

Thus, the definition of a left (right) ample semigroup is broadly similar to that of a left (right) restriction semigroup, but with $\widetilde{\mathscr{R}}_E$ ($\widetilde{\mathscr{L}_E}$) replaced by $\mathscr{R}^*$ ($\mathscr{L}^*$). The notable omission from the left (right) ample definition, however, is the left (right) congruence condition; in fact, $\mathscr{R}^*$ ($\mathscr{L}^*$) is always a left (right) congruence, so this need not be demanded explicitly. Moreover, we have $\widetilde{\mathscr{R}} = \mathscr{R}^*$ ($\widetilde{\mathscr{L}} = \mathscr{L}^*$) in a left (right) ample semigroup, so there is no ambiguity in our use of a^+ (a^*) to denote the idempotent which is $\mathscr{R}^*$- ($\mathscr{L}^*$-)related to $a \in S$.

We of course obtain the notion of a (two-sided) ample semigroup by combining the preceding two definitions:

Definition 7. *We call a semigroup S* (two-sided) ample *if it is both left ample and right ample.*

It is clear from the definitions that any ample semigroup is a restriction semigroup, and so everything we have said about restriction semigroups may be applied to ample semigroups. In particular, Lemma 1 holds in any ample semigroup, and such a semigroup possesses a partial order defined by (2), (3) or (4). Once again, any inverse semigroup is ample, with $a^+ = aa^{-1}$ and $a^* = a^{-1}a$. In contrast to the situation with restriction semigroups, left/right ample semigroups form only a quasi-variety of algebras of type (2,1), whilst two-sided ample semigroups form a quasi-variety of algebras of type (2,1,1).

As a final comment on the ample case, we observe that, unlike for restriction semigroups, we have not defined ample semigroups with respect to a distinguished subsemilattice. This is because there is no need to do so: any left/right/two-sided ample semigroup is necessary full in the sense of Definition 2 (3). To see this, we suppose that we have defined a "left E-ample semigroup" S by replacing all occurrences of "$E(S)$" in Definition 5 by "E", where E is some distinguished subsemilattice of S. We take an arbitrary idempotent $e \in S$ and observe, using the second condition of Definition 5, together with Lemma 1, that $ee^+ = (ee^+)^+e = (ee)^+e = e^+e = e$. However, since $ee = e^+e$ and $e\,\mathscr{R}^*\,e^+$, we have $ee^+ = e^+e^+$, hence $e = e^+$, from which we conclude that $E = E(S)$. A similar argument may be made for the right-hand version of these semigroups. The notion of a "left/right/two-sided E-ample semigroup" is therefore redundant.

4. Inductive Categories

Having defined the semigroups of interest, we now turn our attention to the definition of a category (in sense (♣) of the Introduction). Let C be a class and let $\cdot$ be a partial binary operation on C, i.e., an operation which is not necessarily defined for all pairs $(x, y) \in C \times C$; whenever the product $x \cdot y$ *is* defined, we denote the fact by "$\exists x \cdot y$". When we write expressions such as "$\exists (x \cdot y) \cdot z$", for example, we mean that $\exists x \cdot y$ *and* $\exists (x \cdot y) \cdot z$. An element $e \in C$ is termed *idempotent* if

$$\exists e \cdot e \qquad \text{and} \qquad e \cdot e = e.$$

The *identities* of C are those idempotents e which satisfy the following conditions, for any $x \in C$:

$$\exists e \cdot x \Longrightarrow e \cdot x = x;$$

$$\exists x \cdot e \Longrightarrow x \cdot e = x.$$

We denote the subset of identities of C by C_o ("o" for "objects").

Definition 8. *Let C be a class and let $\cdot$ be a partial binary operation on C. The pair $(C, \cdot)$ is a* category *if the following conditions hold:*

(Ca1) $\exists x \cdot (y \cdot z) \Longleftrightarrow \exists (x \cdot y) \cdot z$, *in which case* $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

(Ca2) $\exists x \cdot (y \cdot z) \Longleftrightarrow \exists x \cdot y$ *and* $\exists y \cdot z$;

(Ca3) for each $x \in C$, there exist unique identities $\mathbf{d}(x), \mathbf{r}(x) \in C_o$ such that $\exists \mathbf{d}(x) \cdot x$ and $\exists x \cdot \mathbf{r}(x)$.

If C is simply a set, then we call $(C, \cdot)$ a small *category.*

Whenever the partial multiplication in a category $(C, \cdot)$ is clear, we will refer simply to "the category C". The identity $\mathbf{d}(x)$ is called the *domain* of x and $\mathbf{r}(x)$ is the *range* of x. Notice that by the definition of identities, $\mathbf{d}(x) \cdot x = x$ and $x \cdot \mathbf{r}(x) = x$. Moreover, for any identity e, $\mathbf{d}(e) = \mathbf{r}(e) = e$. All categories considered from this point of view (i.e., in sense (♣)) will be small categories.

Lemma 2 ([F]). *Let $(C, \cdot)$ be a category. Then*

$$\exists x \cdot y \Longleftrightarrow \mathbf{r}(x) = \mathbf{d}(y).$$

Lemma 3 ([F]). *Let C be a category. If $\exists x \cdot y$, then $\mathbf{d}(x \cdot y) = \mathbf{d}(x)$ and $\mathbf{r}(x \cdot y) = \mathbf{r}(y)$.*

Let $(C, \cdot)$ be a category. For $e, f \in C_o$, we define the set $\mathrm{mor}(e, f)$ by

$$\mathrm{mor}(e, f) = \{x \in C : \mathbf{d}(x) = e,\ \mathbf{r}(x) = f\}.$$

It is easy to see that if we put $e = f$, then we have a monoid $\mathrm{mor}(e, e)$ with identity e: since $\mathbf{d}(x) = \mathbf{r}(x) = e$, for all elements x, it follows that all products are defined and that the multiplication is associative, thanks to (Ca1). We call $\mathrm{mor}(e, e)$ the *local submonoid* of C at e. Thus, if $(C, \cdot)$ is a category with precisely one identity, then it is necessarily a monoid. In this way, we can regard a category as a generalisation of a monoid, as noted in the Introduction.

Using the notion of a local submonoid, we introduce a special type of category which is to appear in the next section:

Definition 9 ([26]). *A* unipotent *category is a category in which all local submonoids are unipotent (i.e., contain precisely one idempotent).*

In other words, a category is unipotent if and only if all of its idempotents are identities.

We also have the following further special type of category:

Definition 10 ([1]). *A* cancellative category *$(C, \cdot)$ is a category in which the following additional conditions hold for all $x, y, z \in C$:*

(Ca4) (i) *if* $\exists x \cdot z$, $\exists y \cdot z$ *and* $x \cdot z = y \cdot z$, *then* $x = y$;
(ii) *if* $\exists z \cdot x$, $\exists z \cdot y$ *and* $z \cdot x = z \cdot y$, *then* $x = y$.

Lemma 4 ([F]). *Any cancellative category is unipotent.*

As we have already observed in Section 2, Ehresmann demonstrated the usefulness of an order structure on a category, so, returning to the general case of an arbitrary category $(C, \cdot)$, we now introduce an ordering on C:

Definition 11. *Let* $(C, \cdot)$ *be a category and let* C *be partially ordered by* $\leq$. *The triple* $(C, \cdot, \leq)$ *is an* ordered category *if the following conditions hold:*

(Or1) *if* $a \leq c$, $b \leq d$, $\exists a \cdot b$ *and* $\exists c \cdot d$, *then* $a \cdot b \leq c \cdot d$;

(Or2) *if* $a \leq b$, *then* $\mathbf{r}(a) \leq \mathbf{r}(b)$ *and* $\mathbf{d}(a) \leq \mathbf{d}(b)$;

(Or3) (i) *for each* $f \in C_o$ *and* $a \in C$ *with* $f \leq \mathbf{r}(a)$, *there exists an element of* C, *denoted by* $a|f$, *which is the unique element with the properties* $a|f \leq a$ *and* $\mathbf{r}(a|f) = f$;
(ii) *for each* $f \in C_o$ *and* $a \in C$ *with* $f \leq \mathbf{d}(a)$, *there exists an element of* C, *denoted by* $f|a$, *which is the unique element with the properties* $f|a \leq a$ *and* $\mathbf{d}(f|a) = f$.

An *ordered unipotent (cancellative) category* is a unipotent (cancellative) category which is also an ordered category in the sense of Definition 11.

The element $a|f$ of condition (Or3)(i) is called the *corestriction* (of a to f), whilst the element $f|a$ of condition (Or3)(ii) is called the *restriction* (of f to a). Note that whenever $e|f$ is defined both as a restriction and as a corestriction, for $e, f \in C_o$, it follows that we must have $e = f$, since we require $e \leq f$ for the restriction to be defined, and $f \leq e$ for the corestriction to be defined.

