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Existence Results for Generalized Vector Equilibrium Problems on Unbounded Sets

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Abstract. We provide existence results for generalized (set-valued) vector equilibrium problems on unbounded sets, based on a coercivity condition recently proposed for the scalar and vector cases. Several formulations of the generalized vector equilibrium problem are taken into account, thus covering most cases considered in the literature.
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1. Introduction

Many problems of practical interest in optimization, economics and engineering involve equilibrium in their description; this fact has motivated researchers to establish general results on the existence of solutions for equilibrium problems, see e.g. [2, 3]. Indeed there is a vast literature on equilibrium problems and their treatment in optimization, variational and quasivariational inequalities, and complementarity problems. Many authors investigated different equilibrium models, extending scalar equilibrium problems to the vector-valued and set-valued cases, see e.g. [4, 11–13].

In the case of a set-valued bifunction, the general equilibrium problem can be formulated in several (non equivalent) ways. Our results will cover the following cases. Given topological vector spaces X, Y, a nonempty, closed and convex set $K \subseteq X$, a set-valued mapping $V : K \to \Pi(Y)$ that maps each $x \in K$ to a nonempty set $V(x) \subseteq Y$ ($\Pi(Y)$ denotes the power set of Y) and a set-valued bifunction $\Phi : K \times K \to \Pi(Y)$, we consider two main formulations

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of the generalized vector equilibrium problem, namely the problems of finding $\overline{x} \in K$ such that

(GVEP1) $\Phi(\overline{x}, y) \not\subseteq V(\overline{x}), \quad \forall y \in K;$

(GVEP2) $\Phi(\overline{x}, y) \cap V(\overline{x}) = \emptyset, \quad \forall y \in K.$

In particular, given a set-valued mapping $C : K \to \Pi(Y)$ that maps each $x \in K$ to a pointed (and solid, if necessary) cone $C(x) \subseteq Y$, we can recover the main formulations of the generalized vector equilibrium problem that have been considered in the literature [1, 6, 9–13], namely,

(GVEP1a) $\Phi(\overline{x}, y) \not\subseteq -\text{int } C(\overline{x}), \quad \forall y \in K;$

(GVEP1b) $\Phi(\overline{x}, y) \not\subseteq -C(\overline{x}) \backslash \{0\}, \quad \forall y \in K;$

(GVEP2a) $\Phi(\overline{x}, y) \cap -\text{int } C(\overline{x}) = \emptyset, \quad \forall y \in K;$

(GVEP2b) $\Phi(\overline{x}, y) \cap (-C(\overline{x}) \backslash \{0\}) = \emptyset, \quad \forall y \in K.$

For $i = 1, 2$, or, more specifically, $i = 1a, 1b, 2a, 2b$, we will denote by S_K^i the set of solutions to problem (GVEPi).

Remark 1. *For convenience, we will also denote by S_H^i the set of solutions to problem (GVEPi), with K replaced by a nonempty, closed and convex set $H \subseteq K$ and with Φ and C being replaced by their restrictions to $H \times H$ and H, respectively.*

Note that, if $C(x)$ has nonempty interior for all $x \in K$, then the following inclusions hold

$$\begin{array}{ccc} S_K^{2b} & \subseteq & S_K^{1b} \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ S_K^{2a} & \subseteq & S_K^{1a} \end{array}$$

Anyway, in infinite dimensional spaces the condition int $C(x) \neq \emptyset$ can be a restrictive assumption (for instance, in the Lebesgue space L^2, the cone of functions that are nonnegative almost everywhere is closed and convex, but with empty interior). This justifies our interest in problems (GVEP1b) and (GVEP2b), as they may have a solution even in this case.

In most of the papers on the existence of solutions of GVEPs, either boundedness of the feasible set or a certain coercivity condition is assumed. The purpose of the present paper is to provide some existence theorems concerning solutions of generalized vector equilibrium problems on an unbounded set with set-valued maps defined on reflexive Banach spaces, exploiting a new coercivity condition, which was introduced in [7] for scalar bifunctions and in [8] for vector-valued functions.

In [7], in particular, it is shown how several coercivity conditions proposed in the literature are stronger than this new condition, in the sense that, if the former hold, then the latter holds as well. Thus, employing this weak coercivity condition can yield more general results in the field of equilibrium problems.

