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On the Number of Representation of Integers by the Direct Sum of BQFs with Discriminant −191

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Abstract. In this study we find a basis of the space $S_4(\Gamma_0(191))$ and derive explicit formulae for the number of representation of positive integers by all possible direct sum of 13 quadratic forms from the representatives $x_1^2+x_1x_2+48x_2^2$, $2x_1^2+x_1x_2+24x_2^2$, $3x_1^2+x_1x_2+16x_2^2$, $4x_1^2+x_1x_2+12x_2^2$, $5x_1^2+3x_1x_2+10x_2^2$, $6x_1^2+x_1x_2+8x_2^2$, $6x_1^2+5x_1x_2+9x_2^2$ of the class group of equivalence classes of quadratic forms with discriminant −191.

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1. Introduction

The problem of determining which positive integers are represented by quadratic forms has been studied extensively since earlier mathematicians. Fermat's assertion of 1640 about representation of integers by the binary quadratic form $x_1^2+x_2^2$ was proved by Euler. With Lagrange's four square theorem which states that, the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2$ represent all positive integers, the theory of universal quadratic forms has been started in 1770. It has been proved by Legendre in 1798 that the quadratic form $x_1^2+x_2^2+x_3^2$ represent all positive integers except precisely the numbers of the form $4^a(8k+7)$. Legendre also gives a general theory of binary quadratic forms in his study *Theorie des Nombres* in 1830. In 1930, Mordell [7] proved the five squares theorem, which states that the quadratic form $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2$ represent all positive definite binary quadratic forms. In 1997, Conway and Schneeberger proved that a positive definite integral quadratic form represents every positive integer if and only if it represents the integers $1,2,3,5,6,7,10,14$, and 15. It is known as the *15 theorem* and later has been proved by Bhargava [1] by a simpler method. Bhargava and Hanke [2] have shown in their studies that every integer valued quadratic form is universal if and only if it represent every integer less than 290. For a general information about the theory of quadratic forms one can see [6].

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Determination of positive integers represented by a given quadratic form Q is an interesting problem, but it is also interesting to ask in how many different ways is the integer n represented by Q? If we let $r(n,Q)$ count the number of ways of representing n by Q, then we are asking for a description of the function $r(n,Q)$. In these terms, the question of which integers n can be represented by Q means, for which n, $r(n,Q)$ is > 0? Finding exact formulas for $r(n,Q)$ is a classical problem in number theory. In some cases formulas can be obtained for $r(n,Q)$, but such formulas are quite rare [3].

For instance, if we consider the quadratic form $Q = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $n > 0$, then we have the following Jacobi's result [3]:

$$r(n,Q) = 8 \sum_{\substack{d \backslash n \\ 4 \nmid d > 0}} d.$$

Peterson [8], for the first time, considered the problem of representation of numbers by the direct sum of some binary quadratic forms. Kendirli [4] has given the number of representations of positive integers by some direct sum of binary quadratic forms with discriminant -79. In this study we obtain a basis of the space $S_4(\Gamma_0(191))$ and formulae for the number of representations of positive integers by some direct sums of binary quadratic forms with discriminant -191 which are all quadratic forms 8 variables.

There exist 13 inequivalent classes of binary quadratic forms with discriminant -191. These are:

$$\begin{aligned}
F_1 &= x_1^2 + x_1x_2 + 48x_2^2 \\
\Phi_1 &= 2x_1^2 + x_1x_2 + 24x_2^2, \Phi_1' = 2x_1^2 - x_1x_2 + 24x_2^2 \\
\Psi_1 &= 3x_1^2 + x_1x_2 + 16x_2^2, \Psi_1' = 3x_1^2 - x_1x_2 + 16x_2^2 \\
\Lambda_1 &= 4x_1^2 + x_1x_2 + 12x_2^2, \Lambda_1' = 4x_1^2 - x_1x_2 + 12x_2^2, \\
\Upsilon_1 &= 5x_1^2 + 3x_1x_2 + 10x_2^2, \Upsilon_1' = 5x_1^2 - 3x_1x_2 + 10x_2^2, \\
\Omega_1 &= 6x_1^2 + x_1x_2 + 8x_2^2, \Omega_1' = 6x_1^2 - x_1x_2 + 8x_2^2, \\
\Pi_1 &= 6x_1^2 + 5x_1x_2 + 9x_2^2, \Pi_1' = 6x_1^2 - 5x_1x_2 + 9x_2^2,
\end{aligned}$$

Here Φ_1', Ψ_1', Λ_1', Υ_1', Ω_1', and Π_1' are respectively the inverses of Φ_1, Ψ_1, Λ_1, Υ_1, Ω_1, and Π_1. Therefore the theta series of Φ_1', Ψ_1', Λ_1', Υ_1', Ω_1', and Π_1' are respectively same with the theta series of Φ_1, Ψ_1, Λ_1, Υ_1, Ω_1, and Π_1. Here F_1 is the identity element and these quadratic forms form a group of order 13 which can be described as:

$$\begin{aligned}
\Phi_1, \Phi_1^2 &= \Lambda_1, \Phi_1^3 = \Omega_1', \Phi_1^4 = \Psi_1', \Phi_1^5 = \Pi_1, \Phi_1^6 = \Upsilon_1', \Phi_1^7 = \Upsilon_1, \Phi_1^8 = \Pi_1', \Phi_1^9 = \Psi_1, \\
\Phi_1^{10} &= \Omega_1, \Phi_1^{11} = \Lambda_1', \Phi_1^{12} = \Phi_1', \Phi_1^{13} = F_1.
\end{aligned}$$

Since 191 is prime number then there is only one genus, i.e., the principal genus.

For any quadratic form Q_1 let $Q_k = Q_1 + \ldots + Q_1$ (k times) be kth direct sum of this quadratic form. In the present paper we obtain formulas $r(n,Q)$ for any of the quadratic forms

$$
\begin{aligned}
Q \quad = & F_4, \Phi_4, \Psi_4, \Lambda_4, \Upsilon_4, \Omega_4, \Pi_4, F_i \oplus \Phi_j, F_i \oplus \Psi_j, F_i \oplus \Lambda_j, F_i \oplus \Upsilon_j, F_i \oplus \Omega_j, F_i \oplus \Pi_j, \\
& \Phi_i \oplus \Psi_j, \Phi_i \oplus \Lambda_j, \Phi_i \oplus \Upsilon_j, \Phi_i \oplus \Omega_j, \Phi_i \oplus \Pi_j, \Psi_i \oplus \Lambda_j, \Psi_i \oplus \Upsilon_j, \Psi_i \oplus \Omega_j, \Psi_i \oplus \Pi_j, \\
& \Lambda_i \oplus \Upsilon_j, \Lambda_i \oplus \Omega_j, \Lambda_i \oplus \Pi_j, \Upsilon_i \oplus \Omega_j, \Upsilon_i \oplus \Pi_j, \Omega_i \oplus \Pi_j, F_i \oplus \Phi_j \oplus \Psi_l, F_i \oplus \Phi_j \oplus \Lambda_l, \\
& F_i \oplus \Phi_j \oplus \Upsilon_l, F_i \oplus \Phi_j \oplus \Omega_l, F_i \oplus \Phi_j \oplus \Pi_l, \Phi_i \oplus \Psi_j \oplus \Lambda_l, \Phi_i \oplus \Psi_j \oplus \Upsilon_l, \Phi_i \oplus \Psi_j \oplus \Omega_l, \\
& \Phi_i \oplus \Psi_j \oplus \Pi_l, \Psi_i \oplus \Lambda_j \oplus \Upsilon_l, \Psi_i \oplus \Lambda_j \oplus \Omega_l, \Psi_i \oplus \Lambda_j \oplus \Omega_l, \Psi_i \oplus \Lambda_j \oplus \Pi_l, \Lambda_i \oplus \Upsilon_j \oplus \Omega_l, \\
& \Lambda_i \oplus \Upsilon_j \oplus \Pi_l, \Upsilon_i \oplus \Omega_j \oplus \Pi_l, F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Lambda_1, F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Upsilon_1, F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Omega_1, \\
& F_1 \oplus \Phi_1 \oplus \Psi_1 \oplus \Pi_1, \Phi_1 \oplus \Psi_1 \oplus \Lambda_1 \oplus \Upsilon_1, \Phi_1 \oplus \Psi_1 \oplus \Lambda_1 \oplus \Omega_1, \Phi_1 \oplus \Psi_1 \oplus \Lambda_1 \oplus \Pi_1, \\
& \Psi_1 \oplus \Lambda_1 \oplus \Upsilon_1 \oplus \Omega_1, \Psi_1 \oplus \Lambda_1 \oplus \Upsilon_1 \oplus \Pi_1, \text{and } \Lambda_1 \oplus \Upsilon_1 \oplus \Omega_1 \oplus \Pi_1
\end{aligned} \tag{1}
$$

(where $i, j, l, m \geq 1$ and in any direct sum the sum of the indices is 4). In these direct sums one can replace the quadratic forms Φ_1, Ψ_1, Λ_1, Υ_1, Ω_1, and Π_1 by their inverses.

