Special Involute-Evolute Partner D -Curves in E^3

ÖZCAN BEKTAŞ * SALİM YÜCE *

Abstract

In this paper, we take into account the opinion of involute-evolute curves which lie on fully surfaces and by taking into account the Darboux frames of them we illustrate these curves as special involute-evolute partner D-curves in E^3. Besides, we find the relations between the normal curvatures, the geodesic curvatures and the geodesic torsions of these curves. Finally, some consequences and examples are given.

Keywords: Involute-evolute, Darboux Frame, normal curvature, geodesic curvature, geodesic torsion.

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1 Introduction

In differential geometry, there are many important consequences and properties of curves. Researchers follow labours about the curves. In the light of the existing studies, authors always introduce new curves. Involute-evolute curves are one of them. C. Huggens discovered involutes while trying to build a more accurate clock, [1]. Later, the relations Frenet apparatus of involute-evolute curve couple in the space E^3 were given in [2]. A. Turgut examined involute-evolute curve couple in E^n, [3].

In this study, we consider the notion of the involute-evolute curves lying on the surfaces for a special situation. We determine the special involute-evolute partner D-curves in E^3. By using the Darboux frame of the curves we obtain the necessary and sufficient conditions between κ_g ,τ_g,κ_n and κ_n^* for a curve to be the special involute partner D-curve. κ_g^* and τ_g^* of this special involute partner D-curve are found. Finally, some special case and examples are given.

*Yildiz Technical University, Faculty of Arts And Sciences, Department of Mathematics, 34210, Esenler, Istanbul, Turkey, ozcanbektas1986@hotmail.com, sayuce@yildiz.edu.tr.

2 Preliminaries

In this section, we give informations about Involute-evolute curves and Darboux frame. Let $\alpha(s)$ be a curve on an oriented surface M. Since the curve $\alpha(s)$ is also in space, there exists Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ at each points of the curve where $\mathbf{T}$ is unit tangent vector, $\mathbf{N}$ is principal normal vector and $\mathbf{B}$ is binormal vector, respectively. The Frenet equations of the curve $\alpha(s)$ is given by

$$\begin{cases} \mathbf{T}' = \kappa\mathbf{N} \\ \mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B} \\ \mathbf{B}' = -\tau\mathbf{N} \end{cases}$$

where κ and τ are curvature and torsion of the curve $\alpha(s)$, respectively. Since the curve $\alpha(s)$ lies on the surface M there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$. In this frame $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{n}$ is the unit normal of the surface M and $\mathbf{g}$ is the unit vector given by $\mathbf{g}=\mathbf{n}\times\mathbf{T}$. Since the unit tangent $\mathbf{T}$ is common in both Frenet frame and Darboux frame, the vectors $\mathbf{N},\mathbf{B},\mathbf{g}$, $\mathbf{n}$ lie on the same plane. So that the relations between these frames can be given as follows

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{2.1}$$

where φ is the angle between the vectors $\mathbf{g}$ and $\mathbf{n}$. The derivative formulae of the Darboux frame is

$$\begin{bmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{g}} \\ \dot{\mathbf{n}} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \tag{2.2}$$

where , κ_g is the geodesic curvature, κ_n is the normal curvature and τ_g is the geodesic torsion of $\alpha(s)$. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

The relations between κ_g , κ_n , τ_g and κ , τ are given as follows

$$\kappa_g = \kappa\cos\varphi,\ \kappa_n = \kappa\sin\varphi\ ,\ \tau_g = \tau + \frac{d\varphi}{ds}. \tag{2.3}$$

Furthermore, the geodesic curvature κ_g and geodesic torsion τ_g of the curve $\alpha(s)$ can be calculated as follows

$$\kappa_g = \left\langle \frac{d\alpha}{ds}, \frac{d^2\alpha}{ds^2} \times \mathbf{n} \right\rangle, \quad \tau_g = \left\langle \frac{d\alpha}{ds}, \mathbf{n} \times \frac{d\mathbf{n}}{ds} \right\rangle \tag{2.4}$$

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the followings are well-known

i) $\alpha(s)$ is a geodesic curve $\Leftrightarrow \kappa_g = 0$,

ii) $\alpha(s)$ is an asymptotic line $\Leftrightarrow \kappa_n = 0$,

iii) $\alpha(s)$ is a principal lineprincipal line $\Leftrightarrow \tau_g = 0$, [7].