2. Preliminaries

In this paper we will consider partial orderings on vector spaces induced by cones. We agree that any cone contains the origin, according to the following definition.

Definition 1. *Let X be a vector space and $C \subseteq X$ be nonempty. C is a* cone *if, for all $k \in C$ and $\lambda \geq 0$, $\lambda k \in C$.*

If $-C \cap C = \{0\}$, then the cone C is *pointed*, while it is *solid* if $\operatorname{int} C \neq \emptyset$.

In the next section we will need existence results for generalized vector equilibrium problems defined on compact sets. These results can be based on the following lemma, which is an easy adaptation of [5, Lemma 4], a consequence of the well-known Ky Fan's lemma [5, Lemma 1]. In the following, given a set S, we denote by $\Pi(S)$ the power set of S.

Lemma 1. *Let K be a nonempty compact convex set in a topological vector space and let $A : K \to \Pi(K \times K)$ be such that:*

(i) $(x, x) \notin A(x)$ for all $x \in K$;

(ii) for any fixed $x \in K$, the set $\{y \in K : (x, y) \in A(x)\}$ is convex;

(iii) for any fixed $y \in K$, the set $\{x \in K : (x, y) \notin A(x)\}$ is closed.

Then there exists a point $\overline{x} \in K$ such that $(\overline{x}, y) \notin A(\overline{x})$ for all $y \in K$.

Proof. By *(iii)*, for any fixed $y \in K$ the set $F(y) = \{x \in K : (x, y) \notin A(x)\}$ is closed in K, hence compact. Moreover, F is a KKM-map, i.e. such that the convex hull of any finite subset $\{y_1, \cdots, y_n\}$ of K is contained in $\bigcup_{i=1}^n F(y_i)$. Indeed, suppose by contradiction that there exist $(\lambda_1, \cdots, \lambda_n) \in [0, 1]^n$ such that $\lambda_1 + \cdots + \lambda_n = 1$ and

$$\left(\sum_{j=1}^n \lambda_j y_j, y_i \right) \in A \left(\sum_{j=1}^n \lambda_j y_j \right), \quad \forall i = 1, \cdots, n,$$

i.e. $y_i \in \{y \in K : (\sum_{j=1}^n \lambda_j y_j, y) \in A(\sum_{j=1}^n \lambda_j y_j)\}$. Then, by *(ii)*, $\sum_{j=1}^n \lambda_j y_j$ belongs to the same set, so that

$$\left(\sum_{j=1}^n \lambda_j y_j, \sum_{j=1}^n \lambda_j y_j \right) \in A \left(\sum_{j=1}^n \lambda_j y_j \right),$$

a contradiction to *(i)*.

Therefore, it follows from [5, Lemma 1] that

$$\bigcap_{y \in K} F(y) \neq \emptyset,$$

i.e., there exists $\overline{x} \in K$ such that $(\overline{x}, y) \notin A(\overline{x})$ for all $y \in K$. □

As an immediate consequence of the preceding lemma, one obtains the following existence results for generalized vector equilibrium problems (GVEP1) and (GVEP2) on a compact set K. The proofs are based on Lemma 1, with $A(x) = \Phi^+(V(x))$ and $A(x) = \Phi^-(V(x))$ for all $x \in K$, respectively, where the upper and lower inverse mappings of Φ are defined as

$$\Phi^+(Z) = \{(x,y) \in K \times K : \ \Phi(x,y) \subseteq Z\},$$

$$\Phi^-(Z) = \{(x,y) \in K \times K : \ \Phi(x,y) \cap Z \neq \emptyset\},$$

for all $Z \subseteq Y$.

Theorem 1. *Let X be a reflexive Banach space, Y be a Banach space, K be a nonempty, closed, convex and bounded subset of X and let $V : K \to \Pi(Y)$ map any $x \in K$ to a nonempty set $V(x) \subseteq Y$ such that $0 \notin V(x)$. Moreover, let $\Phi : K \times K \to \Pi(Y)$ be such that:*

- *(i)* $\Phi(x,x) = \{0\}$, *for all $x \in K$;*
- *(ii) for any fixed $x \in K$, the set $\{y \in K : \Phi(x,y) \subseteq V(x)\}$ is convex;*
- *(iii) for any fixed $y \in K$, the set $\{x \in K : \Phi(x,y) \nsubseteq V(x)\}$ is closed with respect to the topology induced on K by the weak topology of X.*

Then, $S_K^1 \neq \emptyset$.