2. The Positive Definite Quadratic Forms

In this section we give some definitions, an important theorem and evaluation of the quadratic forms.

Definition 1. *Let $Q : \mathbb{Z}^{2k} \to \mathbb{Z}$ be a positive definite integer-valued form of 2k variables,*

$$Q = \sum_{1 \leq i \leq j \leq 2k}^{2k} b_{ij} x_i x_j, b_{ij} \in \mathbb{Z}$$

and the matrix A is defined by

$$a_{ii} = 2b_{ii}, a_{ji} = a_{ij} = b_{ij} \text{ for } i < j.$$

Let D be the discriminant of the quadratic form

$$2Q = \sum_{i,j=1}^{2k} a_{ij} x_i x_j$$

i.e., the determinant of the matrix A. Let A_{ij} be the cofactors of a_{ij} for $1 \leq i \leq j \leq 2k$. If $\delta = \gcd(\frac{A_{ii}}{2}, A_{ij}$, for $1 \leq i \leq j \leq 2k)$, then $N := \dfrac{D}{\delta}$ is the smallest positive integer, called the level of Q, for which

NA^{-1} is again an even integral matrix like A.

$\Delta = (-1)^k D$ is called the discriminant of the form Q.

Theorem 1. *Let $Q : \mathbb{Z}^{2k} \to \mathbb{Z}$ be positive definite integer-valued form of $2k$ variables of level N and discriminant Δ. Then*

1 *The theta function*

$$\Theta_Q(q) = \sum_{(n_1,n_2,\dots,n_k)\in\mathbb{Z}\times\mathbb{Z}\times\dots\times\mathbb{Z}} q^{Q(n_1,n_2,\dots,n_k)} = 1 + \sum_{n=1}^{\infty} r(n;Q)q^n, q = e^{2\pi i z}$$

is a modular form on $\Gamma_0(N)$ *of weight* k *and character* χ_d, *i.e.,* $\Theta_Q \in M_k(\Gamma_0(N), \chi_d)$, *where*

$$\chi_\Delta(d) := \left(\frac{\Delta}{d}\right), d \in (\mathbb{Z}/N\mathbb{Z})^\times, \left(\frac{\Delta}{d}\right) \text{ is the Kronecker Character.}$$

2 *The homogeneous quadratic polynomials in* $2k$ *variables*

$$\varphi_{ij} = x_i x_j - \frac{1}{2k}\frac{A_{ij}}{D}2Q, 1 \le i \le j \le 2k \tag{2}$$

are spherical functions of second order with respect to Q.

3 *The theta series*

$$\Theta_{Q,\varphi_{ij}}(q) = \sum_{n=1}^{\infty}\left(\sum_{Q=n}\varphi_{ij}\right)q^n \tag{3}$$

is a cusp form in $S_{k+2}(\Gamma_0(N), \chi_d)$.

4 *If two quadratic forms* Q_1, Q_2 *have the same level* N *and the characteristic are* $\chi_1(d)$, $\chi_2(d)$ *respectively, then the direct sum* $Q_1 \oplus Q_2$ *of the quadratic forms has the same level* N *and the character* $\chi_1(d)$, $\chi_2(d)$.

Proof. See [7].

Now let's look at the positive definite quadratic forms of discriminant -191.
For the quadratic form $F_1 = x_1^2 + x_1x_2 + 48x_2^2$,

$$2F_1 = 2x_1^2 + 2x_1x_2 + 96x_2^2 = (x_1, x_2)\begin{pmatrix} 2 & 1 \\ 1 & 96 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

the determinant $D = 191, A_{22} = 2$, so $\delta = 1, N = D = 191$ and the discriminant is $\Delta = (-1)^{2/2}191 = -191$. Similarly, it can be easily seen that for any of the quadratic forms Φ_1, Ψ_1, Λ_1, Υ_1, Ω_1 and Π_1 the determinant, the discriminant and the character respectively are

$$D = 191, \Delta = -191, \chi(d) = \left(\frac{-191}{d}\right).$$

Consequently F_1, Φ_1, Ψ_1, Λ_1, Υ_1, Ω_1 and Π_1 are quadratic forms whose theta series are in

$$M_1(\Gamma_0(191), \left(\frac{-191}{d}\right)).$$

Hence by Theorem 1 F_2, Φ_2, Ψ_2, Λ_2, Υ_2, Ω_2, Π_2, $F_1 \oplus \Phi_1$, $F_1 \oplus \Psi_1$, $F_1 \oplus \Lambda_1$, $F_1 \oplus \Upsilon_1$, $F_1 \oplus \Omega_1$, $F_1 \oplus \Pi_1$, $\Phi_1 \oplus \Psi_1$, $\Phi_1 \oplus \Lambda_1$, $\Phi_1 \oplus \Upsilon_1$, $\Phi_1 \oplus \Omega_1$, $\Phi_1 \oplus \Pi_1$, $\Psi_1 \oplus \Lambda_1$, $\Psi_1 \oplus \Upsilon_1$, $\Psi_1 \oplus \Omega_1$,

$\Psi_1 \oplus \Pi_1$, $\Lambda_1 \oplus \Upsilon_1$, $\Lambda_1 \oplus \Omega_1$, $\Lambda_1 \oplus \Pi_1$, $\Upsilon_1 \oplus \Omega_1$, $\Upsilon_1 \oplus \Pi_1$, $\Omega_1 \oplus \Pi_1$ are quadratic forms whose theta series are in

$$M_2(\Gamma_0(191)).$$

Theorem 2. *Let Q be a positive definite quadratic form of $2k$ variables,*

$$k = 4, 6, 8, \ldots$$

whose theta series Θ_Q is in $M_k(\Gamma_0(p))$, p prime, then the Eisenstein part of Θ_Q is

$$E(q:Q) = 1 + \sum_{n=1}^{\infty}(\alpha\sigma_{k-1}(n)q^n + \beta\sigma_{k-1}(n)q^{pn}),$$

where

$$\alpha = \frac{i^k}{\rho_k}\frac{p^{k/2} - i^k}{p^k - 1}, \beta = \frac{1}{\rho_k}\frac{p^k - i^k p^{k/2}}{p^k - 1}, \rho_k = (-1)^{k/2}\frac{(k-1)!}{(2\pi)^k}\zeta(k).$$

Proof. See [7].

We immediately obtain the following corollary.