Let α and β be two curves in the Euclidean space E^3. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}^*\}$ be Frenet frames of α and β, respectively. Then the curve β is called the involute of the curve α, if the tangent vector of the curve α at the points $\alpha(s)$ passes through the tangent vector of the curve β at the point $\beta(s)$ and

$$\langle \mathbf{T}, \mathbf{T}^* \rangle = 0.$$

Also, the curve α is called the evolute of the curve β.The pair $\{\alpha, \beta\}$ is said to be a special involute-evolute pair.

3 Special Involute-Evolute Partner D -Curves in E^3

In this section, by considering the Darboux frame, we define involute evolute partner D-curves and give the characterizations of these curves.

Definition 1. Let M and N be oriented surfaces in three dimensional Euclidean space E^3 and the arc length parameter curves $\alpha(s)$ and $\beta(s^*)$ lying fully on M and N, respectively. Denote the Darboux frames of $\alpha(s)$ and $\beta(s^*)$ by $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$, respectively. If there exists a corresponding relationship between the curves α and β such that, at the corresponding points of the curves, the Darboux frame element $\mathbf{T}$ of α coincides with the Darboux frame element $\mathbf{g}^*$ of β, then α is called a special evolute D-curve of β and β is a special involute D -curve of α. Then, the pair $\{\alpha, \beta\}$ is said to be a special involute evolute D -pair.

Theorem 1. Let $\alpha(s)$ and $\beta(s^*)$ be two curves in the Euclidean space E^3. If the pair $\{\alpha, \beta\}$ is a special involute-evolute D-pair, then

$$\beta(s) = \alpha(s) + (c - s)\,\mathbf{T}(s).$$

Proof. Suppose that the pair $\{\alpha, \beta\}$ is a special involute evolute D-pair. From definition of special involute-evolute D-pair, we know

$$\beta(s)=\alpha(s)+\lambda(s)\,\mathbf{T}(s)\,. \tag{3.1}$$

Differentiating both sides of the equation (3.1) with respect to s and use the Darboux formulas, we obtain

$$\mathbf{T}^*(s^*)\,\frac{ds^*}{ds}=\mathbf{T}(s)+\dot{\lambda}(s)\mathbf{T}(s)+\kappa_g(s)\,\lambda(s)\,\mathbf{g}(s)+\kappa_n(s)\,\lambda(s)\,\mathbf{n}(s)\,.$$

Since the direction of $\mathbf{T}$ coincides with the direction of $\mathbf{g}^*$, we get

$$\dot{\lambda}(s)=-1 \tag{3.2}$$

and

$$\lambda(s)=c-s \tag{3.3}$$

where is c constant. Thus, the equality (3.1) can be written as follows

$$\beta(s^*)=\alpha(s)+(c-s)\,\mathbf{T}(s)\,. \tag{3.4}$$

Corollary 1. Let $\alpha(s)$ and $\beta(s^*)$ be two curves in the Euclidean space E^3. If the pair $\{\alpha,\beta\}$ is a special involute-evolute D-pair, then the distance between the curves $\alpha(s)$ and $\beta(s^*)$ is constant.

