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The Drazin Inverses of Combinations of Two Idempotents

Tao Xie*, Kezheng Zuo

Math Department, Hubei Normal University, Hubei, Huangshi, 435002, China

Abstract. By using the methods of splitting operator's matrix into blocks and space decompositions, the existence and calculation formulas of Drazin inverse of the combinations $aP+bQ+cPQ+dQP$ of two idempotent operators P and Q on a Hilbert space are obtained under the conditions $PQP=0, PQP=P$ and $PQP=PQ$ respectively. These generalized the related results of Deng's work, which characterized the Drazin inverse of the sum and difference of two idempotents.

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1. Preliminaries

Let $\mathscr{H}$ be a Hilbert space, the set of all bounded linear operators on $\mathscr{H}$ is denoted by $\mathbf{B}(\mathscr{H})$. For an operator $T\in\mathbf{B}(\mathscr{H})$, $\mathscr{N}(T)$ and $\mathscr{R}(T)$ denote the null space and the range of T, respectively. An operator $P\in\mathbf{B}(\mathscr{H})$ is said to be idempotent if $P^2=P$. If P satisfies $P^2=P=P^*$ then P is called orthogonal projector, where P^* is the conjugate operator of $P\in\mathbf{B}(\mathscr{H})$. Let $T\in\mathbf{B}(\mathscr{H})$, if there exists an operator $T^D\in\mathbf{B}(\mathscr{H})$ and nonnegative integer k such that

$$TT^D=T^DT,\ \ T^DTT^D=T^D,\ \ T^{k+1}T^D=T^k,$$

then T^D is called a Drazin inverse of T. The least integer k such that the above identities are hold is called the index of T, which is denoted by $\operatorname{ind}(T)=k$. Specifically, if $k=0$, then T is invertible and $T^D=T^{-1}$. For Drazin invertible operator $T\in\mathbf{B}(\mathscr{H})$, the Drazin inverse T^D of T is unique [13].

The set of all idempotents in $\mathbf{B}(\mathscr{H})$ is invariant under similarity, that is, if P is an idempotent operator and $S\in\mathbf{B}(\mathscr{H})$ is an invertible operator, then $S^{-1}PS$ is also an idempotent operator. Moreover the Drazin invertibility is also invariant under similarity, that is, if T is Drazin invertible and S is invertible, then $S^{-1}TS$ is Drazin invertible and $(S^{-1}TS)^D=S^{-1}T^DT$. Two facts are well known on a Hilbert space, one is that the orthogonal operator P is Drazin invertible and $P^D=P$, another is that for any idempotent operator P, there exists an invertible

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[*] Corresponding author.

Email addresses: `xietao_1294@163.com` (T. Xie), `xiangzuo28@yahoo.cn` (K. Zuo)

operator S such that $S^{-1}PS$ is an orthogonal projector [13]. In the following discussion, given two idempotent operators P and Q on $\mathscr{H}$, without loss of generality, we may assume that P is orthogonal.

The concept of a Drazin inverse was shown to be very useful in various applied mathematical settings which can be found in references [2, 7, 9, 10].

The problem of finding the Drazin inverse $(P \pm Q)^D$ of the sum and difference of two idempotents P and Q was first considered by Drazin in 1958 in his celebrated paper [3]. Herein, it was proved that

$$(P+Q)^D = P^D + Q^D \text{ provided } PQ = QP = 0.$$

The general question of how to express $(P+Q)^D$ as a function of P, Q, P^D, Q^D, without side condition, is very difficult and remains open [8].

In 2009, Deng extended Drazin's result to the three different cases

$$(i) PQP = 0;\ \ (ii) PQP = P;\ \ (iii) PQP = PQ,$$

see [4]. These cases are useful in several applications, such as in the splitting of operators and iteration theory. Zhang and Wu discussed the Drazin inverse of the linear combinations of two idempotents in a Banach algebras and represent the Drazin inverse as a function of P, Q, PQ, QP, PQP, QPQ [14].