Corollary 1. *Let Q be a positive definite quadratic form of 8 variables whose theta series Θ_Q is in*

$$M_4(\Gamma_0(191))$$

then the Eisenstein part of Θ_Q is

$$E(q:Q) = 1 + \sum_{n=1}^{\infty}(\alpha\sigma_3(n)q^n + \beta\sigma_3(n)q^{191n}),$$

where

$$\rho_4 = \frac{3!}{(2\pi)^4}\zeta(4) = \frac{3!}{(2\pi)^4}.\frac{\pi^4}{90} = \frac{1}{240}, \ \alpha = 240\frac{191^2 - 1}{191^4 - 1} = 240\frac{1}{191^2 + 1} = \frac{120}{18241}$$

$$\beta = 240\frac{191^4 - 191^2}{191^4 - 1} = 240\frac{191^2}{191^2 + 1} = 191^2\frac{120}{18241}$$

and for any Q in (1)

$$E(q:Q) = 1 + \frac{120}{18241}\sum_{n=1}^{\infty}(q^n + 191^2 q^{191n})\sigma_3(n) = \frac{120}{18241}\sum_{n=1}^{\infty}\sigma_3^*(n)q^n$$

where

$$\sigma_3^*(n) = \begin{cases} \sigma_3(n) & \text{if } n \geq 1 \text{ and } 191 \nmid n \\ \sigma_3(n) + 191^2\sigma_3(n/191) & \text{if } 191 \mid n \end{cases}$$

3. The Selection of Spherical Functions

Here we will select 47 spherical functions such that the corresponding generalized theta series span all the generalized theta series of (3) induced by spherical functions of the form (2).

1 For

$$2F_2 = 2x_1^2 + 2x_1x_2 + 96x_2^2 + 2x_3^2 + 2x_3x_4 + 96x_4^2$$
$$= (x_1, x_2, x_3, x_4) \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 96 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 96 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

the determinant $D = 191^2$,$A_{11} = 76.191$. By putting $2k = 4$, $Q = F_2$, and appropriate i, j in Theorem 1, we get the spherical function of second order with respect to F_2 as:

$$\varphi_{12} = x_1^2 - \frac{1}{4}\frac{96.191}{191^2}2F_2 = x_1^2 - \frac{48}{191}F_2,$$

2 For $2\Phi_2 = 4x_1^2+2x_1x_2+48x_2^2+4x_3^2+2x_3x_4+48x_4^2$, by taking $A_{11} = 48.191, A_{12} = -191$ we get

$$\varphi_{11} = x_{11}^2 - \frac{1}{4}\frac{48.191}{191^2}2\Phi_2 = x_{11}^2 - \frac{24}{191}\Phi_2,$$
$$\varphi_{12} = x_1x_2 + \frac{1}{4}\frac{191}{191^2}2\Phi_2 = x_1x_2 + \frac{1}{2.191}\Phi_2,$$

which will be spherical functions of second order with respect to Φ_2.

3 For $2\Psi_2 = 6x_1^2+2x_1x_2+32x_2^2+6x_3^2+2x_3x_4+32x_4^2$, by taking $A_{22} = 6.191, A_{33} = 32.191$ we get;

$$\varphi_{22} = x_2^2 - \frac{3}{191}\Psi_2, \varphi_{33} = x_3^2 - \frac{16}{191}\Psi_2,$$

which will be spherical functions of second order with respect to Ψ_2.

4 For $2\Lambda_2 = 8x_1^2+2x_1x_2+24x_2^2+8x_3^2+2x_3x_4+24x_4^2$, by taking $A_{11} = 24.191, A_{12} = -191$ we get;

$$\varphi_{11} = x_1^2 - \frac{12}{191}\Lambda_2, \varphi_{12} = x_1x_2 + \frac{1}{2.191}\Lambda_2,$$

which will be spherical functions of second order with respect to Λ_2.

5 For $2\Upsilon_2 = 10x_1^2 + 6x_1x_2 + 20x_2^2 + 10x_3^2 + 6x_3x_4 + 20x_4^2$ by taking $A_{11} = 20.191$, $A_{22} = 10.191$ we have;

$$\varphi_{11} = x_1^2 - \frac{10}{191}\Upsilon_2, \varphi_{22} = x_2^2 - \frac{5}{191}\Upsilon_2,$$

which will be spherical functions of second order with respect to Λ_2.

6 For $2\Omega_2 = 12x_1^2 + 2x_1x_2 + 16x_2^2 + 12x_3^2 + 2x_3x_4 + 16x_4^2$, by taking $A_{12} = -191$, $A_{22} = 12.191$ we get;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}\Omega_2, \varphi_{22} = x_2^2 - \frac{6}{191}\Omega_2,$$

which will be spherical functions of second order with respect to Ω_2.

7 For $2\Pi_2 = 12x_1^2 + 10x_1x_2 + 18x_2^2 + 12x_3^2 + 10x_3x_4 + 18x_4^2$, by taking $A_{11} = 18.191$, $A_{22} = 12.191$ we get,

$$\varphi_{11} = x_1^2 - \frac{9}{191}\Pi_2, \varphi_{22} = x_2^2 - \frac{6}{191}\Pi_2,$$

which will be spherical functions of second order with respect to Π_2.

8 For $2(F_1 \oplus \Phi_1) = 2x_1^2 + 2x_1x_2 + 96x_2^2 + 4x_3^2 + 2x_3x_4 + 48x_4^2$ the determinant $D = 191^2$, $A_{12} = -191$, $A_{33} = 48.191$, the spherical functions of second order with respect to $F_1 \oplus \Phi_1$ are;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}(F_1 \oplus \Phi_1), \varphi_{33} = x_3^2 - \frac{24}{191}(F_1 \oplus \Phi_1),$$

9 For $2(F_1 \oplus \Psi_1) = 2x_1^2 + 2x_1x_2 + 96x_2^2 + 6x_3^2 + 2x_3x_4 + 32x_4^2$ the determinant $D = 191^2$, $A_{22} = 2.191, A_{34} = -191$, the spherical functions of second order with respect to $F_1 \oplus \Psi_1$ are;

$$\varphi_{22} = x_2^2 - \frac{1}{191}(F_1 \oplus \Psi_1), \varphi_{34} = x_3x_4 + \frac{1}{2.191}(F_1 \oplus \Psi_1),$$

10 For $2(F_1 \oplus \Lambda_1) = 2x_1^2 + 2x_1x_2 + 96x_2^2 + 8x_3^2 + 2x_3x_4 + 24x_4^2$ the determinant $D = 191^2$, $A_{12} = -191, A_{44} = 8.191$, the spherical functions of second order with respect to $F_1 \oplus \Lambda_1$ are;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}(F_1 \oplus \Lambda_1), \varphi_{44} = x_4^2 - \frac{4}{191}(F_1 \oplus \Lambda_1),$$

11 For $2(F_1 \oplus \Upsilon_1) = 2x_1^2 + 2x_1x_2 + 96x_2^2 + 10x_3^2 + 6x_3x_4 + 20x_4^2$ the determinant $D = 191^2$, $A_{11} = 96.191$, $A_{34} = -3.191$, the spherical functions of second order with respect to $F_1 \oplus \Upsilon_1$ are;

$$\varphi_{11} = x_1^2 - \frac{48}{191}(F_1 \oplus \Upsilon_1), \varphi_{34} = x_3x_4 + \frac{3}{2.191}(F_1 \oplus \Upsilon_1),$$

12 For $2(F_1 \oplus \Omega_1) = 2x_1^2 + 2x_1x_2 + 96x_2^2 + 12x_3^2 + 2x_3x_4 + 16x_4^2$ the determinant $D = 191^2$, $A_{12} = -191$, $A_{33} = 20.191$, the spherical functions of second order with respect to $F_1 \oplus \Omega_1$ are;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}(F_1 \oplus \Omega_1), \varphi_{33} = x_3^2 - \frac{10}{191}(F_1 \oplus \Omega_1),$$

13 For $2(F_1 \oplus \Pi_1) = 2x_1^2 + 2x_1x_2 + 96x_2^2 + 12x_3^2 + 10x_3x_4 + 18x_4^2$ the determinant $D = 191^2$, $A_{22} = 2.191$, $A_{34} = -5.191$, the spherical functions of second order with respect to $F_1 \oplus \Pi_1$ are;

$$\varphi_{22} = x_2^2 - \frac{1}{191}(F_1 \oplus \Pi_1), \varphi_{34} = x_3x_4 + \frac{5}{2.191}(F_1 \oplus \Pi_1),$$

14 For $2(\Phi_1 \oplus \Psi_1) = 4x_1^2 + 2x_1x_2 + 48x_2^2 + 6x_3^2 + 2x_3x_4 + 32x_4^2$ the determinant $D = 191^2$, $A_{11} = 48.191$, $A_{22} = 4.191$, the spherical functions of second order with respect to $\Phi_1 \oplus \Psi_1$ are;

$$\varphi_{11} = x_1^2 - \frac{24}{191}(\Phi_1 \oplus \Psi_1), \varphi_{22} = x_2^2 - \frac{2}{191}(\Phi_1 \oplus \Psi_1),$$