Theorem 2. Let M and N be oriented surfaces in three dimensional Euclidean space E^3 and the arc length parameter curves $\alpha(s)$ and $\beta(s^*)$ lying fully on M and N , respectively. $\beta(s^*)$ is special involute D-curve of $\alpha(s)$ if and only if the normal curvature κ_n^* of $\beta(s^*)$ and the geodesic curvature κ_g , the normal curvature κ_n and the geodesic torsion τ_g of $\alpha(s)$ satisfy the following equation

$$\dot{\kappa_n}=\left(\frac{\kappa_n^2+\kappa_g^2}{\kappa_g}\right)\left(\frac{\lambda\kappa_n^*\kappa_g}{\cos\theta}-\tau_g\right)+\frac{\dot{\kappa_g}\kappa_n}{\kappa_g}.$$

for some nonzero constants λ, where θ is the angle between the vectors $\mathbf{n}$ and $\mathbf{n}^*$ at the corresponding points of $\alpha(s)$ and $\beta(s^*)$.

Proof. Suppose that M and N are oriented surfaces in three dimensional Euclidean space E^3 and the arc length parameter curves $\alpha(s)$ and $\beta(s^*)$ lying fully on M and N, respectively. Denote the Darboux frames of $\alpha(s)$ and $\beta(s^*)$ by $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$ and $\{\mathbf{T}^*,\mathbf{g}^*,\mathbf{n}^*\}$, respectively. Then by the definition we can assume that

$$\beta(s)=\alpha(s)+\lambda(s)\,\mathbf{T}(s) \tag{3.5}$$

for some function $\lambda(s)$. By taking derivative of (3.5) with respect to s and applying the Darboux formulas (2.2) we have

$$\mathbf{T}^* \frac{ds^*}{ds} = \left(1+\dot{\lambda}\right)\mathbf{T} + \lambda\kappa_g \mathbf{g} + \lambda\kappa_n \mathbf{n} \tag{3.6}$$

From (3.2) we get

$$\mathbf{T}^* \frac{ds^*}{ds} = \lambda\kappa_g \mathbf{g} + \lambda\kappa_n \mathbf{n}. \tag{3.7}$$

On the other hand we have

$$\mathbf{T}^* = \cos\theta \mathbf{g} - \sin\boldsymbol{\theta}\mathbf{n}. \tag{3.8}$$

Differentiating (3.8) with respect to s, we obtain

$$\left(\kappa_g^* \mathbf{g}^* + \kappa_n^* \mathbf{n}^*\right)\frac{ds^*}{ds} = \left(\kappa_g \cos\theta - \boldsymbol{\kappa}_\mathbf{n} \sin\boldsymbol{\theta}\right)\mathbf{T} + \left(\tau_g - \dot{\theta}\right)\sin\boldsymbol{\theta}\mathbf{g} + \left(\tau_g - \dot{\theta}\right)\cos\boldsymbol{\theta}\mathbf{n}$$

From the last equation and the fact that

$$\mathbf{n}^* = \sin\theta\mathbf{g} + \cos\boldsymbol{\theta}\mathbf{n}$$

we have

$$\left(\kappa_g^* \mathbf{g}^* + \kappa_n^* \sin\boldsymbol{\theta}\mathbf{g} + \boldsymbol{\kappa}_\mathbf{n}^* \cos\boldsymbol{\theta}\mathbf{n}\right)\frac{ds^*}{ds} = \left(\boldsymbol{\kappa}_\mathbf{n} \sin\boldsymbol{\theta} - \boldsymbol{\kappa}_\mathbf{g} \cos\boldsymbol{\theta}\right)\mathbf{T} + \left(\tau_g - \dot{\theta}\right)\sin\boldsymbol{\theta}\mathbf{g} + \left(\tau_g - \dot{\theta}\right)\cos\boldsymbol{\theta}\mathbf{n}.$$

Since the direction of $\mathbf{T}$ is coincident with $\mathbf{g}^*$ we have

$$\dot{\theta} = \tau_g - \boldsymbol{\kappa}_\mathbf{n}^* \frac{ds^*}{ds}. \tag{3.9}$$

From (3.6) and (3.8) we obtain

$$\frac{ds^*}{ds} = \frac{\lambda\kappa_g}{\cos\theta} = -\frac{\lambda\kappa_n}{\sin\theta} \tag{3.10}$$

and

$$-\lambda\kappa_n = \lambda\kappa_g \tan\theta \tag{3.11}$$

By taking the derivative of this equation and applying (3.9) we get

$$\dot{\kappa_n} = \left(\frac{\kappa_n^2 + \kappa_g^2}{\kappa_g}\right)\left(\frac{\lambda\kappa_n^*\kappa_g}{\cos\theta} - \tau_g\right) + \frac{\dot{\kappa_g}\kappa_n}{\kappa_g}. \tag{3.12}$$

that is desired.