The introduction of such an ordering on our category has many useful consequences:

Lemma 5 ([1, Lemma 3.4]* and [29, Theorem 4.1.3]*). *Let* $(C, \cdot, \leq)$ *be an ordered category. Let* $a, b, x, y, z \in C$ *and* $e, f \in C_o$. *Then*

(a) *if* $a \leq b$, *then* $\mathbf{d}(a)|b = a = b|\mathbf{r}(a)$;

(b) *for* $f \leq \mathbf{r}(a)$, $\exists (a|f) \cdot f$ *with* $(a|f) \cdot f = a|f$, *and for* $f \leq \mathbf{d}(a)$, $\exists f \cdot (f|a)$ *with* $f \cdot (f|a) = f|a$;

(c) *if there exists* $c \in C$ *such that* $a \leq c$ *and* $b \leq c$, *and either* $\mathbf{r}(a) = \mathbf{r}(b)$ *or* $\mathbf{d}(a) = \mathbf{d}(b)$, *then* $a = b$;

(d) *if* $e \leq f \leq \mathbf{r}(a)$, *then* $(a|f)|e = a|e$, *hence* $a|e \leq a|f$; *similarly, if* $e \leq f \leq \mathbf{d}(a)$, *then* $e|(f|a) = e|a$, *hence* $e|a \leq f|a$;

(e) *if* $\exists x \cdot y$ *and* $e \leq \mathbf{r}(x \cdot y) = \mathbf{r}(y)$, *then* $(x \cdot y)|e = \big(x|\mathbf{d}(y|e)\big) \cdot (y|e)$; *similarly, if* $\exists x \cdot y$ *and* $e \leq \mathbf{d}(x \cdot y) = \mathbf{d}(x)$, *then* $e|(x \cdot y) = (e|x) \cdot \big(\mathbf{r}(e|x)|y\big)$.

Proof. (a) If $a \leq b$, then $\mathbf{r}(a) \leq \mathbf{r}(b)$ and $\mathbf{d}(a) \leq \mathbf{d}(b)$, by (Or2). Therefore the restriction $\mathbf{d}(a)|b$ and the corestriction $b|\mathbf{r}(a)$ are both defined. The first of these is defined to be the unique element x such that $x \leq b$ and $\mathbf{d}(x) = \mathbf{d}(a)$. Notice, however, that a also satisfies these conditions. Thus, by uniqueness, $a = \mathbf{d}(a)|b$. Similarly, $b|\mathbf{r}(a) = a$.

(b) By definition of $a|f$, we have $\mathbf{r}(a|f) = f$, and so $a|f = (a|f) \cdot \mathbf{r}(a|f) = (a|f) \cdot f$. Similarly, $f \cdot (f|a) = f|a$.

(c) Suppose that $\mathbf{r}(a) = \mathbf{r}(b)$. Then, since $a \leq c$, we have $a = c|\mathbf{r}(a)$, by (a). Also by (a), $b = c|\mathbf{r}(b)$. Thus $a = c|\mathbf{r}(a) = c|\mathbf{r}(b) = b$. Similarly if $\mathbf{d}(a) = \mathbf{d}(b)$.

(d) Suppose that $e \leq f \leq \mathbf{r}(a)$. Then, by definition of corestrictions, $a|e \leq a$ and $(a|f)|e \leq a|f \leq a$. Also, $\mathbf{r}(a|e) = e$ and $\mathbf{r}[(a|f)|e] = e$. Therefore, by (c), $a|e = (a|f)|e \leq a|f$. Similarly, $e|a = e|(f|a) \leq f|a$.

(e) Suppose that $\exists x \cdot y$ and $e \leq \mathbf{r}(x \cdot y) = \mathbf{r}(y)$. Then the corestrictions $(x \cdot y)|e$ and $y|e$ are both defined. Note that the corestriction $x|\mathbf{d}(y|e)$ is also defined, since $\mathbf{d}(y|e) \leq \mathbf{d}(y) = \mathbf{r}(x)$, using (Or2), together with Lemma 2. The product $\big(x|\mathbf{d}(y|e)\big) \cdot (y|e)$ is certainly defined. Observe that $x|\mathbf{d}(y|e) \leq x$ and that $y|e \leq y$, so $\big(x|\mathbf{d}(y|e)\big) \cdot (y|e) \leq x \cdot y$, by (Or1). We also have $(x \cdot y)|e \leq x \cdot y$. Moreover,

$$\mathbf{r}\left((x \cdot y)|e\right) = e = \mathbf{r}\left(\big(x|\mathbf{d}(y|e)\big) \cdot (y|e)\right).$$

It therefore follows from (c) that $(x \cdot y)|e = \big(x|\mathbf{d}(y|e)\big) \cdot (y|e)$. The second part is similar.

We note two important consequences of Lemma 5(a):

Corollary 1. *Let $(C, \cdot, \leq)$ be an ordered category. Let $a \in C$ and $e, f \in C_o$. Then*

(a) $a|\mathbf{r}(a) = a = \mathbf{d}(a)|a$;

(b) *if $e \leq f$, then $e|f = e = f|e$, where $e|f$ is regarded as a restriction and $f|e$ as a corestriction.*

Proof. (a) Put $a = b$ in Lemma 5(a).

(b) If $e \leq f$, then, since $e = \mathbf{d}(e) = \mathbf{r}(e)$, we have $e|f = \mathbf{d}(e)|f = e = f|\mathbf{r}(e) = f|e$, by Lemma 5(a).

We also record the following for future use:

Lemma 6 ([29, Proposition 4.1.3(5)]*)**.** *Let $(C, \cdot, \leq)$ be an ordered category, and suppose that $x, y, z \in C$. If $\exists x \cdot y$ and $z \leq x \cdot y$, then there exist $x', y' \in C$ with $x' \leq x$ and $y' \leq y$ such that $\exists x' \cdot y'$ and $z = x' \cdot y'$.*

Proof. By (Or2), we have $\mathbf{r}(z) \leq \mathbf{r}(x \cdot y)$, so the corestriction $(x \cdot y)|\mathbf{r}(z)$ is defined. Moreover, $z = (x \cdot y)|\mathbf{r}(z)$, by uniqueness of corestrictions. Then

$$z = (x \cdot y)|\mathbf{r}(z) = \big(x|\mathbf{d}(y|\mathbf{r}(z))\big) \cdot (y|\mathbf{r}(z)),$$

by Lemma 5(e). We put $x' = x|\mathbf{d}(y|\mathbf{r}(z))$ and $y' = y|\mathbf{r}(z)$.

We now turn our attention specifically to the ordering of identities in an ordered category:

Lemma 7 ([1, Lemma 3.5]*). *Let $(C,\cdot,\leq)$ be an ordered category, and suppose that $a \in C$ and $e \in C_o$. If $a \leq e$, then a is an identity.*

Proof. If a and e are such that $a \leq e$, then $a = e|\mathbf{r}(a)$, by Lemma 5(a). By uniqueness of restrictions, $a = \mathbf{r}(a)$, i.e., a is an identity.

In an ordered category $(C,\cdot,\leq)$, if the greatest lower bound of two identities e, f exists (with respect to $\leq$), then we denote it by $e \wedge f$. It follows from Lemma 7 that if $e \wedge f$ exists, then it is an identity.

Definition 12. *An* inductive *category $(C,\cdot,\leq)$ is an ordered category in which the following additional condition holds:*

(In) if $e, f \in C_o$, then $e \wedge f$ exists in C_o.

An *inductive unipotent (cancellative) category* is a unipotent (cancellative) category which is also an inductive category in the sense of Definition 12.

As we already know from the historical comments made earlier, it is inductive categories which will be of the greatest interest in the sequel, since it is these which correspond to restriction semigroups in the appropriate manner. However, for the final few paragraphs of this section, we will continue to work with the notion of an ordered category, since almost everything we have to say is applicable in this more general case.

We use the order structure of an ordered category to define a notion which will be of great significance in the following section. Let $(C,\cdot,\leq)$ be an ordered category. The *pseudoproduct* $\otimes$ in $(C,\cdot,\leq)$ is the binary operation given by[‖]

$$a \otimes b = [a|\mathbf{r}(a) \wedge \mathbf{d}(b)] \cdot [\mathbf{r}(a) \wedge \mathbf{d}(b)|b]. \tag{6}$$

Notice that if $\mathbf{r}(a) \wedge \mathbf{d}(b)$ is defined, then $\mathbf{r}(a) \wedge \mathbf{d}(b) \leq \mathbf{r}(a)$ and $\mathbf{r}(a) \wedge \mathbf{d}(b) \leq \mathbf{d}(b)$, so it makes sense to write "$a|\mathbf{r}(a) \wedge \mathbf{d}(b)$" and "$\mathbf{r}(a) \wedge \mathbf{d}(b)|b$". The product of these latter two clearly exists. Moreover:

Lemma 8 ([1, p. 327]*). *If both $a \otimes b$ and $a \cdot b$ are defined in C, then they are equal.*

Proof. If $\exists a \cdot b$, then $\mathbf{r}(a) = \mathbf{d}(b)$, by Lemma 2, so $a \otimes b = (a|\mathbf{r}(a)) \cdot (\mathbf{d}(b)|b) = a \cdot b$, by Corollary 1(a).

The only bar to $\otimes$ being an everywhere-defined operation in C is the fact that $\mathbf{r}(a) \wedge \mathbf{d}(b)$ may not be defined. Indeed, $a \otimes b$ exists *if and only if* $\mathbf{r}(a) \wedge \mathbf{d}(b)$ does. We see therefore that in an *inductive* category, $\otimes$ is fully defined. Remaining for the time being in the more general case of an ordered category, we note the following pair of propositions:

Proposition 1 ([29, Lemma 4.1.5]*). *Let $(C,\cdot,\leq)$ be an ordered category and define the following subset of $C \times C$:*

$$\langle x, y \rangle = \{(x', y') \in C \times C : \mathbf{r}(x') = \mathbf{d}(y'),\ x' \leq x,\ y' \leq y\}.$$

[‖]Note that we are omitting brackets here and writing "$a|\mathbf{r}(a) \wedge \mathbf{d}(b)$" for "$a|(\mathbf{r}(a) \wedge \mathbf{d}(b))$"; "$a|\mathbf{r}(a) \wedge \mathbf{d}(b)$" should not be read as "$(a|\mathbf{r}(a)) \wedge \mathbf{d}(b)$".