15 For $2(\Phi_1 \oplus \Lambda_1) = 4x_1^2 + 2x_1x_2 + 48x_2^2 + 8x_3^2 + 2x_3x_4 + 24x_4^2$, $D = 191^2$, $A_{12} = -191$, $A_{33} = 24.191$, the spherical functions of second order with respect to $\Phi_1 + \Lambda_1$ are;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}(\Phi_1 \oplus \Lambda_1), \varphi_{33} = x_3^2 - \frac{12}{191}(\Phi_1 \oplus \Lambda_1),$$

16 For $2(\Phi_1 \oplus \Upsilon_1) = 4x_1^2 + 2x_1x_2 + 48x_2^2 + 10x_3^2 + 6x_3x_4 + 20x_4^2$, $D = 191^2$, $A_{12} = -191$, $A_{22} = 4.191$, the spherical functions of second order with respect to $\Phi_1 + \Upsilon_1$ are;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}(\Phi_1 \oplus \Upsilon_1), \varphi_{22} = x_2^2 - \frac{2}{191}(\Phi_1 \oplus \Upsilon_1).$$

17 For $2(\Phi_1 \oplus \Omega_1) = 4x_1^2 + 2x_1x_2 + 48x_2^2 + 12x_3^2 + 2x_3x_4 + 16x_4^2$, $D = 191^2$, $A_{33} = 16.191$, $A_{34} = -191$, the spherical functions of second order with respect to $\Phi_1 \oplus \Omega_1$ are;

$$\varphi_{33} = x_3^2 - \frac{8}{191}(\Phi_1 \oplus \Omega_1), \varphi_{34} = x_3x_4 + \frac{1}{2.191}(\Phi_1 \oplus \Omega_1).$$

18 For $2(\Phi_1 \oplus \Pi_1) = 4x_1^2 + 2x_1x_2 + 48x_2^2 + 12x_3^2 + 10x_3x_4 + 18x_4^2$, $D = 191^2$, $A_{11} = 48.191$, $A_{33} = 18.191$, the spherical functions of second order with respect to $\Phi_1 \oplus \Pi_1$ are;

$$\varphi_{11} = x_1^2 - \frac{24}{191}(\Phi_1 \oplus \Pi_1), \varphi_{33} = x_3^2 - \frac{9}{191}(\Phi_1 \oplus \Pi_1).$$

19 For $2(\Psi_1 \oplus \Lambda_1) = 6x_1^2 + 2x_1x_2 + 32x_2^2 + 8x_3^2 + 2x_3x_4 + 24x_4^2$, $D = 191^2$, $A_{12} = -191$, $A_{22} = 6.191$, the spherical functions of second order with respect to $\Psi_1 \oplus \Lambda_1$ are;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}(\Psi_1 \oplus \Lambda_1), \varphi_{22} = x_2^2 - \frac{3}{191}(\Psi_1 \oplus \Lambda_1),$$

20 For $2(\Psi_1 \oplus \Upsilon_1) = 6x_1^2 + 2x_1x_2 + 32x_2^2 + 10x_3^2 + 6x_3x_4 + 20x_4^2$, $D = 191^2$, $A_{33} = 20.191$, $A_{44} = 10.191$, the spherical functions of second order with respect to $\Psi_1 \oplus \Upsilon_1$ are;

$$\varphi_{33} = x_3^2 - \frac{10}{191}(\Psi_1 \oplus \Upsilon_1), \varphi_{44} = x_4^2 - \frac{5}{191}(\Psi_1 \oplus \Upsilon_1).$$

21 For $2(\Psi_1 \oplus \Omega_1) = 6x_1^2 + 2x_1x_2 + 32x_2^2 + 12x_3^2 + 2x_3x_4 + 16x_4^2$, $D = 191^2$, $A_{12} = -191$, $A_{33} = 16.191$, the spherical functions of second order with respect to $\Psi_1 \oplus \Omega_1$ are;

$$\varphi_{12} = x_1x_2 + \frac{1}{2.191}(\Psi_1 \oplus \Omega_1), \varphi_{33} = x_3^2 - \frac{8}{191}(\Psi_1 \oplus \Omega_1).$$

22 For $2(\Psi_1 \oplus \Pi_1) = 6x_1^2 + 2x_1x_2 + 32x_2^2 + 12x_3^2 + 10x_3x_4 + 18x_4^2$, $D = 191^2$, $A_{11} = 32.191$, $A_{34} = -5.191$, the spherical functions of second order with respect to $\Psi_1 \oplus \Pi_1$ are;

$$\varphi_{11} = x_1^2 - \frac{16}{191}(\Psi_1 \oplus \Pi_1), \varphi_{34} = x_3x_4 + \frac{5}{2.191}(\Psi_1 \oplus \Pi_1).$$

23 For $2(\Lambda_1 \oplus \Upsilon_1) = 8x_1^2 + 2x_1x_2 + 24x_2^2 + 10x_3^2 + 6x_3x_4 + 20x_4^2$, $D = 191^2$, $A_{11} = 24.191$, $A_{22} = 8.191$, the spherical functions of second order with respect to $\Lambda_1 \oplus \Upsilon_1$ are;

$$\varphi_{11} = x_1^2 - \frac{12}{191}(\Lambda_1 \oplus \Upsilon_1), \varphi_{22} = x_2^2 - \frac{4}{191}(\Lambda_1 \oplus \Upsilon_1).$$

24 For $2(\Lambda_1 + \Omega_1) = 8x_1^2 + 2x_1x_2 + 24x_2^2 + 12x_3^2 + 2x_3x_4 + 16x_4^2$, $D = 191^2$, $A_{33} = 16.191$, $A_{44} = 12.191$, the spherical functions of second order with respect to $\Lambda_1 \oplus \Omega_1$ are;

$$\varphi_{33} = x_3^2 - \frac{8}{191}(\Lambda_1 \oplus \Omega_1), \varphi_{44} = x_4^2 - \frac{6}{191}(\Lambda_1 \oplus \Omega_1).$$

4. The Solutions of $Q = n$ and the Theta Series Associated to the Quadratic Forms

The equation $F_1 = x_1^2 + x_1x_2 + 48x_2^2 = n$ have the following solutions:

$$\begin{aligned} n &= 1 \Rightarrow x_1 = \pm 1, x_2 = 0 \\ n &= 4 \Rightarrow x_1 = \pm 4, x_2 = 0 \\ n &= 9 \Rightarrow x_1 = \pm 3, x_2 = 0 \\ n &= 16 \Rightarrow x_1 = \pm 4, x_2 = 0 \\ n &= 25 \Rightarrow x_1 = \pm 5, x_2 = 0 \\ n &= 36 \Rightarrow x_1 = \pm 6, x_2 = 0 \end{aligned}$$

and there is no integral solutions for: $n =$ 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47. Thus the theta series of F_1 is given by

$$\Theta_{F_1}(q) = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + 2q^{25} + 2q^{36} + \ldots$$

By a similar way theta series of Φ_1, Ψ_1, Λ_1, Υ_1, Ω_1, and Π_1 are obtained as follows:

$$\Theta_{\Phi_1}(q) = 1 + 2q^2 + 2q^8 + 2q^{18} + 2q^{24} + 2q^{25} + 2q^{27} + 2q^{30} + 2q^{32} + 2q^{34} + 2q^{39} + 2q^{45} + \ldots$$

$$\Theta_{\Psi_1}(q) = 1 + 2q^3 + 2q^{12} + 2q^{16} + 2q^{18} + 2q^{20} + 2q^{26} + 2q^{27} + 2q^{30} + 2q^{40} + 2q^{46} + \ldots$$
$$\Theta_{\Lambda_1}(q) = 1 + 2q^4 + 2q^{12} + 2q^{15} + 2q^{16} + 2q^{17} + 2q^{26} + 2q^{30} + 2q^{36} + 2q^{45} + \ldots$$
$$\Theta_{\Upsilon_1}(q) = 1 + 2q^5 + 2q^{10} + 2q^{12} + 2q^{18} + 2q^{20} + 2q^{24} + 2q^{36} + 2q^{39} + 2q^{40} + 2q^{45} + 2q^{46} + \ldots$$
$$\Theta_{\Omega_1}(q) = 1 + 2q^6 + 2q^8 + 2q^{13} + 2q^{15} + 2q^{24} + 2q^{30} + 2q^{32} + 2q^{34} + 2q^{36} + 2q^{40} + \ldots$$
$$\Theta_{\Pi_1}(q) = 1 + 2q^6 + 2q^9 + 2q^{10} + 2q^{20} + 2q^{23} + 2q^{24} + 2q^{32} + 2q^{36} + 2q^{40} + 2q^{43} + \ldots$$