Conversely, assume that the equation (3.12) holds for some nonzero constants λ. Then by using (3.10), (3.11) and (3.12) gives us

$$\boldsymbol{\kappa}_\mathbf{n}^* \left(\frac{ds^*}{ds}\right)^3 = \lambda^2 \dot{\kappa_n}\kappa_g - \lambda^2 \dot{\kappa_g}\kappa_n + \lambda^2\left(\kappa_n^2 + \kappa_g^2\right)\tau_g \tag{3.13}$$

Let define a curve

$$\beta(s) = \alpha(s) + \lambda(s)\,\mathbf{T}(s).$$

By taking the derivative of the last equation with respect to s twice, we get

$$\mathbf{T}^* \frac{ds^*}{ds} = \lambda\kappa_g \mathbf{g} + \lambda\kappa_n \mathbf{n} \tag{3.14}$$

and

$$\left(\kappa_g^* \mathbf{g}^* + \kappa_n^* \mathbf{n}^*\right)\left(\frac{ds^*}{ds}\right)^2 + \mathbf{T}^* \frac{d^2 s^*}{ds^2} = -\lambda\left(\kappa_n^2 + \kappa_g^2\right)\mathbf{T} + \left(\lambda\dot{\kappa_g} - \kappa_g - \lambda\kappa_n\tau_g\right)\mathbf{g} + \left(\boldsymbol{\lambda}\dot{\boldsymbol{\kappa}_{\mathbf{n}}} - \boldsymbol{\kappa}_{\mathbf{n}} - \boldsymbol{\lambda}\boldsymbol{\kappa}_{\mathbf{g}}\boldsymbol{\tau}_{\mathbf{g}}\right)\mathbf{n} \tag{3.15}$$

respectively. Taking the cross product of (3.14) with (3.15) we have

$$\left(\kappa_g^* \mathbf{n}^* - \kappa_n^*\ \mathbf{g}^*\right)\left(\frac{ds^*}{ds}\right)^2 = \left[\lambda^2\left(\kappa_g\dot{\kappa_n} - \kappa_n\dot{\kappa_g} + \kappa_g^2\tau_g + \kappa_n^2\tau_g\right)\right]\mathbf{T} - \lambda^2\left(\kappa_n^3 + \kappa_n\kappa_g^2\right)\mathbf{g} + \lambda^2\left(\kappa_g^3 + \kappa_g\kappa_n^2\right)\mathbf{n}. \tag{3.16}$$

By substituting (3.13) in (3.16) we get

$$\left(\kappa_g^* \mathbf{n}^* - \boldsymbol{\kappa}_{\mathbf{n}}^* \mathbf{g}^*\right)\left(\frac{ds^*}{ds}\right)^3 = -\kappa_n^*\left(\frac{ds^*}{ds}\right)^3 \mathbf{T} - \lambda^2\left(\kappa_n^3 + \kappa_n\kappa_g^2\right)\mathbf{g} + \lambda^2\left(\kappa_g^3 + \kappa_g\kappa_n^2\right)\mathbf{n}. \tag{3.17}$$