We specify an ordering on $\langle x, y\rangle$ *by*

$$(a,b) \lhd (c,d) \Longleftrightarrow a \le c \text{ and } b \le d \text{ in } C.$$

Then $\exists x \otimes y$ *if and only if* $\langle x, y\rangle$ *has a maximum element* (x', y') *with respect to* $\lhd$*, and in this case* $x \otimes y = x' \cdot y'$.

Proof. ($\Rightarrow$) Suppose that $\exists x \otimes y$. Then $\mathbf{r}(x) \wedge \mathbf{d}(y) =: e$ exists and

$$x|e \le x, \quad e|y \le y \quad \text{and} \quad \mathbf{r}(x|e) = e = \mathbf{d}(e|y),$$

so $(x|e, e|y) \in \langle x, y\rangle$. Now let $(u, v) \in \langle x, y\rangle$. Then

$$u \le x, \quad v \le y \quad \text{and} \quad \mathbf{r}(u) = \mathbf{d}(v) =: f,$$

so, by uniqueness of restrictions and corestrictions, we have $u = x|f$ and $v = f|y$. It follows from (Or2) that $\mathbf{r}(u) \le \mathbf{r}(x)$ and $\mathbf{d}(v) \le \mathbf{d}(y)$. Thus $f = \mathbf{r}(u) = \mathbf{d}(v)$ is a lower bound for $\mathbf{r}(x)$ and $\mathbf{d}(y)$, in which case, $f \le e$. Then $u = x|f \le x|e$ and $v = f|y \le e|y$, by Lemma 5(d). It follows that $(x|e, e|y)$ is a maximum element in $\langle x, y\rangle$ and $x \otimes y = (x|e)\cdot(e|y)$.

($\Leftarrow$) Suppose that $\langle x, y\rangle$ has a maximum element (x', y'). We put $e = \mathbf{r}(x') = \mathbf{d}(y')$ so that $e \le \mathbf{r}(x), \mathbf{d}(y)$. Let $f \in C_o$ be such that $f \le \mathbf{r}(x), \mathbf{d}(y)$. Then $x|f$ and $f|y$ are defined, with

$$x|e \le x, \quad f|y \le y \quad \text{and} \quad \mathbf{r}(x|f) = f = \mathbf{d}(f|y).$$

Thus $(x|f, f|y) \in \langle x, y\rangle$ and so $(x|f, f|y) \lhd (x', y')$. It follows that $f \le e$, hence $e = \mathbf{r}(x) \wedge \mathbf{d}(y)$ and $\exists x \otimes y$.

Proposition 2 ([29, Lemma 4.1.6]*)**.** *Let* $(C, \cdot, \le)$ *be an ordered category. If both* $x \otimes (y \otimes z)$ *and* $(x \otimes y) \otimes z$ *are defined, then they are equal.*

Proof. We put $(x \otimes y) \otimes z = a \cdot z'$, where $(a, z') = \max\langle x \otimes y, z\rangle$, and also $x \otimes y = x' \cdot y'$, where $(x', y') = \max\langle x, y\rangle$. Then

$$a \le x \otimes y, \quad z' \le z, \quad x' \le x \quad \text{and} \quad y' \le y.$$

By Lemma 6, since $a \le x' \cdot y'$, there exist elements $x'' \le x'$ and $y'' \le y'$ such that $a = x'' \cdot y''$. Thus

$$(x \otimes y) \otimes z = (x'' \cdot y'') \cdot z' = x'' \cdot (y'' \cdot z').$$

Since $\exists y'' \cdot z'$, we have $\mathbf{r}(y'') = \mathbf{d}(z')$. Moreover, $y'' \le y' \le y$ and $z' \le z$, so $(y'', z') \in \langle y, z\rangle$. Let $(b, c) = \max\langle y, z\rangle$, so that $y'' \le b$ and $z' \le c$, whence $y'' \cdot z' \le b \cdot c = y \otimes z$. Similarly, $(x'', y'' \cdot z') \in \langle x, y \otimes z\rangle$ and so $(x \otimes y) \otimes z = x'' \cdot (y'' \cdot z') \le x \otimes (y \otimes z)$. The reverse inequality is similar.

Corollary 2. *In an inductive category* $(C, \cdot, \le)$*,* $\otimes$ *is an everywhere-defined, associative binary operation.*

By way of concluding this section, we record the following properties of the pseudoproduct for later use:

Lemma 9 ([1, Lemma 3.8]*). *Let $(C, \cdot, \leq)$ be an inductive category and let $a \in C$ and $e \in C_o$. Then*

$$e \otimes a = e \wedge \mathbf{d}(a)|a \quad \textit{and} \quad a \otimes e = a|\mathbf{r}(a) \wedge e.$$

Proof. We demonstrate the first equality; the second is similar. By definition, we have:

$$\begin{aligned} e \otimes a &= [e|e \wedge \mathbf{d}(a)] \cdot [e \wedge \mathbf{d}(a)|a] \\ &= e \wedge \mathbf{d}(a) \cdot [e \wedge \mathbf{d}(a)|a] \quad \text{(by Corollary 1(b), since } e \wedge \mathbf{d}(a) \leq e) \\ &= e \wedge \mathbf{d}(a)|a, \end{aligned}$$

by Lemma 5(b), as required.

5. Inductive Categories and Restriction Semigroups

In this section, we show that an inductive category may be constructed from a restriction semigroup, and vice versa.

Given a restriction semigroup S, we define the *restricted product* in S by

$$a \cdot b = \begin{cases} ab & \text{if } a^* = b^+; \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{7}$$

We then have the following result, originally proved by Lawson [28, Theorem 5.7]:

Theorem 2. *Let S be a restriction semigroup with respect to some subsemilattice E and with natural partial order $\leq$. Then $(S, \cdot, \leq)$ is an inductive category with $S_o = E$, $\mathbf{d}(x) = x^+$ and $\mathbf{r}(x) = x^*$, where $\cdot$ is the restricted product of (7). Restrictions, corestrictions and meets in $(S, \cdot, \leq)$ are equal to the corresponding products in S.*

Proof. We begin by showing that the idempotents in E are the identities of $(S, \cdot)$. Let $e \in E$ and suppose that $\exists e \cdot x$. Then $e^* = e = x^+$ and $e \cdot x = ex = x^+x = x$. Similarly, if $\exists x \cdot e$, then $x^* = e$ and $x \cdot e = xe = xx^* = x$.

It is easy to see that $\exists x^+ \cdot x$, since $(x^+)^* = x^+$. In this case, $x^+ \cdot x = x^+x = x$, so $\mathbf{d}(x) = x^+$. Similarly, $\exists x \cdot x^*$ and $x \cdot x^* = x$, so $\mathbf{r}(x) = x^*$.

(Ca1) Suppose that $\exists x \cdot (y \cdot z)$, i.e., $x^* = (yz)^+$ and $y^* = z^+$. Then

$$x^* = (yz)^+ = (yz^+)^+ = (yy^*)^+ = y^+,$$

by Lemma 1, so $\exists x \cdot y$. Also by Lemma 1,

$$(xy)^* = (x^*y)^* = (y^+y)^* = y^* = z^+,$$

so $\exists (x \cdot y) \cdot z$. It is easy to see that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$. The converse is similar.

(Ca2) This part is proved in much the same way as (Ca1): suppose that $\exists x \cdot y$ and $\exists y \cdot z$, i.e., $x^* = y^+$ and $y^* = z^+$. Then

$$(yz)^+ = y^+ = x^*,$$

again by Lemma 3, so $\exists x \cdot (y \cdot z)$.

Conversely, suppose that $\exists x \cdot (y \cdot z)$. This tells us implicitly that $\exists y \cdot z$. By (Ca1), $\exists (x \cdot y) \cdot z$, in which case $\exists x \cdot y$ also.

(Ca3) As already observed, $\mathbf{d}(x) = x^+$ and $\mathbf{r}(x) = x^*$.

We have shown that $(S, \cdot)$ is a category. We must now deal with the "ordered" and "inductive" parts.

(Or1) Suppose that $\exists a \cdot b = ab$ and $\exists c \cdot d = cd$, and that $a \leq c$ and $b \leq d$. We have $a = ec$ and $b = fd$, for some $e, f \in E$. Then

$$ab = ecfd = e(cf)^+ cd \leq cd,$$

using the left ample identity, so $a \cdot b \leq c \cdot d$.

(Or2) Suppose that $a \leq b$. We apply $^+$ to $a = eb$ to obtain

$$a^+ = (eb)^+ = (eb^+)^+ = eb^+ \leq b^+,$$

using Lemma 1, whence $\mathbf{d}(a) \leq \mathbf{d}(b)$. Similarly, applying * to $a = bf$ $(f \in E)$ gives $a^* \leq b^*$ and so $\mathbf{r}(a) \leq \mathbf{r}(b)$.