Here as an example we will compute the theta series of F_4. Theta series $\Theta_Q(q)$ for any quadratic form in (1) are obtained in a similar way.

$$\begin{aligned}\Theta_{F_4}(q) =& \Theta_{F_1}(q).\Theta_{F_1}(q).\Theta_{F_1}(q)\cdot\Theta_{F_1}(q) = 1 + 8q + 24q^2 + 32q^3 + 24q^4 + 48q^5 + 96q^6 \\ &+ 64q^7 + 24q^8 + 104q^9 + 144q^{10} + 96q^{11} + 96q^{12} + 112q^{13} + 192q^{14} + 192q^{15} \\ &+ 24q^{16} + 144q^{17} + 312q^{18} + 160q^{19} + 144q^{20} + 256q^{21} + 288q^{22} + 192q^{23} \\ &+ 96q^{24} + 248q^{25} + 336q^{26} + 320q^{27} + 192q^{28} + 240q^{29} + 576q^{30} + 256q^{31} \\ &+ 24q^{32} + 384q^{33} + 432q^{34} + 384q^{35} + 312q^{36} + 304q^{37} + 480q^{38} + 448q^{39} \\ &+ 144q^{40} + 336q^{41} + 768q^{42} + 352q^{43} + 288q^{44} + 624q^{45} + 576q^{46} + 384q^{47} + \ldots\end{aligned}$$

Theorem 3. *The following system of generalized fourfold theta-series is a basis of* $S_4(\Gamma_0(191))$*,*

$$\Theta_{F_2,\varphi_{11}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_2=n}(191x_1^2 - 48F_2)q^n,$$

$$\Theta_{\Phi_2,\varphi_{11}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(191x_1^2 - 24\Phi_2)q^n,$$

$$\Theta_{\Phi_2,\varphi_{12}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_2=n}(191x_1x_2 + \frac{1}{2}\Phi_2)q^n,$$

$$\Theta_{\Psi_2,\varphi_{22}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(191x_2^2 - 3\Psi_2)q^n,$$

$$\Theta_{\Psi_2,\varphi_{33}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_2=n}(191x_3^2 - 16\Psi_2)q^n,$$

$$\Theta_{\Lambda_2,\varphi_{11}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Lambda_2=n}(191x_1^2 - 12\Lambda_2)q^n,$$

$$\Theta_{\Lambda_2,\varphi_{12}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Lambda_2=n}(191x_1x_2 + \frac{1}{2}\Lambda_2)q^n,$$

$$\Theta_{\Upsilon_2,\varphi_{11}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Upsilon_2=n}(191x_1^2 - 10\Upsilon_2)q^n,$$

$$\Theta_{\Upsilon_2,\varphi_{22}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Upsilon_2=n}(191x_2^2 - 5\Upsilon_2)q^n,$$

$$\Theta_{\Omega_2,\varphi_{12}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Omega_2=n}(191x_1x_2 + \frac{1}{2}\Omega_2)q^n,$$

$$\Theta_{\Omega_2,\varphi_{22}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Omega_2=n}(191x_2^2 - 6\Omega_2)q^n,$$

$$\Theta_{\Pi_2,\varphi_{11}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Pi_2=n}(191x_1^2 - 9\Pi_2)q^n,$$

$$\Theta_{\Pi_2,\varphi_{22}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Pi_2=n}(191x_2^2 - 6\Pi_2)q^n,$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(191x_1x_2 + \frac{1}{2}(F_1\oplus\Phi_1))q^n,$$

$$\Theta_{F_1\oplus\Phi_1,\varphi_{33}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Phi_1=n}(191x_3^2 - 24(F_1\oplus\Phi_1))q^n,$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{22}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}(191x_2^2 - (F_1\oplus\Psi_1))q^n,$$

$$\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Psi_1=n}(191x_3x_4 + \frac{1}{2}(F_1\oplus\Psi_1))q^n,$$

$$\Theta_{F_1\oplus\Lambda_1,\varphi_{12}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Lambda_1=n}(191x_1x_2 + \frac{1}{2}(F_1\oplus\Lambda_1))q^n,$$

$$\Theta_{F_1\oplus\Lambda_1,\varphi_{44}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Lambda_1=n}(191x_4^2 - 4(F_1\oplus\Lambda_1))q^n,$$

$$\Theta_{F_1\oplus\Upsilon_1,\varphi_{11}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Upsilon_1=n}(191x_1^2 - 48(F_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{F_1\oplus\Upsilon_1,\varphi_{34}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Upsilon_1=n}(191x_3x_4 + \frac{3}{2}(F_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{F_1\oplus\Omega_1,\varphi_{12}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Omega_1=n}(191x_1x_2 + \frac{1}{2}(F_1\oplus\Omega_1))q^n,$$

$$\Theta_{F_1\oplus\Omega_1,\varphi_{33}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Omega_1=n}(191x_3^2 - 10(F_1\oplus\Omega_1))q^n,$$

$$\Theta_{F_1\oplus\Pi_1,\varphi_{22}}(q) = \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Pi_1=n}(191x_2^2 - (F_1\oplus\Pi_1))q^n,$$

$$\Theta_{F_1\oplus\Pi_1,\varphi_{34}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_1\oplus\Pi_1=n}(191x_3x_4+\frac{5}{2}(F_1\oplus\Pi_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{11}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}(191x_1^2-24(\Phi_1\oplus\Psi_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Psi_1=n}(191x_2^2-2(\Phi_1\oplus\Psi_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Lambda_1,\varphi_{12}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Lambda_1=n}(191x_1x_2+\frac{1}{2}(\Phi_1\oplus\Lambda_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Lambda_1,\varphi_{33}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Lambda_1=n}(191x_3^2-12(\Phi_1\oplus\Lambda_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Upsilon_1,\varphi_{12}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Upsilon_1=n}(191x_1x_2+\frac{1}{2}(\Phi_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Upsilon_1,\varphi_{22}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Upsilon_1=n}(191x_2^2-2(\Phi_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Omega_1,\varphi_{33}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Omega_1=n}(191x_3^2-8(\Phi_1\oplus\Omega_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Omega_1,\varphi_{34}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Omega_1=n}(191x_3x_4+\frac{1}{2}(\Phi_1\oplus\Omega_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Pi_1,\varphi_{11}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Pi_1=n}(191x_1^2-24(\Phi_1\oplus\Pi_1))q^n,$$

$$\Theta_{\Phi_1\oplus\Pi_1,\varphi_{33}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Phi_1\oplus\Pi_1=n}(191x_3^2-9(\Phi_1\oplus\Pi_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Lambda_1,\varphi_{12}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Lambda_1=n}(191x_1x_2+\frac{1}{2}(\Psi_1\oplus\Lambda_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Lambda_1,\varphi_{22}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Lambda_1=n}(191x_2^2-3(\Psi_1\oplus\Lambda_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Upsilon_1,\varphi_{33}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Upsilon_1=n}(191x_3^2-10(\Psi_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Upsilon_1,\varphi_{44}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Upsilon_1=n}(191x_4^2-5(\Psi_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Omega_1,\varphi_{12}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Omega_1=n}(191x_1x_2+\frac{1}{2}(\Psi_1\oplus\Omega_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Omega_1,\varphi_{33}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Omega_1=n}(191x_3^2-8(\Psi_1\oplus\Omega_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Pi_1,\varphi_{11}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Pi_1=n}(191x_1^2-16(\Psi_1\oplus\Pi_1))q^n,$$