Taking the cross product of (3.14) with (3.17) we have

$$\left(-\kappa_n^* \mathbf{n}^* - \boldsymbol{\kappa}_{\mathbf{g}}^* \mathbf{g}^*\right)\left(\frac{ds^*}{ds}\right)^4 = -\lambda^3\left(\kappa_n^2 + \kappa_g^2\right)\mathbf{T} + \lambda\kappa_n\kappa_n^*\left(\frac{ds^*}{ds}\right)^3 \mathbf{g} - \lambda\kappa_g\kappa_n^*\left(\frac{ds^*}{ds}\right)^3 \mathbf{n}. \tag{3.18}$$

From (3.17) and (3.18) we have

$$\begin{aligned} -\left(\kappa_n^{*2} + \boldsymbol{\kappa}_{\mathbf{g}}^{*\mathbf{2}}\right)\left(\frac{ds^*}{ds}\right)^4 \mathbf{n}^* \quad = \quad & \left[-\kappa_n^* \boldsymbol{\kappa}_{\mathbf{g}}^* \left(\frac{ds^*}{ds}\right)^4 + \lambda^3\kappa_n^*\left(\kappa_n^2 + \kappa_g^2\right)^2\right]\mathbf{T} + \\ & \kappa_n\left\{\left[\lambda^2\kappa_g^*\left(\kappa_n^2 + \kappa_g^2\right)\right]\left(\frac{ds^*}{ds}\right) + \lambda\kappa_n^{*2}\left(\frac{ds^*}{ds}\right)^3\right\}\mathbf{g} - \\ & \kappa_g\left\{\left[\lambda^2\kappa_g^*\left(\kappa_n^2 + \kappa_g^2\right)\right]\left(\frac{ds^*}{ds}\right) + \lambda\kappa_n^{*2}\left(\frac{ds^*}{ds}\right)^3\right\}\mathbf{n}. \end{aligned} \tag{3.19}$$

Furthermore, from (3.14) and (3.17) we get

$$\begin{cases} \left(\frac{ds^*}{ds}\right)^2 = \lambda^2\left(\kappa_n^2 + \kappa_g^2\right) \\ \boldsymbol{\kappa}_{\mathbf{g}}^{*\mathbf{2}}\left(\frac{ds^*}{ds}\right)^2 = \left(\kappa_n^2 + \kappa_g^2\right) \end{cases} \tag{3.20}$$

respectively. Substituting (3.20) in (3.19) we obtain

$$\begin{aligned} -\left(\kappa_n^{*2} + \boldsymbol{\kappa}_{\mathbf{g}}^{*\mathbf{2}}\right)\left(\frac{ds^*}{ds}\right)^4 \mathbf{n}^* \quad = \quad & \kappa_n\left\{\left[\lambda^2\kappa_g^*\left(\kappa_n^2 + \kappa_g^2\right)\right]\left(\frac{ds^*}{ds}\right) + \lambda\kappa_n^{*2}\left(\frac{ds^*}{ds}\right)^3\right\}\mathbf{g} + \\ & \kappa_g\left\{\left[\lambda^2\kappa_g^*\left(\kappa_n^2 + \kappa_g^2\right)\right]\left(\frac{ds^*}{ds}\right) + \lambda\kappa_n^{*2}\left(\frac{ds^*}{ds}\right)^3\right\}\mathbf{n}. \end{aligned} \tag{3.21}$$

Equality (3.14) and (3.21) shows that the vectors $\mathbf{T}^*$ and $\mathbf{n}^*$ lie on the plane $\mathrm{Sp}\{\mathbf{g},\mathbf{n}\}$. So, at the corresponding points of the curves, the Darboux frame element $\mathbf{T}$ of α coincides with the Darboux frame element $\mathbf{g}^*$ of β . Thus, the proof is completed.

