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Skew Polynomial Rings over Weak σ-rigid Rings and $\sigma(*)$-rings

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Abstract. Let R be a ring and σ an endomorphism of R. Recall that R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R. We also recall that R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$, where $N(R)$ is the set of nilpotent elements of R.
In this paper we give a relation between a $\sigma(*)$-ring and a weak σ-rigid ring. We also give a necessary and sufficient condition for a Noetherian ring to be a weak σ-rigid ring. Let σ be an endomorphism of a ring R. Then σ can be extended to an endomorphism (say $\overline{\sigma}$) of $R[x;\sigma]$. With this we show that if R is a Noetherian ring and σ an automorphism of R, then R is a weak σ-rigid ring if and only if $R[x;\sigma]$ is a weak $\overline{\sigma}$-rigid ring.

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1. Introduction

A ring R always means an associative ring with identity $1 \neq 0$. The ring of integers is denoted by $\mathbb{Z}$, and the set of positive integers is denoted by $\mathbb{N}$. The set of prime ideals of R is denoted by $Spec(R)$. The sets of minimal prime ideals of R is denoted by $Min.Spec(R)$. The prime radical and the nil radical of R are denoted by $P(R)$ and $N(R)$ respectively.

Now let R be a ring and σ an endomorphism of R. Recall that the skew polynomial ring $R[x;\sigma]$ is the set of polynomials

$$\{\sum_{i=0}^{n} x^i a_i,\ a_i \in R,\ n \in \mathbb{N}\}$$

with usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a)$ for all $a \in R$. We take any $f(x) \in R[x;\sigma]$ to be of the form $f(x) = \sum_{i=0}^{n} x^i a_i$, $n \in \mathbb{N}$ as followed in McConnell and Robson [12]. We denote $R[x;\sigma]$ by $S(R)$.

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Skew-polynomial rings have been of interest to many authors. For example [1, 2, 5, 7, 10, 11, 13].

The classical study of any commutative Noetherian ring is done by studying its primary decomposition and this forms the fundamental edifice on which any such ring is studied. Further there are other structural properties of rings, for example the existence of quotient rings or more particularly the existence of Artinian quotient rings etc. which can be nicely tied to primary decomposition of a Noetherian ring. The notion of the quotient ring of a ring, the contractions and extensions of ideals arising thereby appear in Chapter 9 of [7].

The first important result in the theory of non commutative Noetherian rings was proved in 1958 (Goldie's Theorem) which gives an analogue of field of fractions for factor rings R/P, where R is a Noetherian ring and P is a prime ideal of R. In 1959 the one sided version was proved by Lesieur and Croisot [Theorem 5.12 of 7] and in 1960 Goldie generalized the result for semiprime rings [Theorem 5.10 of 7].

In [5] it is shown that if R is embeddable in a right Artinian ring and if characteristic of R is zero, then the differential operator ring $R[x;\delta]$ embeds in a right Artinian ring. It is also shown in [5] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew-polynomial ring $R[x;\sigma]$ embeds in an Artinian ring.

A non commutative analogue of associated prime ideals of a Noetherian ring has also been discussed. We would like to note that a considerable work has been done in the investigation of prime ideals (in particular minimal prime ideals and associated prime ideals) of skew polynomial rings (K. R. Goodearl and E. S. Letzter [8], C. Faith [6], S. Annin [1], Leroy and Matczuk [11], Nordstrom [13]) and Bhat [2].

Another related area of interest since recent past has been the study of 2-primal rings. This involves the notions of prime radical and the set of nilpotent elements of a ring. Furthermore the concept of completely prime ideals and the completely semiprime ideals are also studied in this area. Krempa in [9] introduced σ-rigid rings; Kwak in [10] introduced $\sigma(*)$-rings and Ouyang in [14] introduced weak σ-rigid rings, where σ is an endomorphism of ring R. These rings are related to 2-primal rings. In this paper we study these rings and find a relation between them. Towards this we prove the following theorem:

Let R be a ring. Let σ be an endomorphism of R such that R is a $\sigma(*)$-ring. Then R is a weak σ-rigid ring. Conversely a 2-primal weak σ-rigid ring is a $\sigma(*)$-ring. (This is proved in Theorem 2).

We also discuss skew polynomial rings over weak σ-rigid rings.

We note that if σ is an endomorphism of a ring R, then it can be extended to an endomorphism $\overline{\sigma}$ of $S(R) = R[x;\sigma]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. With this we prove the following theorem:

Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring if and only if $S(R) = R[x;\sigma]$ is a weak $\overline{\sigma}$-rigid ring. (This is proved in Theorem 3).