(Or3)(i) We note that $a|f = af$ has the desired properties: $af \leq a$ and $(af)^* = (a^* f)^* = a^* f = f$, since $f \leq a^* = \mathbf{r}(a)$. To show uniqueness, suppose that there is another element g which satisfies the conditions of (O3)(i), i.e., $g \leq a$ and $g^* = f$. Then $g = ag^* = af$. Hence $a|f$ is uniquely defined. Similarly, put $f|a = fa$ for part (ii).

(In) Note that $ef \leq e$ and $ef \leq f$, so $ef \leq e \wedge f$. Now suppose that g is an idempotent lower bound for e and f. Then $g = g^2 \leq ef$, by compatibility of $\leq$. Thus $ef = e \wedge f$.

We deduce some corollaries in the full restriction and ample cases:

Corollary 3 ([26, Theorem 3.15(i)]). *Let S be a full restriction semigroup with natural partial order $\leq$. Then $(S, \cdot, \leq)$ is an inductive unipotent category with $S_o = E(S)$, $\mathbf{d}(x) = x^+$ and $\mathbf{r}(x) = x^*$, where $\cdot$ is the restricted product of* (7).

Proof. By Theorem 2, $(S, \cdot, \leq)$ is an inductive category with $S_o = E(S)$. If e is idempotent with respect to $\cdot$, then it is also idempotent with respect to multiplication in S, so $e \in E(S)$. But $E(S) = S_o$, so e is an identity, and $(S, \cdot, \leq)$ is unipotent.

Corollary 4 ([1, Theorem 3.9]). *Let S be an ample semigroup with natural partial order $\leq$. Then $(S, \cdot, \leq)$ is an inductive cancellative category with $S_o = E(S)$, $\mathbf{d}(x) = x^+$ and $\mathbf{r}(x) = x^*$, where $\cdot$ is the restricted product of* (7).

Proof. By Corollary 3, $(S, \cdot, \leq)$ is an inductive unipotent category. Suppose that $\exists x \cdot z$, $\exists y \cdot z$ and that $x \cdot z = y \cdot z$. Then $x^* = z^+ = y^*$. Since $z \mathscr{R}^* z^+$, we can take the equality $xz = yz$ and replace z by z^+: $xz^+ = yz^+$. But $x^* = z^+ = y^*$, so $xx^* = yy^*$, hence $x = y$. We have shown that (Ca4)(i) holds; part (ii) is similar.

We now turn our attention to the converse construction whereby we obtain a restriction semigroup from an inductive category; the construction makes use of the pseudoproduct $\otimes$ introduced in (6). The result is due originally to Lawson [28, Theorem 5.7], but we give a slightly shorter proof, using Propositions 1 and 2, based upon that given in [29, Proposition 4.1.7] for the inverse case.

Theorem 3. *If $(C,\cdot,\leq)$ is an inductive category, then $(C,\otimes)$ is a restriction semigroup with respect to C_o.*

Proof. We first note that $(C,\otimes)$ is indeed a semigroup, by Corollary 2.

We aim to construct a restriction semigroup, so we must define a^+ and a^*, for each $a \in C$. Let us put $a^+ = \mathbf{d}(a)$ and $a^* = \mathbf{r}(a)$. We must now show that a^* is in fact the unique idempotent in the $\widetilde{\mathscr{L}}_{C_o}$-class of a, and that a^+ is the unique idempotent in the $\widetilde{\mathscr{R}}_{C_o}$-class of a. We have

$$a \otimes a^* = a|a^* \wedge a^* = a|a^* = a,$$

by Lemmas 1(a) and 9. Thus a^* is a right identity for a.

Suppose now that $a \otimes e = a$, for some $e \in S_o$. We need to show that $a^* \otimes e = a^*$. By Lemma 9, we have

$$a \otimes e = a|a^* \wedge e = a.$$

By applying * to both sides, we obtain $[a|a^* \wedge e]^* = a^*$, whence $a^* \wedge e = a^*$. Then

$$a^* \otimes e = a^*|a^* \wedge e = a^*|a^* = a^*,$$

as required. Therefore $a^* \, \widetilde{\mathscr{L}}_{C_o} \, a$. It may be shown in a similar way that a^+ is $\widetilde{\mathscr{R}}_{C_o}$-related to a.

We now show that idempotents in C_o commute (with respect to $\otimes$); it will then follow that the idempotents a^+ and a^* are the unique idempotents in the $\widetilde{\mathscr{R}}_{C_o}$- and $\widetilde{\mathscr{L}}_{C_o}$-classes of a, respectively. Let $e, f \in C_o$. Then

$$e \otimes f = [e|e \wedge f] \cdot [e \wedge f|f].$$

Note that $e|e \wedge f = e \wedge f = e \wedge f|f$, by Corollary 1. Thus $e \otimes f = (e \wedge f) \cdot (e \wedge f) = e \wedge f$. Similarly, $f \otimes e = f \wedge e = e \wedge f$, hence idempotents in C_o commute with respect to $\otimes$.

We show that $\widetilde{\mathscr{R}}_{C_o}$ is a left congruence. First note that $a \, \widetilde{\mathscr{R}}_{C_o} \, b$ if and only if $\mathbf{d}(a) = \mathbf{d}(b)$. Let $a, b \in C$ be such that $a \, \widetilde{\mathscr{R}}_{C_o} \, b$. For any $c \in C$, we have

$$\mathbf{d}(c \otimes a) = \mathbf{d}\,(c|\mathbf{r}(c) \wedge \mathbf{d}(a)) = \mathbf{d}\,(c|\mathbf{r}(c) \wedge \mathbf{d}(b)) = \mathbf{d}(c \otimes b).$$

Hence $c \otimes a \, \widetilde{\mathscr{R}}_{C_o} \, c \otimes b$, i.e., $\widetilde{\mathscr{R}}_{C_o}$ is a left congruence. It may be shown in a similar way that $\widetilde{\mathscr{L}}_{C_o}$ is a right congruence.

We must show that the ample identities hold:

$$a \otimes e = (a \otimes e)^+ \otimes a \qquad \text{and} \qquad e \otimes a = a \otimes (e \otimes a)^*.$$

We consider the $^+$ identity. By Lemma 9, $a \otimes e = a|(a^* \wedge e)$ and

$$(a \otimes e)^+ \otimes a = [(a \otimes e)^+ \wedge a^+]|a = \left[(a|a^* \wedge e)^+ \wedge a^+\right]|a.$$

Since $a|a^* \wedge e \leq a$, we have $[a|a^* \wedge e]^+ \leq a^+$, hence

$$\left[(a|a^* \wedge e)^+ \wedge a^+\right]|a = (a|a^* \wedge e)^+|a.$$

So $a \otimes e = a|a^* \wedge e \leq a$ and

$$(a \otimes e)^+ \otimes a = (a|a^* \wedge e)^+|a \leq a.$$

Also,

$$[(a \otimes e)^+ \otimes a]^+ = \left[(a|a^* \wedge e)^+|a\right]^+ = [a|a^* \wedge e]^+ = (a \otimes e)^+.$$

Thus, by Lemma 5(c), $a \otimes e = (a \otimes e)^+ \otimes a$, hence $(C, \otimes)$ is a left restriction semigroup with respect to C_o. The * identity is shown in a similar way.

We finally confirm that the ordering $\leq$ in the original inductive category becomes the usual ordering (4) of a restriction semigroup in $(C, \otimes)$. Suppose that $a \leq b$. Then $\mathbf{d}(a) \leq \mathbf{d}(b)$, by (Or2). Also, by Lemma 5(a), $a = \mathbf{d}(a)|b$. Consider $a^+ \otimes b$. By Lemma 9, we have

$$a^+ \otimes b = \mathbf{d}(a) \otimes b = \mathbf{d}(a) \wedge \mathbf{d}(b)|b = \mathbf{d}(a)|b,$$

so $a = a^+ \otimes b$, as required.

Now suppose that $a \leq b$ in $(C, \otimes)$, so that $a = e \otimes b$, for some idempotent $e \in E = C_o$. Then, using Lemma 9,

$$a = e \otimes b = e \wedge \mathbf{d}(b)|b \leq b,$$

in $(C, \cdot, \leq)$. Thus $(C, \cdot, \leq)$ and $(C, \otimes)$ have the same ordering.

Once again, we can write down corollaries in the full restriction and ample cases:

Corollary 5 ([26, Theorem 3.15(ii)]). *If $(C, \cdot, \leq)$ is an inductive unipotent category, then $(C, \otimes)$ is a full restriction semigroup.*

Proof. This follows easily from the fact that the only idempotents in the category are its identities.

Corollary 6 ([1, Theorem 3.9]). *If $(C, \cdot, \leq)$ is an inductive cancellative category, then $(C, \otimes)$ is an ample semigroup.*

Proof. By Corollary 5, $(C, \otimes)$ is a full restriction semigroup. It only remains to prove that $a^* = \mathbf{r}(a)$ is the unique idempotent which is $\mathscr{L}^*$-related to a and that $a^+ = \mathbf{d}(a)$ is the unique idempotent which is $\mathscr{R}^*$-related to a. We already know that a^* is a left identity for a, so, following (5), we need to prove that

$$a \otimes x = a \otimes y \Longrightarrow a^* \otimes x = a^* \otimes y,$$

for all $x, y \in (C, \otimes)^1$ (that is, $(C, \otimes)$ with identity adjoined). In fact, it is sufficient to show this for $x, y \in (C, \otimes)$.