Proof. Let $A : K \to \Pi(K \times K)$ be defined as $A(x) = \Phi^+(V(x))$ for all $x \in K$. By *(i)*, $\Phi(x,x) \nsubseteq V(x)$, i.e. $(x,x) \notin A(x)$. By *(ii)*, the set $\{y \in K : (x,y) \in A(x)\}$ is convex for all $x \in K$, while, by *(iii)*, $\{x \in K : (x,y) \notin A(x)\}$ is closed for all $y \in K$. Then, by Lemma 1, there exists $\overline{x} \in K$ such that $\Phi(\overline{x}, y) \nsubseteq V(\overline{x})$ for all $y \in K$. □

Theorem 2. *Let X be a reflexive Banach space, Y be a Banach space, K be a nonempty, closed, convex and bounded subset of X and let $V : K \to \Pi(Y)$ map any $x \in K$ to a set $V(x) \subseteq Y$ such that $0 \notin V(x)$. Moreover, let $\Phi : K \times K \to \Pi(Y)$ be such that:*

- *(i)* $\Phi(x,x) = \{0\}$, *for all $x \in K$;*
- *(ii) for any fixed $x \in K$, the set $\{y \in K : \Phi(x,y) \cap V(x) \neq \emptyset\}$ is convex;*
- *(iii) for any fixed $y \in K$, the set $\{x \in K : \Phi(x,y) \cap V(x) = \emptyset\}$ is closed with respect to the topology induced on K by the weak topology of X.*

Then, $S_K^2 \neq \emptyset$.

Proof. Let $A : K \to \Pi(K \times K)$ be defined as $A(x) = \Phi^-(V(x))$ for all $x \in K$. By *(i)*, $\Phi(x,x) \cap V(x) = \emptyset$, i.e. $(x,x) \notin A(x)$. By *(ii)*, the set $\{y \in K : (x,y) \in A(x)\}$ is convex for all $x \in K$, while, by *(iii)*, $\{x \in K : (x,y) \notin A(x)\}$ is closed for all $y \in K$. Then, by Lemma 1, there exists $\overline{x} \in K$ such that $\Phi(\overline{x}, y) \cap V(\overline{x}) = \emptyset$ for all $y \in K$. □

Remark 2. *Ky Fan's lemma [5, Lemma 1] is the standard tool to derive existence results for equilibrium problems on bounded sets (see e.g. [9] and [10]). While in [9] existence of solutions is obtained based on a duality approach and generalized monotonicity properties, adapting [5, Lemma 4] enables us to follow a more direct reasoning, that avoids both duality and monotonicity. Our approach in formulating the preceding theorems is more in the spirit of [10]. Anyway, [10] directly extends existence results for generalized vector equilibrium problems to the case in which the set K is unbounded, while we will pursue this task separately in the following sections by means of an apt coercivity condition.*

3. Existence Results for (GVEP1a) and (GVEP1b)

The main goal of this section is to prove an existence result for the generalized vector equilibrium problem (GVEP1a) on unbounded sets, conditional on available results for the existence of solutions to the same problem on closed, convex and bounded sets. This task is pursued in Theorem 3 below, which generalizes similar results of [7, 8] to set-valued bifunctions, while Corollary 1 combines this theorem with Theorem 1.

The main tool in our proof will be a new coercivity condition introduced in [7, 8] for scalar and vector equilibrium problems, which is weaker than standard coercivity conditions in the literature. Given a metric space X, $K \subseteq X$ and a function $\mu : X \to \mathbb{R}$, we will use the following notation for lower level sets of μ restricted to K. For any $r \in \mathbb{R}$,

$$W_r := \{x \in K : \mu(x) \leq r\} \text{ and } U_r := \{x \in K : \mu(x) < r\}.$$

Definition 2. *Let X be a set and $K \subseteq X$ be nonempty. A function $\mu : X \to \mathbb{R}$ is* weakly coercive *with respect to the set K if there exists $r \in \mathbb{R}$ such that W_r is nonempty and bounded.*

Remark 3. *If a function μ is lower semicontinuous and strongly convex, or it is coercive in the usual sense (i.e., $\mu(x) \to +\infty$ as $\|x\| \to +\infty$), then μ is weakly coercive with respect to any nonempty set. Moreover, if μ is convex and weakly coercive, then W_ϱ is bounded for each $\varrho \in \mathbb{R}$ [14, Chapter 3, Theorem 3.14].*

In Theorem 3, we will assume the following coercivity condition, which extends that of [7, 8] to the case of set-valued bifunctions.