In 2010, Zuo considered a special combination $aP + bQ - cPQ$ of two idempotent matrices over complex numbers, and obtained that

$$r(aP + bQ - cPQ) = \begin{cases} r(P-Q), & \text{when} c = a+b \\ r(P+Q), & \text{when} c \neq a+b, \end{cases}$$

where $r(A)$ represents the rank of the matrix A [15]. Later, Xie and Zuo found that the Fredholmness of $aP + bQ - cPQ$ is independent of choices of scalars $a, b, c \in \mathbb{C}$ with $ab \neq 0$ [12]. After that, Liu, Wu and Yu discussed the group invertibility of combinations of two idempotents and represent the group inverse as a function of P, Q, PQ, QP, PQP, QPQ [11].

Under the above works, we consider the Drazin invertibility of combinations $aP + bQ + cPQ + dQP$ of two idempotent operators P and Q on $\mathscr{H}$. Under the conditions $PQP = 0$, $PQP = P$ and $PQP = PQ$, the representations for the Drazin inverse of $aP + bQ + cPQ + dQP$ as a functions of P, Q, PQ, QP, PQP, QPQ are obtained by using the technique of splitting matrices into blocks and space decompositions.

The following two Lemmas which were proved for a bounded linear operator [5] and for arbitrary elements in a Banach algebra [1].

Lemma 1. *Let $A \in \mathbf{B}(X), B \in \mathbf{B}(Y)$ and $C \in \mathbf{B}(Y,X)$. If A and B are Drazin invertible, then*

$$M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \quad N = \begin{pmatrix} B & 0 \\ C & A \end{pmatrix}$$

are Drazin invertible and

$$M^D = \begin{pmatrix} A^D & X \\ 0 & B^D \end{pmatrix}, \quad N^D = \begin{pmatrix} B^D & 0 \\ X & A^D \end{pmatrix},$$

where $X = (A^D)^2[\sum_{i=0}^{\infty}(A^D)^i C B^i](I - BB^D) + (I - AA^D)[\sum_{i=0}^{\infty} A^i C (B^D)^i](B^D)^2 - A^D C B^D$.

Lemma 2. *Let* $A \in \mathbf{B}(X), B \in \mathbf{B}(Y)$ *and* $C \in \mathbf{B}(Y,X)$*. If* A *is invertible and* $B^k = 0$*, then*

$$M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$$

are Drazin invertible and

$$M^D = \begin{pmatrix} A^{-1} & 0 \\ X & 0 \end{pmatrix},$$

where $X = \sum_{i=0}^{k-1} B^{k-1-i} C A^{i-k-1}$.

Lemma 3 (see [6])**.** *Let* $A, B \in \mathbf{B}(\mathscr{H})$*. Then the following conditions are equivalent.*

(i) $\mathscr{R}(B) \subseteq \mathscr{R}(A)$*;*

(ii) There exists $D \in \mathbf{B}(\mathscr{H})$ *such that* $B = AD$*.*

2. Main results

Theorem 1. *Let* P *and* Q *be two idempotents in* $B(\mathscr{H})$*, and* $a, b, c, d \in \mathbb{C}, ab \neq 0$*. If* $PQP = 0$*, then* $aP + bQ + cPQ + dQP$ *is Drazin invertible and*

$$\begin{aligned}(aP + bQ + cPQ + dQP)^D =& \frac{1}{a}P + \frac{1}{b}Q - (\frac{1}{a} + \frac{1}{b} + \frac{c}{ab})PQ - (\frac{1}{a} + \frac{1}{b} + \frac{d}{ab})QP \\ &+ (\frac{1}{a} + \frac{2}{b} + \frac{c}{ab} + \frac{d}{ab} + \frac{cd}{ab^2})QPQ.\end{aligned}$$

Proof. Let P and Q be two idempotent operators in $B(\mathscr{H})$. With out loss of generality, we assume that P is an orthogonal projector. By Lemma 3, the condition $PQP = 0$ implies that $\mathscr{R}(QP) \subseteq \mathscr{N}(P)$ and $\mathscr{R}(QP) \subseteq \mathscr{R}(Q)$. Observing that $Q(\overline{\mathscr{R}(QP)} \oplus \mathscr{R}(P)) \subseteq \overline{\mathscr{R}(QP)}$, the space $\mathscr{H}$ can be decomposed as

$$\mathscr{H} = \overline{\mathscr{R}(QP)} \oplus \mathscr{R}(P) \oplus (\mathscr{R}(QP)^{\perp} \ominus \mathscr{R}(P)).$$