Suppose that $a \otimes x = a \otimes y$. Then $(a \otimes x)^+ = (a \otimes y)^+$, i.e.,

$$\left[(a|a^* \wedge x^+) \cdot (a^* \wedge x^+|x)\right]^+ = \left[(a|a^* \wedge y^+) \cdot (a^* \wedge y^+|y)\right]^+,$$

whence

$$\left(a|a^* \wedge x^+\right)^+ = \left(a|a^* \wedge y^+\right)^+, \tag{8}$$

by Lemma 3. Notice that $a|a^* \wedge x^+ \leq a$ and $a|a^* \wedge y^+ \leq a$ so, using (8), we deduce that

$$a|a^* \wedge x^+ = a|a^* \wedge y^+,$$

by Lemma 5(c). Then

$$a \otimes x = [a|a^* \wedge x^+] \cdot [a^* \wedge x^+|x] = [a|a^* \wedge y^+] \cdot [a^* \wedge x^+|x].$$

But $a \otimes x = a \otimes y = [a|a^* \wedge y^+] \cdot [a^* \wedge y^+|y]$, by assumption, so

$$[a|a^* \wedge y^+] \cdot [a^* \wedge x^+|x] = [a|a^* \wedge y^+] \cdot [a^* \wedge y^+|y],$$

whence $a^* \wedge x^+|x = a^* \wedge y^+|y$, by cancellation. Thus, using Lemma 9,

$$a^* \otimes x = a^* \wedge x^+|x = a^* \wedge y^+|y = a^* \otimes y.$$

Therefore $a \mathscr{L}^* a^*$ in $(S, \otimes)$. Similarly, $a \mathscr{R}^* a^+$ in $(S, \otimes)$.

Let S be a restriction semigroup. We will denote the inductive category associated with S by $\mathbf{C}(S)$. Similarly, if C is an inductive category, then we will denote its associated restriction semigroup by $\mathbf{S}(C)$.

Theorem 4 (Implicit in [28])**.** *Let S be a restriction semigroup and C be an inductive category. Then $\mathbf{S}(\mathbf{C}(S)) = S$ and $\mathbf{C}(\mathbf{S}(C)) = C$.*

Proof. Let the operation in S be denoted by juxtaposition. By Theorem 2, $\mathbf{C}(S)$ is an inductive category under the restricted product $\cdot$ of (7). Further, in $\mathbf{C}(S)$, we have $e|a = ea$, $a|e = ae$ and $e \wedge f = ef$.

We now construct $\mathbf{S}(\mathbf{C}(S))$ by defining the pseudoproduct $\otimes$ of (6). By Theorem 3, $\mathbf{S}(\mathbf{C}(S))$ is a restriction semigroup under $\otimes$. It is clear that S and $\mathbf{S}(\mathbf{C}(S))$ share the same underlying set. Observe further that

$$a \otimes b = [a|\mathbf{r}(a) \wedge \mathbf{d}(b)] \cdot [\mathbf{r}(a) \wedge \mathbf{d}(b)|b] = (aa^* b^+) \cdot (a^* b^+ b) = ab,$$

so the operations in S and $\mathbf{S}(\mathbf{C}(S))$ are the same. Hence $S = \mathbf{S}(\mathbf{C}(S))$.

We turn now to the second part of the proposition. Let $\cdot$ denote the operation in C. We construct the restriction semigroup $\mathbf{S}(C)$ by defining the pseudoproduct $\otimes$ of (6).

We next define the restricted product:

$$\begin{aligned} a \odot b &= \begin{cases} a \otimes b & \text{if } a^* = b^+; \\ \text{undefined} & \text{otherwise.} \end{cases} \\ &= \begin{cases} [a|\mathbf{r}(a) \wedge \mathbf{d}(b)] \cdot [\mathbf{r}(a) \wedge \mathbf{d}(b)|b] & \text{if } \mathbf{r}(a) = \mathbf{d}(b); \\ \text{undefined} & \text{otherwise.} \end{cases} \end{aligned}$$

$$= \begin{cases} (a|\mathbf{r}(a))\cdot(\mathbf{d}(b)|b) & \text{if } \mathbf{r}(a)=\mathbf{d}(b); \\ \text{undefined} & \text{otherwise.} \end{cases}$$
$$= a\cdot b,$$

using Lemma 5(a). We know that $\mathbf{C}(\mathbf{S}(C))$ is an inductive category under $\odot$, and we see that $\odot$ and $\cdot$ coincide.

It is again clear that C and $\mathbf{C}(\mathbf{S}(C))$ share the same underlying set. We must now show they have the same ordering, restriction and corestriction. We first consider the ordering. By definition, $\mathbf{C}(\mathbf{S}(C))$ has the same ordering as $\mathbf{S}(C)$. We know from Theorem 3 that the ordering in $\mathbf{S}(C)$ is the same as the ordering in C. Hence $\mathbf{C}(\mathbf{S}(C))$ and C have the same ordering.

Let $|$ denote restriction and corestriction in C, and $\|$ denote the same in $\mathbf{C}(\mathbf{S}(C))$. Suppose that $e \leq \mathbf{r}(a)$. We then have

$$a\|e = a\otimes e = a|\mathbf{r}(a)\wedge e = a|e,$$

using Lemma 9. Similarly, if $e\leq \mathbf{d}(a)$, then $e\|a = e|a$. Thus $C = \mathbf{C}(\mathbf{S}(C))$, as required.

In the following sections, we will prove results which establish the isomorphisms of certain categories (in sense (♠)) of restriction semigroups and certain categories (again in sense (♠)) of inductive categories (now in sense (♣)). The arrows of these categories are yet to be defined and considered but Theorems 2, 3 and 4 provide us with the "objects" parts of the upcoming category isomorphisms.

6. $\vee$-Premorphisms and Ordered Functors

The first functions to be considered as the arrows of a category of restriction semigroups are so-called $\vee$-*premorphisms*, which generalise morphisms. These functions were originally introduced in the inverse case; we will see their "inverse version" in Section 8.2.

Definition 13. *Let S and T be restriction semigroups. A* $\vee$-premorphism *is a function $\theta : S\to T$ such that*

($\vee$1) $(st)\theta \leq (s\theta)(t\theta)$;

($\vee$2) $s^+\theta \leq (s\theta)^+$ and $s^*\theta\leq (s\theta)^*$.

We note some useful properties of $\vee$-premorphisms:

Lemma 10 ([22, Lemma 4.5])**.** *Let S and T be restriction semigroups with respect to semilattices E and F, respectively. If $\theta : S\to T$ is a $\vee$-premorphism, then*

(a) *$e\in E(S)\Rightarrow e\theta\in E(T)$;*

(b) *$e\in E\Rightarrow e\theta\in F$;*

(c) *$(s\theta)^+ = s^+\theta$ and $(s\theta)^* = s^*\theta$;*

(d) θ *is order-preserving.*

Using Lemma 10(d), the following is easily verified:

Proposition 3. *The composition of two $\vee$-premorphisms is a $\vee$-premorphism, hence restriction semigroups and $\vee$-premorphisms form a category.*

Regarding restriction semigroups as algebras with one binary operation and two unary operations, we can speak of *(2,1,1)-morphisms* of restriction semigroups: morphisms which respect $^+$ and *. It is clear that any such morphism is a $\vee$-premorphism. The following results will allow us (in the next section) to deduce theorems on (2,1,1)-morphisms from those proved here for $\vee$-premorphisms.

Lemma 11 ([29, Theorem 3.1.5]*). *A $\vee$-premorphism of restriction semigroups respects the restricted product* (7).

Proof. Let $\theta : S \to T$ be a $\vee$-premorphism of restriction semigroups, and suppose that $\exists s \cdot t$ in S, where $\cdot$ denotes the restricted product (7). By definition of $\cdot$, we have $s^* = t^+$ and $s \cdot t = st$. Observe further that

$$s^* = t^+ \Longrightarrow s^*\theta = t^+\theta \Longrightarrow (s\theta)^* = (t\theta)^+,$$

using Lemma 10(c), and so $\exists (s\theta) \cdot (t\theta)$ in T. It follows further from the definition of θ that $(s \cdot t)\theta \le (s\theta) \cdot (t\theta) = (s\theta)(t\theta)$. Thus, applying (4), we have

$$\begin{aligned}
(s \cdot t)\theta &= [(s \cdot t)\theta]^+ (s\theta)(t\theta) \\
&= \left[(s \cdot t)^+\theta\right](s\theta)(t\theta) \\
&= \left[(st)^+\theta\right](s\theta)(t\theta) \\
&= \left[(st^+)^+\theta\right](s\theta)(t\theta) \\
&= \left[(ss^*)^+\theta\right](s\theta)(t\theta) \\
&= \left(s^+\theta\right)(s\theta)(t\theta) \\
&= (s\theta)^+(s\theta)(t\theta) \\
&= (s\theta)(t\theta) = (s\theta) \cdot (t\theta).
\end{aligned}$$

Hence θ respects restricted products.

Lemma 12 ([29, Theorem 3.1.5]*). *Let S and T be restriction semigroups, where S has distinguished subsemilattice of idempotents E. A $\vee$-premorphism $\theta : S \to T$ is a (2,1,1)-morphism if and only if $(e\theta)(f\theta) = (ef)\theta$, for any $e, f \in E$.*

Proof. If θ is a (2,1,1)-morphism, then it is clear that $(e\theta)(f\theta) = (ef)\theta$, for any $e, f \in E$, so we move straight to the converse. We need only deal with the "2" part of "(2,1,1)-morphism", since both "1" parts are taken care of by Lemma 10(c).