(G1a) *There exist a convex and lower semicontinuous function $\mu : X \to \mathbb{R}$, which is weakly coercive with respect to the set K, and a number r such that for any point $\overline{x} \in K \backslash W_r$ with*

$$\Phi\left(\overline{x}, y\right) \not\subseteq -\text{int } C(\overline{x}), \quad \forall y \in W_r, \tag{1}$$

there is a point $z \in K$, $\mu(z) < \mu(\overline{x})$, such that $\Phi(\overline{x}, z) \subseteq -C(\overline{x})$.

Remark 4. *Notice that, when X is a reflexive Banach space, W_r in **(G1a)** must be nonempty. Indeed, otherwise we can take $\overline{x} \in K = K \setminus W_r$ such that, by Weierstrass' theorem (μ is convex and lower semicontinuous, hence lower semicontinuous with respect to the weak topology; moreover,* *since it is weakly coercive, there exists $\bar{r} \in \mathbb{R}$ such that $W_{\bar{r}}$ is nonempty and bounded, hence compact in the weak topology),*

$$\mu(\overline{x}) = \min_{x \in W_{\bar{r}}} \mu(x) = r' > r,$$

but then there exists $z \in K$, $\mu(z) < \mu(\overline{x}) = r'$, a contradiction.

Remark 5. *In what follows, we will assume that X and Y are Banach spaces and that X is reflexive (in order to easily fulfill the assumptions of compactness), though Lemma 2 and Theorem 3 could also be stated in a more general setting. Similarly, the assumptions on C could be weakened. A similar remark holds for the results presented in the following sections.*

In order to prove the main result of this section, we state the following lemma first.

Lemma 2. *Let X be a reflexive Banach space, Y be a Banach space, $K \subseteq X$ be nonempty, closed and convex, $C : K \to \Pi(Y)$ be a set-valued mapping that maps any $x \in K$ to a convex, solid and pointed cone of Y and $\Phi : K \times K \to \Pi(Y)$ be such that*

C1) for all $x, y', y'' \in K$, if $\Phi(x, y') \subseteq -C(x)$ and $\Phi(x, y'') \subseteq -\text{int}\, C(x)$, then $\Phi(x, \alpha y' + (1-\alpha)y'') \subseteq -\text{int}\, C(x)$ for all $\alpha \in]0,1[$.

If there exist $\varrho \in \mathbb{R}$, $x^{\varrho} \in K$ such that

$$\Phi\left(x^{\varrho}, y\right) \nsubseteq -\text{int}\, C(x^{\varrho}), \quad \forall y \in W_{\varrho} \tag{2}$$

and $z \in U_{\varrho}$ such that $\Phi(x^{\varrho}, z) \subseteq -C(x^{\varrho})$, then $x^{\varrho} \in S_K^{1a}$.

Proof. Suppose by contradiction that there exists $y' \in K \setminus W_{\varrho}$ such that

$$\Phi\left(x^{\varrho}, y'\right) \subseteq -\text{int}\, C(x^{\varrho}).$$

Since K is a convex set, μ is a convex function and $z \in U_{\varrho}$, it is easy to prove that there exists $\hat{\alpha} \in]0,1[$ such that $y(\hat{\alpha}) := \hat{\alpha} z + (1-\hat{\alpha})y' \in W_{\varrho}$. Therefore, by hypothesis, we obtain

$$\Phi\left(x^{\varrho}, y(\hat{\alpha})\right) \subseteq -\text{int}\, C(x^{\varrho}),$$

a contradiction to (2). □

As we anticipated, the following theorem provides a general scheme to obtain existence results for the generalized vector equilibrium problem (GVEP1a) on an unbounded set, given coercivity condition **(G1a)** and any arbitrary existence result for problem (GVEP1a) on a closed, convex and bounded set.