Special Case 1. Let $\beta(s^*)$ be an asymptotic special involute D-curve of α.

i) Consider that $\alpha(s)$ is an asymptotic line. Then $\alpha(s)$ is special evolute D-curve of $\beta(s^*)$ if and only if the geodesic curvature κ_g , the geodesic normal κ_n and the geodesic torsion τ_g of $\alpha(s)$ satisfy the following equation,

$$\dot{\kappa_n} = -\tau_g \kappa_g.$$

ii) Consider that $\alpha(s)$ is a principal line. Then $\alpha(s)$ is special evolute D-curve of $\beta(s^*)$ if and only if he geodesic curvature κ_g and the geodesic normal κ_n of $\alpha(s)$ satisfy the following equation,

$$\dot{\kappa_n} = \frac{\kappa_n \dot{\kappa_g}}{\kappa_g}.$$

Theorem 3. Let the pair $\{\alpha,\beta\}$ be a special involute evolute D-pair in the Euclidean space E^3.Then the relation between the geodesic curvature κ_g^* and the geodesic torsion τ_g^* of $\beta(s^*)$ is given as follows

$$\kappa_g^* + \tau_g^* = -\frac{1}{\lambda}$$

for some nonzero constants λ, where θ is the angle between the vectors $\mathbf{n}$ and $\mathbf{n}^*$ at the corresponding points of $\alpha(s)$ and $\beta(s^*)$.

Proof. Let the pair $\{\alpha,\beta\}$ be a special involute-evolute D-pair in the Euclidean space E^3.Then from (3.5) we can write

$$\beta(s) = \alpha(s) + \lambda(s)\,\mathbf{T}(s)$$

for some constants λ. The last equation is written as follows

$$\alpha(s) = \beta(s) - \lambda(s)\,\mathbf{T}(s).$$

Since the direction of $\mathbf{T}$ is coincident with $\mathbf{g}^*$ we have

$$\alpha(s) = \beta(s) - \lambda(s)\,\mathbf{g}^*(s). \tag{3.22}$$

By differentiating (3.22) with respect to s and since the direction of $\mathbf{T}$ is coincident with $\mathbf{g}^*$ we have

$$\kappa_g^* + \tau_g^* = -\frac{1}{\lambda}.$$

Special Case 2. Let the pair $\{\alpha,\beta\}$ be a special involute-evolute D-pair in the Euclidean space E^3.

i) If β is geodesic curve, then

$$\tau_g^* = -\frac{1}{\lambda}.$$

ii) If β is principal line, then

$$\kappa_g^* = -\frac{1}{\lambda}.$$

Theorem 4. Let the pair $\{\alpha,\beta\}$ be a special involute-evolute D-pair in the Euclidean space E^3. Then the following relations hold:

i) $\kappa_n^* = \tau_g \frac{ds}{ds^*} - \frac{d\theta}{ds^*}$

ii) $\kappa_g \frac{ds}{ds^*} = -\kappa_g^* \cos\theta + \tau_g^* \sin\theta$

iii) $\kappa_n \frac{ds}{ds^*} = \kappa_g^* \sin\theta + \tau_g^* \cos\theta$

iv) $\kappa_g^* = (\kappa_n \sin\theta - \kappa_g \cos\theta) \frac{ds}{ds^*}$

Proof.

i) By differentiating the equation $\langle \mathbf{n}, \mathbf{n}^* \rangle = \cos\theta$ with respect to s^* we have

$$\left\langle (-\kappa_n \mathbf{T} - \tau_g \mathbf{g}) \frac{ds}{ds^*}, \mathbf{n}^* \right\rangle + \langle \mathbf{n}, -\boldsymbol{\kappa}_{\mathbf{n}}^* \mathbf{T}^* - \boldsymbol{\tau}_{\mathbf{g}}^* \mathbf{g}^* \rangle = -\sin\theta \frac{d\theta}{ds^*}$$

Using the fact that the direction of $\mathbf{T}$ coincides with the direction of $\mathbf{g}^*$ and

$$\begin{aligned} \mathbf{T}^* &= \cos\theta \mathbf{g} - \sin\theta \mathbf{n} \\ \mathbf{g}^* &= \sin\theta \mathbf{g} + \cos\theta \mathbf{n} \end{aligned}$$

we easily get that

$$\kappa_n^* = \tau_g \frac{ds}{ds^*} - \frac{d\theta}{ds^*}$$

Similarly, other choices are testified.