Suppose that $(e\theta)(f\theta) = (ef)\theta$, for any $e, f \in E$, and take a (semigroup) product $st \in S$. We notice that $st = (se)(et)$, where $e = s^* t^+$. Indeed, we observe further that the restricted product $(se) \cdot (et)$ is defined, since

$$(se)^* = (ss^* t^+)^* = (st^+)^* = (s^* t^+)^* = (s^* t^+)^+ = (s^* t)^+ = (s^* t^+ t)^+ = (et)^+.$$

Thus $st = (se) \cdot (et)$. Then, since θ respects restricted products (by Lemma 11), we have $(st)\theta = (se)\theta \cdot (et)\theta = (se)\theta(et)\theta$.

We next show that $(se)\theta = (s\theta)(e\theta)$; we do so by stepping across into the inductive category $\mathbf{C}(T)$ and employing Lemma 5(c). Notice first of all that $(se)\theta \leq (s\theta)(e\theta)$, by definition of θ, and that, naturally, $(s\theta)(e\theta) \leq (s\theta)(e\theta)$. We now observe that, on the one hand,

$$\mathbf{r}((se)\theta) = ((se)\theta)^* = (se)^*\theta = (s^* e)^*\theta = (s^* e)\theta = e\theta$$

(since $e = s^* t^+ \leq s^*$), whilst on the other,

$$\mathbf{r}((s\theta)(e\theta)) = ((s\theta)(e\theta))^* = \left((s\theta)^*(e\theta)\right)^* = \left((s^*\theta)(e\theta)\right)^* = \left((s^* e)\theta\right)^* = (e\theta)^* = e\theta.$$

Thus $\mathbf{r}((se)\theta) = \mathbf{r}((s\theta)(e\theta))$, and so $(se)\theta = (s\theta)(e\theta)$, by Lemma 5(c). It follows in a similar manner that $(et)\theta = (e\theta)(t\theta)$.

Finally, putting all the pieces together, we have:

$$\begin{aligned}(st)\theta &= (se)\theta(et)\theta = (s\theta)(e\theta)(e\theta)(t\theta) = (s\theta)(e\theta)(t\theta) \\ &= (s\theta)\left((s^* t^+)\theta\right)(t\theta) = (s\theta)(s^*\theta)(t^+\theta)(t\theta) = (s\theta)(s\theta)^*(t\theta)^+(t\theta) = (s\theta)(t\theta).\end{aligned}$$

Hence θ is a (2,1,1)-morphism.

We aim to obtain an isomorphism of categories involving the category of restriction semigroups and $\vee$-premorphisms. We therefore need to decide what the arrows will be in the corresponding category of inductive categories. These will be so-called *ordered functors*. We note that, just like categories, we will be using the term "functor" in two slightly different, though equivalent, senses, depending on how we are regarding the underlying categories: we will have functors between categories of semigroups, say, where the categories are viewed in sense (♠), and functors between categories viewed in sense (♣). It is this latter sense of "functor" which we now define:

Definition 14. *Let C and D be categories. A function $\phi : C \to D$ is called a* functor *if it satisfies the following condition:*

(F) if $\exists x \cdot y$ in C, then $\exists (x\phi) \cdot (y\phi)$ in D and $(x\phi) \cdot (y\phi) = (x \cdot y)\phi$.

Lemma 13 ([F]). *Let $\phi : C \to D$ be a functor between categories C and D. For any $x \in C$, $\mathbf{d}(x)\phi = \mathbf{d}(x\phi)$ and $\mathbf{r}(x)\phi = \mathbf{r}(x\phi)$.*

Definition 15. *Let C and D be ordered categories. A functor $\phi : C \to D$ is called an* ordered functor *if it satisfies the following additional condition:*

(OF) if $x \leq y$ in C, then $x\phi \leq y\phi$ in D.

Lemma 14 ([29, Proposition 4.1.2]). *Let $\phi : C \to D$ be an ordered functor between ordered categories C and D. If $f \in C_o$ is such that $f \leq \mathbf{r}(a)$, for some $a \in C$, then $(a|f)\phi = a\phi|f\phi$. Similarly, if $f \leq \mathbf{d}(a)$, then $(f|a)\phi = f\phi|a\phi$.*

The following is easy to verify:

Proposition 4 (F). *The composition of two ordered functors is also an ordered functor. Consequently, inductive categories (in sense (♣)) and ordered functors form a category (in sense (♠)).*

Before we prove the correspondence between ordered functors and $\vee$-premorphisms, we first record the following useful result:

Lemma 15 ([22, Lemma 4.6]). *Let $\alpha : S \to T$ be an order-preserving function of restriction semigroups. We define $\mathbf{C}(\alpha) : \mathbf{C}(S) \to \mathbf{C}(T)$ to be the same function on the underlying sets. Then $\mathbf{C}(\alpha)$ is order-preserving.*

Let $\beta : C \to D$ be an order-preserving function of inductive categories. We define $\mathbf{S}(\beta) : \mathbf{S}(C) \to \mathbf{S}(D)$ to be the same function on the underlying sets. Then $\mathbf{S}(\beta)$ is order-preserving.

Proposition 5 ([22, Proposition 4.8]). *Let S and T be restriction semigroups with respect to semilattices E and F, respectively. Let $\theta : S \to T$ be a $\vee$-premorphism. We define $\Theta := \mathbf{C}(\theta) : \mathbf{C}(S) \to \mathbf{C}(T)$ to be the same function on the underlying sets. Then Θ is an ordered functor with respect to the restricted products in $\mathbf{C}(S)$ and $\mathbf{C}(T)$.*

Proposition 6 ([22, Proposition 4.9]). *Let $\phi : C \to D$ be an ordered functor of inductive categories. We define $\Phi := \mathbf{S}(\phi) : \mathbf{S}(C) \to \mathbf{S}(D)$ to be the same function on the underlying sets. Then Φ is a $\vee$-premorphism with respect to the pseudoproducts in $\mathbf{S}(C)$ and $\mathbf{S}(D)$.*

It is clear that if $\theta : S \to T$ is a $\vee$-premorphism and $\phi : C \to D$ is an ordered functor, then $\mathbf{S}(\mathbf{C}(\theta)) = \theta$ and $\mathbf{C}(\mathbf{S}(\phi)) = \phi$. Furthermore, if $\theta' : T \to T'$ is another $\vee$-premorphism of restriction semigroups, and $\phi' : D \to D'$ is another ordered functor of inductive categories, then $\mathbf{C}(\theta\theta') = \mathbf{C}(\theta)\mathbf{C}(\theta')$ and $\mathbf{S}(\phi\phi') = \mathbf{S}(\phi)\mathbf{S}(\phi')$. We therefore have the following theorem and its corollaries:

Theorem 5 ([22, Theorem 4.1]). *The category of restriction semigroups and $\vee$-premorphisms is isomorphic to the category of inductive categories and ordered functors.*

Corollary 7. *The category of full restriction semigroups and $\vee$-premorphisms is isomorphic to the category of inductive unipotent categories and ordered functors.*

Corollary 8. *The category of ample semigroups and $\vee$-premorphisms is isomorphic to the category of inductive cancellative categories and ordered functors.*

7. Morphisms and Inductive Functors

We turn now to the second type of arrow to be considered between restriction semigroup: (2,1,1)-morphisms, morphisms which respect $^+$ and *. Again thinking of restriction semigroups as algebras of type (2,1,1), we begin by noting the following fact from universal algebra:

Fact 1 ([F]). *The composition of two (2,1,1)-morphisms is also a (2,1,1)-morphism. Consequently, restriction semigroups and (2,1,1)-morphisms form a category.*

The functions between inductive categories to which (2,1,1)-morphisms will correspond are so-called *inductive functors*, which we define by building upon Definitions 14 and 15:

Definition 16. *Let C and D be inductive categories. An ordered functor $\phi : C \to D$ is called an* inductive functor *if it satisfies the following additional condition:*

(IF) for $e, f \in C_o$, $(e \wedge f)\phi = e\phi \wedge f\phi$.

Lemma 16. *Let $\phi : C \to D$ be an inductive functor between inductive categories C and D. Then $x\phi \otimes y\phi = (x \otimes y)\phi$.*

Proof. We have

$$\begin{aligned} x\phi \otimes y\phi &= \left[x\phi|\mathbf{r}(x\phi) \wedge \mathbf{d}(y\phi)\right] \cdot \left[\mathbf{r}(x\phi) \wedge \mathbf{d}(y\phi)|y\phi\right] \\ &= \left[x\phi|\mathbf{r}(x)\phi \wedge \mathbf{d}(y)\phi\right] \cdot \left[\mathbf{r}(x)\phi \wedge \mathbf{d}(y)\phi|y\phi\right], \text{ by Lemma 13} \\ &= \left[x\phi|(\mathbf{r}(x) \wedge \mathbf{d}(y))\phi\right] \cdot \left[(\mathbf{r}(x) \wedge \mathbf{d}(y))\phi|y\phi\right], \text{ by (IF)} \\ &= (x|\mathbf{r}(x) \wedge \mathbf{d}(y))\phi \cdot (\mathbf{r}(x) \wedge \mathbf{d}(y)|y)\phi, \text{ by Lemma 14} \\ &= \left[(x|\mathbf{r}(x) \wedge \mathbf{d}(y)) \cdot (\mathbf{r}(x) \wedge \mathbf{d}(y)|y)\right]\phi, \text{ by (F)} \\ &= (x \otimes y)\phi, \end{aligned}$$

as required.