Theorem 3. *Let X be a reflexive Banach space, Y be a Banach space, $K \subseteq X$ be nonempty, closed and convex, $C : K \to \Pi(Y)$ be a set-valued mapping that maps any $x \in K$ to a convex, solid and pointed cone of Y and $\Phi : K \times K \to \Pi(Y)$ be such that:*

(i) $\Phi(x, x) = \{0\}$, for all $x \in K$;

(ii) *for all* $x, y', y'' \in K$, *if* $\Phi(x, y') \subseteq -C(x)$ *and* $\Phi(x, y'') \subseteq -\text{int}\, C(x)$, *then* $\Phi(x, \alpha y' + (1-\alpha)y'') \subseteq -\text{int}\, C(x)$, *for all* $\alpha \in]0, 1[$.

Suppose that $S^{1a}_H \neq \emptyset$ *whenever* H *is a nonempty, closed, convex and bounded subset of* K. *If* ***(G1a)*** *holds, then* $S^{1a}_K \neq \emptyset$.

Proof. Let $r \in \mathbb{R}$ be as in **(G1a)** and take $\varrho > r$. By remarks 4 and 3, W_ϱ is nonempty and bounded; furthermore, it is closed and convex, since K is closed and convex and μ is lower semicontinuous and convex. Therefore, by hypothesis, there exists $x^\varrho \in S^{1a}_{W_\varrho}$, i.e. $x^\varrho \in W_\varrho$ satisfying (2). If $x^\varrho \in W_\varrho \backslash W_r$, then by **(G1a)** there exists $z \in K$ such that $\mu(z) < \mu(x^\varrho)$, i.e. $z \in U_\varrho$, and $\Phi(x^\varrho, z) \subseteq -C(x^\varrho)$. On the other hand, if $x^\varrho \in W_r$, setting $z := x^\varrho$, one obtains $\Phi(x^\varrho, z) = \{0\} \subseteq -C(x^\varrho)$. In both cases, the result follows from Lemma 2. □

The existence of solutions to problem (GVEP1a) on nonempty, bounded, closed and convex subsets of K is guaranteed, for instance, by Theorem 1.

Corollary 1. *Let* X *be a reflexive Banach space,* Y *be a Banach space,* K *be a nonempty, closed and convex subset of* X *and let* $C : K \to \Pi(Y)$ *map any* $x \in K$ *to a convex, solid and pointed cone* $C(x) \subseteq Y$. *Moreover, let* $\Phi : K \times K \to \Pi(Y)$ *be such that:*

(i) $\Phi(x, x) = \{0\}$, *for all* $x \in K$;

(ii) *for all* $x, y', y'' \in K$, *if* $\Phi(x, y') \subseteq -C(x)$ *and* $\Phi(x, y'') \subseteq -\text{int}\, C(x)$, *then* $\Phi(x, \alpha y' + (1-\alpha)y'') \subseteq -\text{int}\, C(x)$, *for all* $\alpha \in]0, 1[$;

(iii) *for any fixed* $y \in K$, *the set* $\{x \in K : \Phi(x, y) \nsubseteq -\text{int}\, C(x)\}$ *is closed with respect to the topology induced on* K *by the weak topology of* X.

If ***(G1a)*** *holds, then* $S^{1a}_K \neq \emptyset$.

Proof. Observe that assumption *(ii)* implies that for any fixed $x \in K$, the set $\{y \in K : \Phi(x, y) \subseteq -\text{int}\, C(x)\}$ is convex. Hence, for any nonempty, closed, convex and bounded set $H \subseteq K$, assumptions *(i)-(iii)* of Theorem 1 hold (with K replaced by H and $V(x) = -\text{int}\, C(x)$) for Φ restricted to $H \times H$. Then, by Theorem 1, $S^{1a}_H \neq \emptyset$. Hence, the conclusion follows from Theorem 3. □

Finally, reasoning on the same lines, one can prove analogous results for problem (GVEP1b). To this end, it basically suffices to substitute $C(\overline{x}) \backslash \{0\}$ for int $C(\overline{x})$ (in this case, C does not need to be solid).

For instance, Corollary 1 reads as in the following, where the coercivity assumption **(G1a)** is replaced by

(G1b) *There exist a convex lower semicontinuous function* $\mu : X \to \mathbb{R}$, *which is weakly coercive with respect to the set* K, *and a number* r *such that for any point* $\overline{x} \in K \backslash W_r$ *with*

$$\Phi(\overline{x}, y) \nsubseteq -C(\overline{x}) \backslash \{0\}, \quad \forall y \in W_r, \tag{3}$$

there is a point $z \in K$, $\mu(z) < \mu(\overline{x})$, *such that* $\Phi(\overline{x}, z) \subseteq -C(\overline{x})$.