$$\Theta_{\Psi_1\oplus\Pi_1,\varphi_{34}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Psi_1\oplus\Pi_1=n}(191x_3x_4+\frac{5}{2}(\Psi_1\oplus\Pi_1))q^n,$$

$$\Theta_{\Lambda_1\oplus\Upsilon_1,\varphi_{11}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Lambda_1\oplus\Upsilon_1=n}(191x_1^2-12(\Lambda_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{\Lambda_1\oplus\Upsilon_1,\varphi_{22}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Lambda_1\oplus\Upsilon_1=n}(191x_2^2-4(\Lambda_1\oplus\Upsilon_1))q^n,$$

$$\Theta_{\Lambda_1\oplus\Omega_1,\varphi_{33}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Lambda_1\oplus\Omega_1=n}(191x_3^2-8(\Lambda_1\oplus\Omega_1))q^n,$$

$$\Theta_{\Lambda_1\oplus\Omega_1,\varphi_{44}}(q)=\frac{1}{191}\sum_{n=1}^{\infty}\sum_{\Lambda_1\oplus\Omega_1=n}(191x_4^2-6(\Lambda_1\oplus\Omega_1))q^n.$$

Proof. $F_2=x_1^2+x_1x_2+48x_2^2+x_3^2+x_3x_4+48x_4^2=n$ has the following solutions;
$n=1\Rightarrow$ the solutions are; $(\pm1,0,0,0),(0,0,\pm1,0)$,
$n=2\Rightarrow$ the solutions are;$(\pm1,0,\pm1,0)$,
$n=4\Rightarrow$ the solutions are; $(\pm2,0,0,0),(0,0,\pm2,0)$,
$n=5\Rightarrow$ the solutions are;$(\pm2,0,\pm1,0),(\pm1,0,\pm2,0)$,
$n=8\Rightarrow$ the solutions are;$(\pm2,0,\pm2,0)$,
$n=9\Rightarrow$ the solutions are;$(\pm3,0,0,0),(0,0,\pm3,0)$,
$n=10\Rightarrow$ the solutions are;$(\pm3,0,\pm1,0),(\pm1,0,\pm3,0)$,
$n=13\Rightarrow$ the solutions are;$(\pm3,0,\pm2,0),(\pm2,0,\pm3,0)$,
$n=16\Rightarrow$ the solutions are;$(\pm4,0,0,0),(0,0,\pm4,0)$,
$n=17\Rightarrow$ the solutions are;$(\pm4,0,\pm1,0),(\pm1,0,\pm4,0)$,
$n=18\Rightarrow$ the solutions are;$(\pm3,0,\pm3,0)$,
$n=20\Rightarrow$ the solutions are;$(\pm4,0,\pm2,0),(\pm2,0,\pm4,0)$,
$n=25\Rightarrow$ the solutions are;$(\pm5,0,0,0),(0,0,\pm5,0),(\pm4,0,\pm3,0),(\pm3,0,\pm4,0)$,
$n=26\Rightarrow$ the solutions are;$(\pm5,0,\pm1,0),(\pm1,0,\pm5,0)$,
$n=29\Rightarrow$ the solutions are;$(\pm5,0,\pm2,0),(\pm2,0,\pm5,0)$,
$n=32\Rightarrow$ the solutions are;$(\pm4,0,\pm4,0)$,
$n=34\Rightarrow$ the solutions are;$(\pm5,0,\pm3,0),(\pm3,0,\pm5,0)$,
$n=36\Rightarrow$ the solutions are;$(\pm6,0,0,0),(0,0,\pm6,0)$,
$n=37\Rightarrow$ the solutions are;$(\pm6,0,\pm1,0),(\pm1,0,\pm6,0)$,

$n = 40 \Rightarrow$ the solutions are;$(\pm 6, 0, \pm 2, 0), (\pm 2, 0, \pm 6, 0)$,

$n = 41 \Rightarrow$ the solutions are;$(\pm 5, 0, \pm 4, 0), (\pm 4, 0, \pm 5, 0)$,

$n = 45 \Rightarrow$ the solutions are;$(\pm 6, 0, \pm 3, 0), (\pm 3, 0, \pm 6, 0)$,

and for n =3, 6, 7, 11, 12, 14, 15, 19, 21, 22, 23, 24, 27, 28, 30, 31, 33, 35, 38, 39, 42, 43, 44, 46 there is no integral solution. Hence;

$$
\begin{aligned}
\Theta_{F_2,\varphi_{11}}(q) =& \frac{1}{191}\sum_{n=1}^{\infty}\sum_{F_2=n}(191x_1^2 - 48F_2)q^n \\
=& \frac{1}{191}((191.2 - 48.4)q + (191.1.4 - 48.4.2)q^2 + (191.4.2 - 48.4.4)q^4 \\
& + (191.4.4 + 191.1.4 - 48.8.5)q^5 + (191.4.4 - 48.4.8)q^8 + (191.9.2 - 48.4.9)q^9 \\
& + (191.9.4 + 191.1.4 - 48.8.10)q^{10} + (191.9.4 + 191.4.4 - 48.8.13)q^{13} \\
& + (191.16.2 - 48.4.16)q^{16} + (191.16.4 + 191.1.4 - 48.8.17)q^{17} \\
& + (191.9.4 - 48.4.18)q^{18} + (191.16.4 + 191.4.4 - 48.8.20)q^{20} \\
& + (191.25.2 + 191.16.4 + 191.9.4 - 48.12.25)q^{25} \\
& + (191.25.4 + 191.1.4 - 48.8.26)q^{26} \\
& + (191.25.4 + 191.4.4 - 48.8.29)q^{29} + (191.16.4 - 48.4.32)q^{32} \\
& + (191.25.4 + 191.9.4 - 48.8.34)q^{34} + (191.36.2 - 48.4.36)q^{36} \\
& + (191.36.4 + 191.1.4 - 48.8.37)q^{37} + (191.36.4 + 191.4.4 - 48.8.40)q^{40} \\
& + (191.25.4 + 191.16.4 - 48.8.41)q^{41} + (191.36.4 + 191.9.4 - 48.8.45)q^{45} + \ldots)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Theta_{F_2,\varphi_{11}}(q) =& \frac{1}{191}(190q + 380q^2 + 760q^4 + 1900q^5 + 1520q^8 + 1710q^9 + 3800q^{10} \\
& + 4940q^{13} + 3040q^{16} + 6460q^{17} + 3420q^{18} + 7600q^{20} + 14250q^{25} + 9880q^{26} \\
& + 11020q^{29} + 6080q^{32} + 12920q^{34} + 6840q^{36} + 14060q^{37} + 15200q^{40} \\
& + 15580q^{41} + 17100q^{45} + \ldots)
\end{aligned}
$$

is obtained.

We obtained the remaining theta series by similar calculations. For a complete list of theta series see Table 1 in [5]. The Calculations in this article are done by using the software packages Pari GP and Maple.

The 47−th determinant of the coefficients of Theta Series is

$$
\begin{aligned}
&-541525623777652121697698053409485040217624613147555162030545924237 81\\
&224840137656462040609175707484907271200394042517917403141286962135 04\\
&00000000\frac{1}{191^{47}} \neq 0.
\end{aligned}
$$

So, the Theta series in Theorem 3 is a basis of $S_4(\Gamma_0(191))$.