Theorem 5. Let the pair $\{\alpha,\beta\}$ be a special involute-evolute D-pair in the Euclidean space E^3. Then geodesic curvature κ_g^* of $\beta(s^*)$ is

$$\kappa_g^* = \lambda^2 \left(\kappa_n^2 - \kappa_g^2\right) \left(\frac{ds}{ds^*}\right)^3 (\kappa_g \cos\theta + \kappa_n \sin\theta)$$

where θ is the angle between the vectors $\mathbf{n}$ and $\mathbf{n}^*$ at the corresponding points of $\alpha(s)$ and $\beta(s^*)$.

Proof. Suppose that the pair $\{\alpha, \beta\}$ is a special involute-evolute D-pair in the Euclidean 3 space E^3. From the first equation of (2.4) and by using the fact that $\mathbf{T}$ is coincident with $\mathbf{g}^*$ we have

$$\begin{aligned}\kappa_g^* &= \left\langle \frac{d\beta}{ds^*}, \frac{d^2\beta}{ds^{*2}} \times \mathbf{n}^* \right\rangle \\ &= \lambda^2 \left(\kappa_n^2 - \kappa_g^2\right) \left(\frac{ds}{ds^*}\right)^3 \left(\kappa_g \cos\theta + \kappa_n \sin\theta\right).\end{aligned}$$

Special Case 3. Let the pair $\{\alpha, \beta\}$ be a special involute-evolute D-pair in the Euclidean space E^3.

i) If α is a geodesic curve, then the geodesic curvature κ_g^* of $\beta(s^*)$ is

$$\kappa_g^* = \lambda^2 \kappa_n^3 \left(\frac{ds}{ds^*}\right)^3 \sin\theta.$$

ii) If α is an asymptotic line, then the geodesic curvature κ_g^* of $\beta(s^*)$ is

$$\kappa_g^* = \lambda^2 \kappa_g^3 \left(\frac{ds}{ds^*}\right)^3 \cos\theta.$$

Theorem 6. Let the pair $\{\alpha, \beta\}$ be a special involute evolute D -pair in the Euclidean space E^3.Then geodesic curvature τ_g^* of $\beta(s^*)$ is

$$\tau_g^* = -\lambda \sin\theta \cos\theta \left(\kappa_n^2 + \kappa_g^2\right) \left(\frac{ds}{ds^*}\right)^2 - \lambda \kappa_n \kappa_g \left(\frac{ds}{ds^*}\right)^2.$$

where θ is the angle between the vectors $\mathbf{n}$ and $\mathbf{n}^*$ at the corresponding points of $\alpha(s)$ and $\beta(s^*)$.

Proof. Suppose that the pair $\{\alpha, \beta\}$ is a special involute-evolute D-pair in the Euclidean space E^3. From the first equation of (2.4) and by using the fact that $\mathbf{T}$ is coincident with $\mathbf{g}^*$ we have

$$\begin{aligned}\tau_g^* &= \left\langle \frac{d\beta}{ds^*}, \mathbf{n}^* \times \frac{d\mathbf{n}^*}{ds^*} \right\rangle \\ &= -\lambda \sin\theta \cos\theta \left(\kappa_n^2 + \kappa_g^2\right) \left(\frac{ds}{ds^*}\right)^2 - \lambda \kappa_n \kappa_g \left(\frac{ds}{ds^*}\right)^2.\end{aligned}$$

Corollary 2. Let the pair $\{\alpha, \beta\}$ be a special involute-evolute D-pair in the Euclidean space E^3.

i) If α is a geodesic curve, then the geodesic curvature τ_g^* of $\beta(s^*)$ is

$$\tau_g^* = -\lambda \sin\theta \cos\theta \kappa_n^2 \left(\frac{ds}{ds^*}\right)^2 .$$

ii) If α is an asymptotic line, then the geodesic curvature τ_g^* of $\beta(s^*)$ is

$$\tau_g^* = -\lambda \sin\theta \cos\theta \kappa_g^2 \left(\frac{ds}{ds^*}\right)^2 .$$