Corollary 2. *Let X be a reflexive Banach space, Y be a Banach space, K be a nonempty, closed and convex subset of X and let $C : K \to \Pi(Y)$ map any $x \in K$ to a convex and pointed cone $C(x) \subseteq Y$. Moreover, let $\Phi : K \times K \to \Pi(Y)$ be such that:*

(i) $\Phi(x,x) = \{0\}$, *for all* $x \in K$;

(ii) for all $x, y', y'' \in K$, *if* $\Phi(x,y') \subseteq -C(x)$ *and* $\Phi(x,y'') \subseteq -C(x)\backslash\{0\}$, *then* $\Phi(x, \alpha y' + (1-\alpha)y'') \subseteq -C(x)\backslash\{0\}$, *for all* $\alpha \in]0,1[$;

(iii) for any fixed $y \in K$, *the set* $\{x \in K : \Phi(x,y) \nsubseteq -C(x)\backslash\{0\}\}$ *is closed with respect to the topology induced on* K *by the weak topology of* X.

*If **(G1b)** holds, then* $S_K^{1b} \neq \emptyset$.

4. Existence Results for (GVEP2a) and (GVEP2b)

The goal of this section is to briefly explain how to obtain existence results for problems (GVEP2a) and (GVEP2b) on unbounded sets, analogous to corollaries 1 and 2.

The proofs are on the same lines and we will explicitly propose only those related to problem (GVEP2a). In this case, the coercivity assumption is modified in the following natural way.

(G2a) *There exist a convex lower semicontinuous function $\mu : X \to \mathbb{R}$, which is weakly coercive with respect to the set K, and a number r such that for any point $\overline{x} \in K\backslash W_r$ with*

$$\Phi\left(\overline{x}, y\right) \cap -\mathrm{int}\, C(\overline{x}) = \emptyset, \quad \forall y \in W_r, \tag{4}$$

there is a point $z \in K$, $\mu(z) < \mu(\overline{x})$, such that $\Phi\left(\overline{x}, z\right) \cap -C(\overline{x}) \neq \emptyset$.

Again, as in **(G1a)** the set W_r in **(G2a)** is nonempty and bounded (see remarks 4 and 3).

Lemma 3. *Let X be a reflexive Banach space, Y be a Banach space, $K \subseteq X$ be nonempty, closed and convex, $C : K \to \Pi(Y)$ be a set-valued mapping that maps any $x \in K$ to a convex, solid and pointed cone of Y and $\Phi : K \times K \to \Pi(Y)$ be such that*

C2) for all $x, y', y'' \in K$ *if* $\Phi(x,y') \cap -C(x) \neq \emptyset$ *and* $\Phi(x,y'') \cap -\mathrm{int}\, C(x) \neq \emptyset$, *then* $\Phi(x, \alpha y' + (1-\alpha)y'') \cap -\mathrm{int}\, C(x) \neq \emptyset$ *for all* $\alpha \in]0,1[$.

If there exist $\varrho \in \mathbb{R}$, $x^{\varrho} \in K$ such that

$$\Phi\left(x^{\varrho}, y\right) \cap -\mathrm{int}\, C(x^{\varrho}) = \emptyset, \quad \forall y \in W_{\varrho} \tag{5}$$

and $z \in U_{\varrho}$ such that $\Phi\left(x^{\varrho}, z\right) \cap -C(x^{\varrho}) \neq \emptyset$, then $x^{\varrho} \in S_K^{2a}$.

Proof. Suppose by contradiction that there exists $y' \in K\backslash W_{\varrho}$ such that

$$\Phi\left(x^{\varrho}, y'\right) \cap -\mathrm{int}\, C(x^{\varrho}) \neq \emptyset.$$

Since K is a convex set, μ is a convex function and $z \in U_\varrho$, there exists $\hat{\alpha} \in]0,1[$ such that $y(\hat{\alpha}) := \hat{\alpha} z + (1-\hat{\alpha})y' \in W_\varrho$. Therefore, by hypothesis, we obtain

$$\Phi\left(x^\varrho, y(\hat{\alpha})\right) \cap -\mathrm{int}\ C(x^\varrho) \neq \emptyset,$$

a contradiction to (5). □

The following theorem provides a general scheme to obtain existence results for the generalized vector equilibrium problem (GVEP2a) on an unbounded set, given coercivity condition **(G2a)** and any arbitrary existence result for problem (GVEP2a) on a closed, convex and bounded set.