5. Representation Numbers of n

Proposition 1. *The differences between the Theta series of the quadratic forms in* (1) *(in these direct sums any form can be replaced by its inverse) and the Eisenstein Series*

$$\begin{aligned}
E(q \quad &: \quad Q) = 1 + \frac{120}{18241}\sum_{n=1}^{\infty}(q^n + 191^2 q^{191n})\sigma_3(n) = \frac{120}{18241}\sum_{n=1}^{\infty}\sigma_3^*(n)q^n \\
&= 1 + \tfrac{120}{18241}q + \tfrac{120.9}{18241}q^2 + \tfrac{120.28}{18241}q^3 + \tfrac{120.73}{18241}q^4 + \tfrac{120.126}{18241}q^5 + \tfrac{120.252}{18241}q^6 + \tfrac{120.344}{18241}q^7 \\
&\quad + \tfrac{120.585}{18241}q^8 + \tfrac{120.757}{18241}q^9 + \tfrac{120.1134}{18241}q^{10} + \tfrac{120.1332}{18241}q^{11} + \tfrac{120.2044}{18241}q^{12} + \tfrac{120.2198}{18241}q^{13} + \\
&\quad + \tfrac{120.3096}{18241}q^{14} + \tfrac{120.3528}{18241}q^{15} + \tfrac{120.4681}{18241}q^{16} + \tfrac{120.4914}{18241}q^{17} + \tfrac{120.6813}{18241}q^{18} + \tfrac{120.6860}{18241}q^{19} \\
&\quad + \tfrac{120.9198}{18241}q^{20} + \tfrac{120.9632}{18241}q^{21} + \tfrac{120.11988}{18241}q^{22} + \tfrac{120.12168}{18241}q^{23} + \tfrac{120.16380}{18241}q^{24} \\
&\quad + \tfrac{120.15751}{18241}q^{25} + \tfrac{120.19782}{18241}q^{26} + \tfrac{120.20440}{18241}q^{27} + \tfrac{120.25112}{18241}q^{28} + \tfrac{120.24390}{18241}q^{29} \\
&\quad + \tfrac{120.31752}{18241}q^{30} + \tfrac{120.29792}{18241}q^{31} + \tfrac{120.37449}{18241}q^{32} + \tfrac{120.37296}{18241}q^{33} + \tfrac{120.44226}{18241}q^{34} \\
&\quad + \tfrac{120.43344}{18241}q^{35} + \tfrac{120.55261}{18241}q^{36} + \tfrac{120.50654}{18241}q^{37} + \tfrac{120.61740}{18241}q^{38} + \tfrac{120.61544}{18241}q^{39} \\
&\quad + \tfrac{120.73710}{18241}q^{40} + \tfrac{120.68922}{18241}q^{41} + \tfrac{120.86688}{18241}q^{42} + \tfrac{120.79508}{18241}q^{43} + \tfrac{120.97236}{18241}q^{44} \\
&\quad + \tfrac{120.95382}{18241}q^{45} + \tfrac{120.109512}{18241}q^{46} + \tfrac{120.103824}{18241}q^{47} + \ldots
\end{aligned}$$

are linear combination of the Theta Series in the preceding theorem.

Proof. Now we will consider the case;

$$\begin{aligned}
\Theta_{F_4} - E(q:F_4) =& c_1\Theta_{F_2,\varphi_{11}}(q) + c_2\Theta_{\Phi_2,\varphi_{11}}(q) + c_3\Theta_{\Phi_2,\varphi_{12}}(q) + c_4\Theta_{\Psi_2,\varphi_{22}}(q) \\
&+ c_5\Theta_{\Psi_2,\varphi_{33}}(q) + c_6\Theta_{\Lambda_2,\varphi_{11}}(q) + c_7\Theta_{\Lambda_2,\varphi_{12}}(q) + c_8\Theta_{\Upsilon_2,\varphi_{11}}(q) + c_9\Theta_{\Upsilon_2,\varphi_{22}}(q) \\
&+ c_{10}\Theta_{\Omega_2,\varphi_{12}}(q) + c_{11}\Theta_{\Omega_2,\varphi_{22}}(q) + c_{12}\Theta_{\Pi_2,\varphi_{11}}(q) + c_{13}\Theta_{\Pi_2,\varphi_{22}}(q) \\
&+ c_{14}\Theta_{F_1\oplus\Phi_1,\varphi_{12}}(q) + c_{15}\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q) + c_{16}\Theta_{F_1\oplus\Psi_1,\varphi_{22}}(q) + c_{17}\Theta_{F_1\oplus\Psi_1,\varphi_{34}}(q) \\
&+ c_{18}\Theta_{F_1\oplus\Lambda_1,\varphi_{12}}(q) + c_{19}\Theta_{F_1\oplus\Lambda_1,\varphi_{44}}(q) + c_{20}\Theta_{F_1\oplus\Upsilon_1,\varphi_{11}}(q) + c_{21}\Theta_{F_1\oplus\Upsilon_1,\varphi_{34}}(q) \\
&+ c_{22}\Theta_{F_1\oplus\Omega_1,\varphi_{12}}(q) + c_{23}\Theta_{F_1\oplus\Omega_1,\varphi_{33}}(q) + c_{24}\Theta_{F_1\oplus\Pi_1,\varphi_{22}}(q) + c_{25}\Theta_{F_1\oplus\Pi_1,\varphi_{34}}(q) \\
&+ c_{26}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{11}}(q) + c_{27}\Theta_{\Phi_1\oplus\Psi_1,\varphi_{22}}(q) + c_{28}\Theta_{\Phi_1\oplus\Lambda_1,\varphi_{12}}(q) + c_{29}\Theta_{\Phi_1\oplus\Lambda_1,\varphi_{33}}(q) \\
&+ c_{30}\Theta_{\Phi_1\oplus\Upsilon_1,\varphi_{12}}(q) + c_{31}\Theta_{\Phi_1\oplus\Upsilon_1,\varphi_{22}}(q) + c_{32}\Theta_{\Phi_1\oplus\Omega_1,\varphi_{33}}(q) + c_{33}\Theta_{\Phi_1\oplus\Omega_1,\varphi_{34}}(q) \\
&+ c_{34}\Theta_{\Phi_1\oplus\Pi_1,\varphi_{11}}(q) + c_{35}\Theta_{\Phi_1\oplus\Pi_1,\varphi_{33}}(q) + c_{36}\Theta_{\Psi_1\oplus\Lambda_1,\varphi_{12}}(q) + c_{37}\Theta_{\Psi_1\oplus\Lambda_1,\varphi_{22}}(q) \\
&+ c_{38}\Theta_{\Psi_1\oplus\Upsilon_1,\varphi_{33}}(q) + c_{39}\Theta_{\Psi_1\oplus\Upsilon_1,\varphi_{44}}(q) + c_{40}\Theta_{\Psi_1\oplus\Omega_1,\varphi_{12}}(q) + c_{41}\Theta_{\Psi_1\oplus\Omega_1,\varphi_{33}}(q) \\
&+ c_{42}\Theta_{\Psi_1\oplus\Pi_1,\varphi_{11}}(q) + c_{43}\Theta_{\Psi_1\oplus\Pi_1,\varphi_{34}}(q) + c_{44}\Theta_{\Lambda_1\oplus\Upsilon_1,\varphi_{11}}(q) + c_{45}\Theta_{\Lambda_1\oplus\Upsilon_1,\varphi_{22}}(q) \\
&+ c_{46}\Theta_{\Lambda_1\oplus\Omega_1,\varphi_{33}}(q) + c_{47}\Theta_{\Lambda_1\oplus\Omega_1,\varphi_{44}}(q) \\
=& (145808/18241)q + (436704/18241)q^2 + (580352/18241)q^3 \\
&+ (429024/18241)q^4 + (860448/18241)q^5 + (1720896/18241)q^6 \\
&+ (1126144/18241)q^7 + (367584/18241)q^8 + (1806224/18241)q^9
\end{aligned}$$

$$
\begin{aligned}
&+(2490624/18241)q^{10}+(43008/493)q^{11}+(1505856/18241)q^{12}\\
&+(1779232/18241)q^{13}+(3130752/18241)q^{14}+(3078912/18241)q^{15}\\
&-(123936/18241)q^{16}+(2037024/18241)q^{17}+(4873632/18241)q^{18}\\
&+(2095360/18241)q^{19}+(1522944/18241)q^{20}+(3513856/18241)q^{21}\\
&+(103104/493)q^{22}+(2042112/18241)q^{23}-(214464/18241)q^{24}\\
&+(2633648/18241)q^{25}+(3755136/18241)q^{26}+(3384320/18241)q^{27}\\
&+(488832/18241)q^{28}+(1451040/18241)q^{29}+(6696576/18241)q^{30}\\
&+(1094656/18241)q^{31}-(4056096/18241)q^{32}+(68352/493)q^{33}\\
&+(2572992/18241)q^{34}+(1803264/18241)q^{35}-(940128/18241)q^{36}\\
&-(533216/18241)q^{37}+(1346880/18241)q^{38}+(786688/18241)q^{39}\\
&-(6218496/18241)q^{40}-(2141664/18241)q^{41}+(3606528/18241)q^{42}\\
&-(3120128/18241)q^{43}-(173376/493)q^{44}-(63456/18241)q^{45}\\
&-(2634624/18241)q^{46}-(5454336/18241)q^{47}+\dots.
\end{aligned}
$$

By equating the coefficients of q^n in both sides for $n = 1,2,3,\dots,47$, we get an equation in coefficients c_i for $i = 1,\dots 47$. For the list of coefficients of any form in (1) see Table 2 in [5].