The following is an easy consequence of Proposition 4:

Proposition 7 ([F]). *The composition of two inductive functors is also an inductive functor. Consequently, inductive categories and inductive functors form a category.*

We are now ready to establish a correspondence between (2,1,1)-morphisms and inductive functors, which we achieve through the combination of Lemma 12 with Propositions 5 and 6.

Proposition 8. *Let $\varphi : S \to T$ be a (2,1,1)-morphism between restriction semigroups S and T. We define $\Phi := \mathbf{C}(\varphi) : \mathbf{C}(S) \to \mathbf{C}(T)$ to be the same function on the underlying sets. Then Φ is an inductive functor with respect to the restricted products in $\mathbf{C}(S)$ and $\mathbf{C}(T)$.*

Proof. As a (2,1,1)-morphism, φ is a $\vee$-premorphism, and so, by Proposition 5, Φ is an ordered functor. To see that Φ is inductive, we simply observe that

$$e\Phi \wedge f\Phi = e\varphi \wedge f\varphi = (e\varphi)(f\varphi) = (ef)\varphi = (e \wedge f)\varphi = (e \wedge f)\Phi,$$

for $e, f \in E$.

Proposition 9. *Let $\phi : C \to D$ be an inductive functor of inductive categories C and D. We define $\Phi := \mathbf{S}(\phi) : \mathbf{S}(C) \to \mathbf{S}(D)$ to be the same function on the underlying sets. Then Φ is a (2,1,1)-morphism with respect to the pseudoproducts in $\mathbf{S}(C)$ and $\mathbf{S}(D)$.*

Proof. That Φ respects pseudoproducts follows from Lemma 16. Moreover, since x^+ and x^* in $\mathbf{S}(C)$ correspond to $\mathbf{d}(x)$ and $\mathbf{r}(x)$ in C, we see that Φ must preserve both $^+$ and *, thanks to Lemma 13.

(Alternatively, in place of Lemma 16, we could have included the weaker result that an inductive functor respects pseudoproducts of idempotents. This, together with Lemma 12, would then give the desired result.)

Proposition 10. *If $\varphi : S \to T$ is a (2,1,1)-morphism between restriction semigroups and $\phi : C \to D$ is an inductive functor between inductive categories, then $\mathbf{S}(\mathbf{C}(\varphi)) = \varphi$ and $\mathbf{C}(\mathbf{S}(\phi)) = \phi$.*

Proof. This is an easy consequence of Propositions 8 and 9, together with Theorem 4.

Observe that if $\varphi' : T \to T'$ is another (2,1,1)-morphism of restriction semigroups, and $\phi' : D \to D'$ is another inductive functor of inductive categories, then $\mathbf{C}(\varphi\varphi') = \mathbf{C}(\varphi)\mathbf{C}(\varphi')$ and $\mathbf{S}(\phi\phi') = \mathbf{S}(\phi)\mathbf{S}(\phi')$. Thus $\mathbf{S}(\cdot)$ and $\mathbf{C}(\cdot)$ form a pair of mutually inverse functors[**] between the category of restriction semigroups and (2,1,1)-morphisms, and that of inductive categories and inductive functors. Theorems 2, 3, and 4 and Propositions 8, 9 and 10 can therefore be brought together into the following:

Theorem 6 ([28, Theorem 5.7])**.** *The category of restriction semigroups and (2,1,1)-morphisms is isomorphic to the category of inductive categories and inductive functors.*

Corollaries 3 and 5 enable us to write down the following specialisation of Theorem 6:

Corollary 9 ([26, Theorem 3.16])**.** *The category of full restriction semigroups and (2,1,1)-morphisms is isomorphic to the category of inductive unipotent categories and inductive functors.*

Finally, from Corollaries 4 and 6, we have the following:

Corollary 10. *The category of ample semigroups and (2,1,1)-morphisms is isomorphic to the category of inductive cancellative categories and inductive functors.*

8. Inverse Semigroups and Inductive Groupoids

Now that we have established a series of category isomorphisms for restriction semigroups and inductive categories, we are ready to turn our attention to the special case of inverse semigroups. As noted in Section 3, these may be regarded as full restriction semigroups with $a^+ = aa^{-1}$ and $a^* = a^{-1}a$. The particular type of inductive category to which an inverse semigroup will correspond, under the constructions of Section 5, is a so-called *inductive groupoid*.

[**]Note that these functors are regarded as functions between categories in sense (♠).

8.1. Inductive Groupoids

Definition 17. *Let $(G,\cdot)$ be a small category. We call $(G,\cdot)$ a* groupoid *if it satisfies the following additional condition:*

(G) for every $x \in G$, there exists $x^{-1} \in G$ such that $\exists x \cdot x^{-1}$ and $\exists x^{-1} \cdot x$ with $x \cdot x^{-1} = \mathbf{d}(x)$ and $x^{-1} \cdot x = \mathbf{r}(x)$.

To put this another way: a groupoid is a small category in which every arrow is invertible.

Lemma 17. *A groupoid is cancellative in the sense of Definition 10.*

Proof. Let $(G,\cdot)$ be a groupoid and suppose that $\exists x \cdot z$ and $\exists y \cdot z$ with $x \cdot z = y \cdot z$. Thus $\mathbf{r}(x) = \mathbf{r}(y) = \mathbf{d}(z)$. Note that since $\exists z \cdot z^{-1}$, we may conclude that both $x \cdot (z \cdot z^{-1})$ and $(x \cdot z) \cdot z^{-1}$ are defined. Similarly for $y \cdot (z \cdot z^{-1})$ and $(y \cdot z) \cdot z^{-1}$. Then

$$\begin{aligned} x \cdot z = y \cdot z &\Longrightarrow (x \cdot z) \cdot z^{-1} = (y \cdot z) \cdot z^{-1} \\ &\Longrightarrow x \cdot (z \cdot z^{-1}) = y \cdot (z \cdot z^{-1}) \\ &\Longrightarrow x \cdot \mathbf{d}(z) = y \cdot \mathbf{d}(z) \\ &\Longrightarrow x = y, \end{aligned}$$

since $\mathbf{d}(z) = \mathbf{r}(x) = \mathbf{r}(y)$. The second part is similar.

In particular, a groupoid is necessarily unipotent. We observe also that the inverses in a groupoid behave in the manner in which we would expect them to:

Lemma 18 ([F]). *Let $(G,\cdot)$ be a groupoid. Then*

(a) $\mathbf{r}(x) = \mathbf{d}(x^{-1})$ *and* $\mathbf{d}(x) = \mathbf{r}(x^{-1})$*;*

(b) *for each $x \in G$, x^{-1} is unique;*

(c) $(x^{-1})^{-1} = x$.

Note that in the case of a groupoid, the local submonoids $\mathrm{mor}(e,e)$ are in fact local sub*groups*. Thus, a groupoid with one identity is necessarily a group: a groupoid may be regarded as a generalisation of a group, which perhaps goes some way towards explaining why the notion of a groupoid arose before that of a category, as we saw Section 2.[††]

We must now introduce an ordering onto a groupoid $(G,\cdot)$:

Definition 18. *Let $(G,\cdot)$ be a groupoid and suppose that G is partially ordered by $\leq$. Then we call $(G,\cdot,\leq)$ an* ordered groupoid *if it satisfies conditions (Or1) and (Or3), together with the following in place of (Or2):*

(Or2′) if $a \leq b$, then $a^{-1} \leq b^{-1}$.

[††]See footnote § on page 420.

Notice that an *inductive* category which is also a groupoid necessarily satisfies (Or2′): if $a \leq b$ in the category, then $a \leq b$ in the corresponding semigroup, which means that $a = e \otimes b$, for some idempotent e. But then $a^{-1} = b^{-1} \otimes e$, whence $a^{-1} \leq b^{-1}$.

In fact, (Or2) still holds in an ordered groupoid:

Lemma 19 ([29, Proposition 4.1.3(1)]). *An ordered groupoid $(G, \cdot, \leq)$ satisfies (Or2).*

Proof. Suppose that $a \leq b$. We have that $a^{-1} \leq b^{-1}$, by (Or2′), and we know that $\exists a \cdot a^{-1}$ and $\exists b \cdot b^{-1}$. Therefore, $a \cdot a^{-1} \leq b \cdot b^{-1}$, by (Or1), hence $\mathbf{d}(a) \leq \mathbf{d}(b)$. Similarly, $\mathbf{r}(a) \leq \mathbf{r}(b)$.