Theorem 4. *Let X be a reflexive Banach space, Y be a Banach space, $K \subseteq X$ be nonempty, closed and convex, $C : K \to \Pi(Y)$ be a set-valued mapping that maps any $x \in K$ to a convex, solid and pointed cone of Y and $\Phi : K \times K \to \Pi(Y)$ be such that:*

(i) $\Phi(x,x) = \{0\}$*, for all* $x \in K$*;*

(ii) for all $x, y', y'' \in K$*, if* $\Phi(x,y') \cap -C(x) \neq \emptyset$ *and* $\Phi(x,y'') \cap -\mathrm{int}\ C(x) \neq \emptyset$*, then* $\Phi(x, \alpha y' + (1-\alpha)y'') \cap -\mathrm{int}\ C(x) \neq \emptyset$*, for all* $\alpha \in]0,1[$.

Suppose that $S_H^{2a} \neq \emptyset$ whenever H is a nonempty, closed, convex and bounded subset of K. If ***(G2a)*** *holds, then $S_K^{2a} \neq \emptyset$.*

Proof. Let $r \in \mathbb{R}$ be as in **(G2a)** and take $\varrho > r$. W_ϱ is nonempty, bounded, closed and convex. Thus, by hypothesis, there exists $x^\varrho \in S_{W_\varrho}^{2a}$, i.e. $x^\varrho \in W_\varrho$ satisfying (5). If $x^\varrho \in W_\varrho \backslash W_r$, then by **(G2a)** there exists $z \in K$ such that $\mu(z) < \mu(x^\varrho)$, i.e. $z \in U_\varrho$, and $\Phi(x^\varrho, z) \cap -C(x^\varrho) \neq \emptyset$. On the other hand, if $x^\varrho \in W_r$, setting $z := x^\varrho$, one obtains $\Phi(x^\varrho, z) \cap -C(x^\varrho) = \{0\} \cap -C(x^\varrho) \neq \emptyset$. In both cases, the result follows from Lemma 3. □

The existence of solutions to problem (GVEP2a) on nonempty, bounded, closed and convex subsets of X is guaranteed, for instance, by Theorem 2.

Corollary 3. *Let X be a reflexive Banach space, Y be a Banach space, K be a nonempty, closed and convex subset of X and let $C : K \to \Pi(Y)$ map any $x \in K$ to a convex, solid and pointed cone $C(x) \subseteq Y$. Moreover, let $\Phi : K \times K \to \Pi(Y)$ be such that:*

(i) $\Phi(x,x) = \{0\}$*, for all* $x \in K$*;*

(ii) for all $x, y', y'' \in K$*, if* $\Phi(x,y') \cap -C(x) \neq \emptyset$ *and* $\Phi(x,y'') \cap -\mathrm{int}\ C(x) \neq \emptyset$*, then* $\Phi(x, \alpha y' + (1-\alpha)y'') \cap -\mathrm{int}\ C(x) \neq \emptyset$*, for all* $\alpha \in]0,1[$*;*

(iii) for any fixed $y \in K$*, the set* $\{x \in K : \Phi(x,y) \cap -\mathrm{int}\ C(x) = \emptyset\}$ *is closed with respect to the topology induced on K by the weak topology of X.*

If ***(G2a)*** *holds, then $S_K^{2a} \neq \emptyset$.*

Proof. Assumption *(ii)* implies that for any fixed $x \in K$, the set $\{y \in K : \Phi(x,y) \cap -\text{int } C(x) \neq \emptyset\}$ is convex. Hence, for any nonempty, closed, convex and bounded set $H \subseteq K$, assumptions *(i)-(iii)* of Theorem 2 hold (with K replaced by H and $V(x) = -\text{int } C(x)$) for Φ restricted to $H \times H$. Then, by Theorem 2, $S_H^{2a} \neq \emptyset$. Hence, the conclusion follows from Theorem 4. □

Remark 6. *With a reasoning analogous to that employed in this section, one can prove an existence result for problem (GVEP2b) as well. To this end, it suffices to replace condition C2 in Lemma 3 (and, consequently, condition (ii) of Theorem 4 and Corollary 3) by*