2. Preliminaries

We begin with the following definitions:

Definition 1 (Krempa [9]). *An endomorphism σ of a ring R is said to be rigid if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be σ-rigid if there exists a rigid endomorphism σ of R.*

Definition 2 (Kwak [10]). *Let R be a ring and σ an endomorphism of R. Then R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.*

Example 1 (Kwak [10]). *Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then P(R) $= \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.*

We note that the above ring is not σ-rigid. Let $0 \neq a \in F$. Then

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ but } \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

In [10], Kwak also establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. Recall that a ring R is 2-primal if $N(R) = P(R)$. Also an ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$. Clearly R is a I(*)-ring if and only if R is a 2-primal ring, where I is the identity map on R. The ring in Example 1 is 2-primal.

In [10], the 2-primal property has also been extended to the skew-polynomial ring $R[x;\sigma]$.

We now give an example of a ring R, and an endomorphism σ of R such that R is not a $\sigma(*)$-ring, however R is 2-primal.

Example 2 (Kwak [10]). *Let R = F[x] be the polynomial ring over a field F. Then R is 2-primal with P(R) = 0. Let $\sigma : R \to R$ be an endomorphism defined by $\sigma(f(x)) = f(0)$. Then R is not a $\sigma(*)$-ring. For example consider $f(x) = xa$, $a \neq 0$.*

Let R be a ring and σ an automorphism of R. We now give a necessary and sufficient condition for R to be a $\sigma(*)$-ring in the following proposition:

Proposition 1. *Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring implies that $P(R)$ is completely semiprime.*

Proof. Let R be a $\sigma(*)$-ring. We show that $P(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in P(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R).$$

Therefore $a\sigma(a) \in P(R)$ and hence $a \in P(R)$.

Converse of the above need not be true.

Example 3 (Kwak [10]). *Let K be a field, $R = K \times K$ and the automorphism σ of R defined by $\sigma((a,b)) = (b,a)$, $a,b \in K$. Then R is a reduced ring and so $P(R) = 0$ is completely semiprime. But the ring R is not a $\sigma(*)$-ring since $(1,0)\sigma((1,0)) = (0,0)$ but $(1,0) \notin P(R)$.*

Recall that an ideal P of a ring R is completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for a, $b \in R$. In commutative sense completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

Example 4. *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2(\mathbb{Z})$. If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R, but is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.*

There are examples of rings (noncommutative) in which prime ideals are completely prime.

Example 5. *Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $P_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and $P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ are prime ideals of R and all these are completely prime also.*

A necessary and sufficient condition for a Noetherian ring R to be a $\sigma(*)$-ring (where σ is an automorphism of R) has been given in Theorem 2.4 of [3]:

Theorem 1. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. See Theorem 2.4 of [3]. Please note that in Proposition 2.2 of [3] R should be Noetharian. Proposition 2.2 of [3] has been used to prove Theorem 2.4 of [3].

Proposition 2. *Let R be a Noetherian ring. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring. Then $U \in Min.Spec(R)$ implies that $US(R) = U[x;\sigma]$ is a completely prime ideal of $S(R) = R[x;\sigma]$.*

Proof. Proposition 1 implies that $P(R)$ is completely semiprime ideal of R. Let $U \in Min.Spec(R)$. Then Theorem 1 implies that $\sigma(U) = U$ and U is completely prime. Now we note that σ can be extended to an automorphism $\overline{\sigma}$ of R/U. Now it is well known that $S/US \simeq (R/U)[x;\overline{\sigma}]$ and hence US is a completely prime ideal of S.

3. Skew Polynomial Rings over Weak σ-rigid Rings

Definition 3 (Ouyang [14]). *Let R be a ring. Then R is said to be a weak σ-rigid ring if $a\sigma(a) \in N(R)$ if and only if $a \in N(R)$ for $a \in R$.*

Example 6 (Example 2.1 of Ouyang [14]). *Let σ be an endomorphism of a ring R such that R is a σ-rigid ring. Let*

$$A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,b,c,d \in R \right\}$$

be a subring of $T_3(R)$, the ring of upper triangular matrices over R. Now σ can be extended to an endomorphism $\overline{\sigma}$ of A by $\overline{\sigma}((a_{ij})) = (\sigma(a_{ij}))$. Then it can be seen that A is a weak $\overline{\sigma}$-rigid ring.

We now give a relation between a $\sigma(*)$-ring and a weak σ-rigid ring in the following Theorem:

Theorem 2. *Let R be a Noetherian ring. Let σ be an endomorphism of R such that R is a $\sigma(*)$-ring. Then R is a weak σ-rigid ring. Conversely a 2-primal weak σ-rigid ring is a $\sigma(*)$-ring.*

Proof. Let σ be an endomorphism of R such that R is a $\sigma(*)$-ring. Now R is completely semiprime by Proposition 1. Therefore, R is 2-primal, i.e. $N(R) = P(R)$. Thus $a\sigma(a) \in N(R) = P(R)$ implies that $a \in P(R) = N(R)$. Hence R is weak σ-rigid ring.