Corollary 2. *The representation numbers $r(n,F_4)$ are*

$$
\begin{aligned}
\Theta_{F_4}-E(q:F_4)=&\frac{120}{18241}\sigma_3^*(n)+\frac{1}{191}(c_1\sum_{F_2=n}(191x_1^2-48F_2)+c_2\sum_{\Phi_2=n}(191x_1^2-24\Phi_2)\\
&+c_3\sum_{\Phi_2=n}(191x_1x_2+\frac{1}{2}\Phi_2)+c_4\sum_{\Psi_2=n}(191x_2^2-3\Psi_2)+c_5\sum_{\Psi_2=n}(191x_3^2-16\Psi_2)\\
&+c_6\sum_{\Lambda_2=n}(191x_1^2-12\Lambda_2)+c_7\sum_{\Lambda_2=n}(191x_1x_2+\frac{1}{2}\Lambda_2)+c_8\sum_{\Upsilon_2=n}(191x_1^2-10\Upsilon_2)\\
&+c_9\sum_{\Upsilon_2=n}(191x_2^2-5\Upsilon_2)+c_{10}\sum_{\Omega_2=n}(191x_1x_2+\frac{1}{2}\Omega_2)+c_{11}\sum_{\Omega_2=n}(191x_2^2-6\Omega_2)\\
&+c_{12}\sum_{\Pi_2=n}(191x_1^2-9\Pi_2)+c_{13}\sum_{\Pi_2=n}(191x_2^2-6\Pi_2)\\
&+c_{14}\sum_{F_1\oplus\Phi_1=n}(191x_1x_2+\frac{1}{2}(F_1\oplus\Phi_1))+c_{15}\sum_{F_1\oplus\Phi_1=n}(191x_3^2-24(F_1\oplus\Phi_1))\\
&+c_{16}\sum_{F_1\oplus\Psi_1=n}(191x_2^2-(F_1\oplus\Psi_1))+c_{17}\sum_{F_1\oplus\Psi_1=n}(191x_3x_4+\frac{1}{2}(F_1\oplus\Psi_1))\\
&+c_{18}\sum_{F_1\oplus\Lambda_1=n}(191x_1x_2+\frac{1}{2}(F_1\oplus\Lambda_1))+c_{19}\sum_{F_1\oplus\Lambda_1=n}(191x_4^2-4(F_1\oplus\Lambda_1))
\end{aligned}
$$

$$+ c_{20} \sum_{F_1 \oplus \Upsilon_1 = n} (191x_1^2 - 48(F_1 \oplus \Upsilon_1)) + c_{21} \sum_{F_1 \oplus \Upsilon_1 = n} (191x_3x_4 + \frac{3}{2}(F_1 \oplus \Upsilon_1))$$
$$+ c_{22} \sum_{F_1 \oplus \Omega_1 = n} (191x_1x_2 + \frac{1}{2}(F_1 \oplus \Omega_1)) + c_{23} \sum_{F_1 \oplus \Omega_1 = n} (191x_3^2 - 10(F_1 \oplus \Omega_1))$$
$$+ c_{24} \sum_{F_1 \oplus \Pi_1 = n} (191x_2^2 - (F_1 \oplus \Pi_1)) + c_{25} \sum_{F_1 \oplus \Pi_1 = n} (191x_3x_4 + \frac{5}{2}(F_1 \oplus \Pi_1))$$
$$+ c_{26} \sum_{\Phi_1 \oplus \Psi_1 = n} (191x_1^2 - 24(\Phi_1 \oplus \Psi_1)) + c_{27} \sum_{\Phi_1 \oplus \Psi_1 = n} (191x_2^2 - 2(\Phi_1 \oplus \Psi_1))$$
$$+ c_{28} \sum_{\Phi_1 \oplus \Lambda_1 = n} (191x_1x_2 + \frac{1}{2}(\Phi_1 \oplus \Lambda_1)) + c_{29} \sum_{\Phi_1 \oplus \Lambda_1 = n} (191x_3^2 - 12(\Phi_1 \oplus \Lambda_1))$$
$$+ c_{30} \sum_{\Phi_1 \oplus \Upsilon_1 = n} (191x_1x_2 + \frac{1}{2}(\Phi_1 \oplus \Upsilon_1)) + c_{31} \sum_{\Phi_1 \oplus \Upsilon_1 = n} (191x_2^2 - 2(\Phi_1 \oplus \Upsilon_1))$$
$$+ c_{32} \sum_{\Phi_1 \oplus \Omega_1 = n} (191x_3^2 - 8(\Phi_1 \oplus \Omega_1))) + c_{33} \sum_{\Phi_1 \oplus \Omega_1 = n} (191x_3x_4 + \frac{1}{2}(\Phi_1 \oplus \Omega_1)$$
$$+ c_{34} \sum_{\Phi_1 \oplus \Pi_1 = n} (191x_1^2 - 24(\Phi_1 \oplus \Pi_1)) + c_{35} \sum_{\Phi_1 \oplus \Pi_1 = n} (191x_3^2 - 9(\Phi_1 \oplus \Pi_1))$$
$$+ c_{36} \sum_{\Psi_1 \oplus \Lambda_1 = n} (191x_1x_2 + \frac{1}{2}(\Psi_1 \oplus \Lambda_1)) + c_{37} \sum_{\Psi_1 \oplus \Lambda_1 = n} (191x_2^2 - 3(\Psi_1 \oplus \Lambda_1))$$
$$+ c_{38} \sum_{\Psi_1 \oplus \Upsilon_1 = n} (191x_3^2 - 10(\Psi_1 \oplus \Upsilon_1)) + c_{39} \sum_{\Psi_1 \oplus \Upsilon_1 = n} (191x_4^2 - 5(\Psi_1 \oplus \Upsilon_1))$$
$$+ c_{40} \sum_{\Psi_1 \oplus \Omega_1 = n} (191x_1x_2 + \frac{1}{2}(\Psi_1 \oplus \Omega_1)) + c_{41} \sum_{\Psi_1 \oplus \Omega_1 = n} (191x_3^2 - 8(\Psi_1 \oplus \Omega_1))$$
$$+ c_{42} \sum_{\Psi_1 \oplus \Pi_1 = n} (191x_1^2 - 16(\Psi_1 \oplus \Pi_1)) + c_{43} \sum_{\Psi_1 \oplus \Pi_1 = n} (191x_3x_4 + \frac{5}{2}(\Psi_1 \oplus \Pi_1))$$
$$+ c_{44} \sum_{\Lambda_1 \oplus \Upsilon_1 = n} (191x_1^2 - 12(\Lambda_1 \oplus \Upsilon_1)) + c_{45} \sum_{\Lambda_1 \oplus \Omega_1 = n} (191x_2^2 - 4(\Lambda_1 \oplus \Upsilon_1))$$
$$+ c_{46} \sum_{\Lambda_1 \oplus \Omega_1 = n} (191x_3^2 - 8(\Lambda_1 \oplus \Omega_1)) + c_{47} \sum_{\Lambda_1 \oplus \Omega_1 = n} (191x_4^2 - 6(\Lambda_1 \oplus \Omega_1))).$$

The coefficients are the same coefficients with preceding theorem, see Table 2 in [5].

Proof. It follows from the preceding theorem.

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