C3) for all $x, y', y'' \in K$, if $\Phi(x,y') \cap -C(x) \neq \emptyset$ and $\Phi(x,y'') \cap (-C(x)\backslash\{0\}) \neq \emptyset$, then $\Phi(x, \alpha y' + (1-\alpha)y'') \cap (-C(x)\backslash\{0\}) \neq \emptyset$ for all $\alpha \in]0,1[$,

*and to modify the coercivity condition **(G2a)** replacing* (4) *by*

$$\Phi\left(\overline{x}, y\right) \cap -C(\overline{x})\backslash\{0\} = \emptyset, \quad \forall y \in W_r. \tag{6}$$

5. Applications and Examples

As an application of the theoretical results obtained in the preceding section, one can consider the case in which the order structure in Y is lexicographic. In this case, Y is a finite dimensional vector space, which we will identify with $\mathbb{R}^n$, for simplicity, and the lexicographic order is defined by considering the cone

$$C_{lex} = \{0\} \cup \{x \in \mathbb{R}^n : \exists i \in I_n \; x_i > 0, \forall j < i \; x_j = 0\}, \tag{7}$$

where $I_n = \{1, \ldots, n\}$. Note that C_{lex} is convex, pointed and solid, but it is neither closed, nor open. Moreover, the order is total, since $-C_{lex} \cup C_{lex} = \mathbb{R}^n$.

Though C_{lex} is solid, in this setting it is worthwhile considering formulation (GVEP2b) of the generalized vector equilibrium problem, since, as lexicographic order is total, it coincides with the strong formulation of the problem, i.e., find

$$\overline{x} \in K \text{ such that } \Phi(\overline{x}, y) \subseteq C_{lex}, \quad \forall y \in K. \tag{8}$$

Corollary 2, with $Y = \mathbb{R}^n$ and $C(x)$ replaced by C_{lex} for all $x \in X$, then yields an existence result for the generalized vector equilibrium problem (GVEP2b) on unbounded sets.

Finally, we provide a simple numerical example instantiating the results presented in the previous sections.

Example 1. *Let $X = \mathbb{R}$, $K = [0, +\infty[$, Y be a Banach space, $C \subseteq Y$ be a convex, solid and pointed cone, and $C : K \to \Pi(Y)$ be the constant set-valued mapping defined as $C(x) = C$ for all $x \in K$. Then, the set-valued mapping $\Phi : K \times K \to \Pi(Y)$ defined as*

$$\Phi(x,y) = \begin{cases} -\text{int } C, & \text{if } x > y \\ \{0\}, & \text{if } x = y \\ \text{int } C, & \text{if } x < y, \end{cases}$$

for all $(x,y) \in K \times K$*, satisfies the assumptions of corollaries 1 and 3. We prove it for Corollary 1 only, given that the proof for Corollary 3 is similar. First of all, by definition of* Φ*,* $\Phi(x,x) = \{0\}$ *for all* $x \in [0,+\infty[$*. Next, given a fixed* $y \in [0,+\infty[$*, the set* $\{x \in [0,+\infty[: \Phi(x,y) \not\subseteq -\mathrm{int}\, C\}$ *is the interval* $[0,y]$*, which is closed. Finally, given arbitrary* $x, y', y'' \in [0,+\infty[$ *such that* $\Phi(x,y') \subseteq -C$ *and* $\Phi(x,y'') \subseteq -\mathrm{int}\, C$*, one has* $y' \in [0,x]$ *and* $y'' \in [0,x[$*. Then, for any* $\alpha \in]0,1[$*,* $\alpha y' + (1-\alpha)y''$ *belongs to* $[0,x[$ *and, as a consequence,* $\Phi(x, \alpha y' + (1-\alpha)y'') \subseteq -\mathrm{int}\, C$*.*

An analogous example, with C *a non solid cone and* int C *replaced by* $C \setminus \{0\}$*, can be provided for Corollary 2.*

As a particular case, when $Y = \mathbb{R}$ *and* $K = C = [0,+\infty[$*, we obtain the set-valued mapping* $\Phi : K \times K \to \Pi(\mathbb{R})$ *defined as*

$$\Phi(x,y) = \begin{cases}]-\infty, 0[, & \text{if } x > y \\ \{0\}, & \text{if } x = y \\ ]0, +\infty[, & \text{if } x < y, \end{cases}$$

for all $(x,y) \in K \times K$*, which satisfies the assumptions of corollaries 1, 2 and 3.*

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