Example 1. Let $\alpha(s) = \left(\sin s, \cos s, \sin^3 s - 3\sin s\cos^2 s\right)$ be a curve. This curve lies on the surface $z = x^3 - 3xy^2$ (monkey saddle). The special involute D-curve of the curve $\alpha(s)$ can be given below $\beta(s) = \left(\sin s + (c-s)\cos s, \cos s + (s-c)\sin s, \sin^3 s - 3\sin s\cos^2 s + (1-s)(9\sin^2 s\cos s - 3\cos^3 s)\right)$, $c \in \mathbb{R}$. For specially, $c = 1$ and $s \in [0, 2\pi]$, we can draw special involute-evolute D-pair $\{\alpha, \beta\}$ with helping the programme of Mapple 12 as follow,

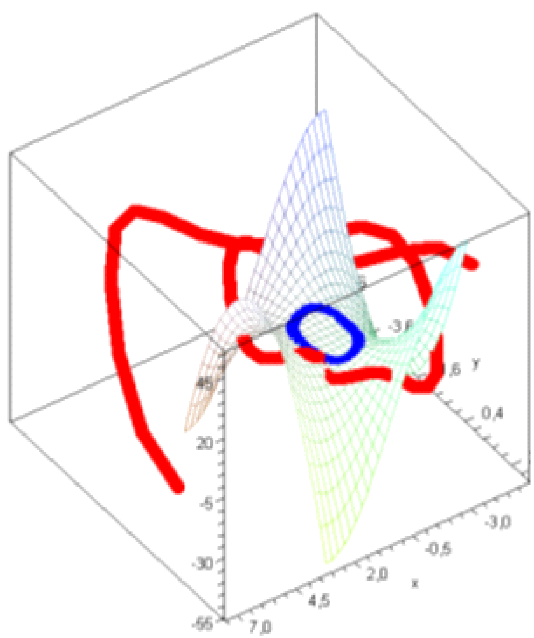

Figure 1. Special Involute-Evolute Partner D-Curves

Example 2. Let $\alpha(s) = \left(s\sin s, s\cos s, s^2\right)$ be a curve. This curve lies on the surface $z = x^2 + y^2$. The special involute D-curve of the curve $\alpha(s)$ can be given below $\beta(s) = \left(s\sin s + (c-s)(\sin s + s\cos s), s\cos s + (c-s)(\cos s - s\sin s), s^2 + 2(c-s)s\right)$, $c \in \mathbb{R}$. This curve lies on the surface $z = -\sqrt{x^2+y^2}$ .For specially, $c = 0$ and $s \in [0, \frac{3}{2}\pi]$, we can draw special involute-evolute D-pair $\{\alpha, \beta\}$with helping the programme of Mapple 12 as follow,

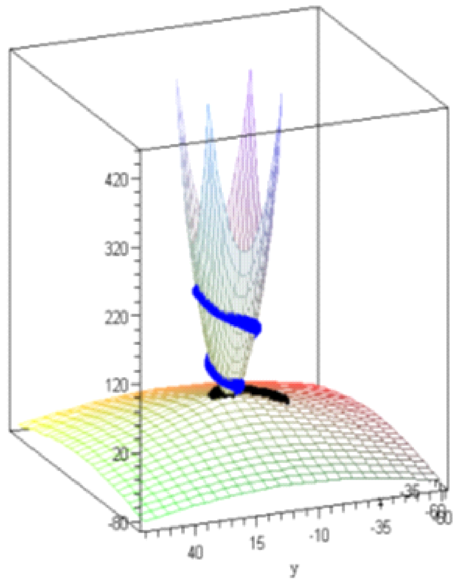

Figure 2. Special Involute-Evolute Partner D-Curves

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