Thus, an ordered groupoid, in the sense of Definition 18, is an ordered category, in the sense of Definition 11. Similarly, if we give the following very natural definition for an inductive groupoid, then an inductive groupoid is an inductive category, in the sense of Definition 12:

Definition 19. *Let $(G, \cdot, \leq)$ be an ordered groupoid. We call $(G, \cdot, \leq)$ an* inductive groupoid *if it also satisfies (In).*

An inductive groupoid has many nice properties, amongst which is the possibility of expressing the corestriction in terms of the restriction, and vice versa:

Lemma 20 ([36, Proposition 3.1]). *Let $(G, \cdot, \leq)$ be an ordered groupoid and suppose that $f \leq \mathbf{d}(a)$, for some $f \in G_o$ and some $a \in G$. Then the restriction $f|a$ may be expressed in terms of the corestriction: $f|a = (a^{-1}|f)^{-1}$. Dually, if $f \leq \mathbf{r}(a)$, then $a|f = (f|a^{-1})^{-1}$.*

We now turn our attention to the specialisation of the results of the preceding sections to the case of inverse semigroups and inductive groupoids. We begin by making the easy observation that the restricted product of (7) may be rewritten as follows in an inverse semigroup:

$$a \cdot b = \begin{cases} ab & \text{if } a^{-1}a = bb^{-1}; \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{9}$$

We have the following corollary to Theorem 2:

Theorem 7 ([45, p. 109]). *Let S be an inverse semigroup with natural partial order $\leq$. Then $(S, \cdot, \leq)$ is an inductive groupoid with $S_o = E(S)$, $\mathbf{d}(x) = xx^{-1}$ and $\mathbf{r}(x) = x^{-1}x$, where $\cdot$ is the restricted product of* (9).

Proof. By Corollary 3, $(S, \cdot, \leq)$ is an inductive unipotent category. We must verify that condition (G) holds. For clarity, let y be the inverse of x in the original semigroup. We observe that $xx^{-1} = y^{-1}y$, so $\exists x \cdot y$ in $(S, \cdot)$. Moreover, $x \cdot y = xx^{-1} = \mathbf{d}(x)$, as required. Similarly, $\exists y \cdot x$ and $y \cdot x = \mathbf{r}(x)$.

Corollary 3 tells us that $(S, \cdot, \leq)$ satisfies condition (Or2) but we must verify that it also satisfies the stronger (Or2′). Suppose that $a \leq b$ in $(S, \cdot, \leq)$. Then $a \leq b$ in $(S, \otimes)$, so $a = eb$, for some $e \in E(S)$. Since ordering and multiplication are compatible in the semigroup, we have $a^{-1} \leq (eb)^{-1} = b^{-1}e \leq b^{-1}$. Therefore $a^{-1} \leq b^{-1}$ in $(S, \cdot, \leq)$.

Conversely, we have the following:

Theorem 8 ([45, Theorem 3.4])**.** *If $(G,\cdot,\leq)$ is an inductive groupoid, then $(G,\otimes)$ is an inverse semigroup, where $\otimes$ is the pseudoproduct of* (6).

Proof. By Corollary 5, $(G,\otimes)$ is a full restriction semigroup. We therefore know that $E(G)=G_o$ and, from the proof of Theorem 3, that these idempotents commute with respect to $\otimes$. It remains to show that the groupoid inverses in $(G,\cdot,\leq)$ serve as semigroup inverses in $(G,\otimes)$. Let x^{-1} be the inverse of $x\in G$ in the sense of condition (G). Then, since $\cdot$ and $\otimes$ coincide whenever the former is defined, we have

$$(x\otimes x^{-1})\otimes x=(x\cdot x^{-1})\otimes x=\mathbf{d}(x)\otimes x=\mathbf{d}(x)\cdot x=x.$$

Similarly, $x^{-1}\otimes x\otimes x^{-1}=x^{-1}$.

Let S be an inverse semigroup. We will denote the inductive groupoid associated with S by $\mathbf{G}(S)$. Similarly, if G is an inductive groupoid, then we will denote its associated inverse semigroup by $\mathbf{I}(G)$. The following is an easy consequence of Theorem 4:

Theorem 9 ([29, Proposition 4.1.7(2),(3)])**.** *Let S be an inverse semigroup and G be an inductive groupoid. Then $\mathbf{I}(\mathbf{G}(S))=S$ and $\mathbf{G}(\mathbf{I}(G))=G$.*

As in the more general case of restriction semigroups and inductive categories, we are aiming to prove an isomorphism of categories for inverse semigroups and inductive groupoids. We have dealt with the "objects" parts, so we must now turn our attention to the arrows; we take each of the previously considered cases in turn.

8.2. $\vee$-Premorphisms and Ordered Functors

The notion of a $\vee$-premorphism was originally introduced in [32] for inverse semigroups with the following definition:

Definition 20. *Let S and T be inverse semigroups. A function $\theta: S\to T$ is called a $\vee$-*premorphism *if $(st)\theta\leq(s\theta)(t\theta)$.*

Given that we included a condition relating to $^+$ and * in the definition of a $\vee$-premorphism for restriction semigroups, it is perhaps a little surprising that we make no mention of inverses in the definition for inverse semigroups. In fact, there is no real omission here:

Lemma 21 ([29, Theorem 3.1.5])**.** *Let $\theta: S\to T$ be a $\vee$-premorphism of inverse semigroups, as in Definition 20. Then θ respects inverses and the natural partial order.*

We have so far been a little sloppy in referring to both the function of Definition 13 and that of Definition 20 as "$\vee$-premorphisms". In fact, there is no ambiguity:

Lemma 22. *Let $\theta: S\to T$ be a function between inverse semigroups. Then θ is a $\vee$-premorphism in the sense of Definition 20 if and only if it is a $\vee$-premorphism in the sense of Definition 13.*

Proof. (⇐) Immediate.

(⇒) Suppose that $\theta : S \to T$ is a $\vee$-premorphism in the sense of Definition 20. Then

$$s^{+}\theta = (ss^{-1})\theta \leq (s\theta)(s^{-1}\theta) = (s\theta)(s\theta)^{-1} = (s\theta)^{+},$$

using Lemma 21. Similarly, $s^{*}\theta \leq (s\theta)^{*}$.

We note the following:

Lemma 23 ([32, Corollary 2.2])**.** *Inverse semigroups and $\vee$-premorphisms form a category.*

We then have an immediate corollary to Theorem 5:

Theorem 10 ([27, Theorem 3.5(i)])**.** *The category of inverse semigroups and $\vee$-premorphisms is isomorphic to the category of inductive groupoids and ordered functors.*

This is of course the first part of the original ESN Theorem (Theorem 1).

8.3. Morphisms and Inductive Functors

We now take the functions between inverse semigroups to be (inverse semigroup) morphisms. We make the following very easy observation:

Proposition 11 ([F])**.** *Inverse semigroups and morphisms form a category.*

The functions between the corresponding inductive groupoids will be inductive functors, just as in the case of restriction semigroups and inductive categories. We first note the following:

Lemma 24. *Let G and H be groupoids and let $\phi : G \to H$ be a functor, in the sense of Definition 14. Then $(g\phi)^{-1} = g^{-1}\phi$.*

Proof. We know that $\exists g \cdot g^{-1}$, so $\exists (g\phi)\cdot(g^{-1}\phi)$, by (F). Moreover, we have $(g\phi)\cdot(g^{-1}\phi) = (g \cdot g^{-1})\phi = \mathbf{d}(g)\phi = \mathbf{d}(g\phi)$, by Lemma 13. Similarly, we have that $\exists (g^{-1}\phi)\cdot(g\phi)$ and $(g^{-1}\phi)\cdot(g\phi) = \mathbf{r}(g\phi)$. The result then follows from Lemma 18(b).

We see then that functors are suitable functions to consider between groupoids; inductive functors are therefore appropriate arrows to consider between inductive groupoids. The following is an easy consequence of Proposition 7:

Proposition 12 ([F])**.** *Inductive groupoids and inductive functors form a category.*

The appropriate specialisations of Propositions 8, 9 and 10 are clear:

Proposition 13. *Let $\varphi : S \to T$ be a morphism between inverse semigroups S and T. We define $\Phi := \mathbf{G}(\varphi) : \mathbf{G}(S) \to \mathbf{G}(T)$ to be the same function on the underlying sets. Then Φ is an inductive functor with respect to the restricted products in $\mathbf{G}(S)$ and $\mathbf{G}(T)$.*

Proposition 14. *Let $\phi : G \to H$ be an inductive functor of inductive groupoids G and H. We define $\Phi := \mathbf{S}(\phi) : \mathbf{I}(G) \to \mathbf{I}(H)$ to be the same function on the underlying sets. Then Φ is a morphism with respect to the pseudoproducts in $\mathbf{I}(G)$ and $\mathbf{I}(H)$.*

Proposition 15. *If $\varphi : S \to T$ is a morphism between inverse semigroups and $\phi : G \to H$ is an inductive functor between inductive groupoids, then $\mathbf{I}(\mathbf{G}(\varphi)) = \varphi$ and $\mathbf{G}(\mathbf{I}(\phi)) = \phi$.*

Moreover, if $\varphi' : T \to T'$ is another morphism of inverse semigroups, and $\phi' : H \to H'$ is another inductive functor of inductive groupoids, then $\mathbf{G}(\varphi\varphi') = \mathbf{G}(\varphi)\mathbf{G}(\varphi')$ and $\mathbf{I}(\phi\phi') = \mathbf{I}(\phi)\mathbf{I}(\phi')$. Thus $\mathbf{I}(\cdot)$ and $\mathbf{G}(\cdot)$ form a pair of mutually inverse functors[‡‡] between the category of inverse semigroups and morphisms, and that of inductive groupoids and inductive functors. We may therefore write down the following corollary to Theorem 6 in the inverse case:

Theorem 11 ([27, Theorem 3.5(ii)]). *The category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors.*

This is, of course, the second half of the original ESN Theorem (Theorem 1).

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