Conversely let R be 2-primal weak σ-rigid ring. Then $N(R) = P(R)$ and $a\sigma(a) \in N(R)$ implies that $a \in N(R)$. Hence R is a $\sigma(*)$-ring.

Corollary 1. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a 2-primal weak σ-rigid ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is completely prime ideal of R.*

Proof. Combine Theorem 1 and Theorem 2.

Let R be a Noetherian ring and σ an automorphism of R. We now give a characterization for R to be a weak σ-rigid ring (an analog of Proposition 1 for weak σ-rigid rings).

Proposition 3. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring implies that $N(R)$ is completely semiprime.*

Proof. First of all we show that $\sigma(N(R)) = N(R)$. We have $\sigma(N(R)) \subseteq N(R)$ as $\sigma(N(R))$ is a nilpotent ideal of R. Now for any $n \in N(R)$, there exists $a \in R$ such that $n = \sigma(a)$. So $I = \sigma^{-1}(N(R)) = \{a \in R$ such that $\sigma(a) = n \in N(R)\}$ is an ideal of R. Now I is nilpotent, therefore $I \subseteq N(R)$, which implies that $N(R) \subseteq \sigma(N(R))$. Hence $\sigma(N(R)) = N(R)$.

Now let R be a weak σ-rigid ring. We will show that $N(R)$ is completely semiprime. Let $a \in R$ be such that $a^2 \in N(R)$. Then

$$a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(N(R)) = N(R).$$

Therefore $a\sigma(a) \in N(R)$ and hence $a \in N(R)$. So $N(R)$ is completely semiprime.

Converse of the above Proposition need not be true (Example 3).

As mentioned earlier, we note that if σ is an endomorphism of a ring R, then it can be extended to an endomorphism $\overline{\sigma}$ of $S(R) = R[x;\sigma]$ by $\overline{\sigma}(\sum_{i=0}^{m} x^i a_i) = \sum_{i=0}^{m} x^i \sigma(a_i)$. We now prove the following:

Theorem 3. *Let R be a Noetherian ring. Let σ be an automorphism of R. Then R is a weak σ-rigid ring if and only if $S(R) = R[x;\sigma]$ is a weak $\overline{\sigma}$-rigid ring.*

Proof. First of all we note that Proposition 2.2 of Bhat [4] implies that $S(N(R)) = N(S(R))$. Now let R be a weak σ-rigid ring. We show that $R[x;\sigma]$ is a weak $\overline{\sigma}$-rigid ring.

Let $f \in S(R)$ (say $f = \sum_{i=0}^{m} x^i a_i$) be such that $f\overline{\sigma}(f) \in N(S(R))$. We use induction on m to prove the Theorem. For $m = 1$, $f = xa_1 + a_0$. Now $f\overline{\sigma}(f) \in N(S(R))$ implies that $(xa_1 + a_0)(x\sigma(a_1) + \sigma(a_0)) \in N(S(R)) = S(N(R))$, i.e.

$$x^2\sigma^2(a_1) + x\sigma(a_0)\sigma(a_1) + xa_1\sigma(a_0) + a_0\sigma(a_0) \in S(N(R)) \tag{1}$$

Therefore, $\sigma^2(a_1) \in N(R)$. Now $\sigma(N(R)) = N(R)$ implies that $a_1 \in N(R)$. So (1) implies that $a_0\sigma(a_0) \in N(R)$ implies that $a_0 \in N(R)$. Therefore, $f \in S(N(R)) = N(S(R))$.

Suppose the result is true for $m = k$. We prove for $m = k + 1$. Now $f\overline{\sigma}(f) \in N(S(R))$ implies that $(x^{k+1}a_{k+1} + \ldots + a_0)(x^{k+1}\sigma(a_{k+1}) + ldots + \sigma(a_0)) \in N(S(R)) = S(N(R))$, i.e.

$$x^{2k+2}\sigma^{k+2}(a_{k+1}) + x^{2k+1}(\sigma^k(a_{k+1})\sigma(a_k) + \sigma^{k+1}(a_k)\sigma(a_{k+1})) + g\overline{\sigma}(g) \in S(N(R)),$$

where $g = \sum_{i=0}^{k} x^i a_i$. Therefore, $\sigma^{k+2}(a_{k+1}) \in N(R)$ implies that $a_{k+1} \in N(R)$. Also $\sigma^k(a_{k+1})\sigma(a_k)+$ $\sigma^{k+1}(a_k)\sigma(a_{k+1}) \in N(R)$ implies that $g\overline{\sigma}(g) \in N(S(R))$, but the degree of g is k, therefore, by induction hypothesis, the result is true for all m.

Converse is obvious.

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