Then P and Q can be represented as

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_{12} & Q_{13} \\ 0 & 0 & Q_{23} \\ 0 & 0 & Q_{33} \end{pmatrix},$$

where $\overline{\mathscr{R}(QP)}$ denotes the closure of $\mathscr{R}(QP)$. On the other hand, $Q^2 = Q$ gives that $Q_{33}^2 = Q_{33}$ and $\mathscr{R}(QP)^\perp \ominus \mathscr{R}(P) = \mathscr{R}(Q_{33}) \oplus \mathscr{R}(Q_{33})^\perp$. It follows that P and Q can be written as

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_{12} & Q'_{13} & Q''_{13} \\ 0 & 0 & Q'_{23} & Q''_{23} \\ 0 & 0 & I & Q''_{33} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

under the space decomposition $\mathscr{H} = \overline{\mathscr{R}(QP)} \oplus \mathscr{R}(P) \oplus \mathscr{R}(Q_{33})\mathscr{R}(Q_{33})^\perp$. The idempotency of Q implies that

$$Q'_{23}Q''_{33} = Q''_{23}, \;\; Q_{12}Q'_{23} + Q'_{13} = 0, \;\; Q_{12}Q''_{23} + Q'_{13}Q''_{33} = 0.$$

Direct calculations show that

$$aP + bQ + cPQ + dQP = \begin{pmatrix} bI & (b+d)Q_{12} & bQ'_{13} & bQ''_{13} \\ 0 & aI & (b+c)Q'_{23} & (b+c)Q''_{23} \\ 0 & 0 & bI & bQ''_{33} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It is clear that the condition $a, b \neq 0$ implies the invertibility of

$$\begin{pmatrix} bI & (b+d)Q_{12} & bQ'_{13} \\ 0 & aI & (b+c)Q'_{23} \\ 0 & 0 & bI \end{pmatrix}$$

on $\overline{\mathscr{R}(QP)} \oplus \mathscr{R}(P) \oplus \mathscr{R}(Q_{33})$ and its inverse is

$$\begin{pmatrix} \frac{1}{b}I & -\frac{b+d}{ab}Q_{12} & -[\frac{(b+c)(b+d)+ab}{ab^2}]Q'_{13} \\ 0 & \frac{1}{a}I & -\frac{b+c}{ab}Q'_{23} \\ 0 & 0 & \frac{1}{b}I \end{pmatrix}.$$

Moreover,

$$\begin{pmatrix} \frac{1}{b}I & -\frac{b+d}{ab}Q_{12} & -[\frac{(b+c)(b+d)+ab}{ab^2}]Q'_{13} \\ 0 & \frac{1}{a}I & -\frac{b+c}{ab}Q'_{23} \\ 0 & 0 & \frac{1}{b}I \end{pmatrix}^2 \begin{pmatrix} bQ''_{13} \\ (b+c)Q''_{23} \\ bQ''_{33} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{b}Q''_{13} - [\frac{(b+c)(b+d)}{ab^2} + \frac{2}{b}]Q'_{13}Q''_{33} \\ -(\frac{1}{a} + \frac{c}{ab})Q''_{23} \\ \frac{1}{b}Q''_{33} \end{pmatrix}.$$

Applying $B = 0$ to the formula of representing Drazin inverse of upper triangle block matrix in Lemma 1, we have

$$(aP + bQ + cPQ + dQP)^D =$$

$$\begin{pmatrix} \frac{1}{b}I & -\frac{b+d}{ab}Q_{12} & -\frac{(b+c)(b+d)+ab}{ab^2}Q'_{13} & \frac{1}{b}Q''_{13}-[\frac{(b+c)(b+d)}{ab^2}+\frac{2}{b}]Q'_{13}Q''_{33} \\ 0 & \frac{1}{a}I & -\frac{b+c}{ab}Q'_{23} & -\frac{b+c}{ab}Q''_{23} \\ 0 & 0 & \frac{1}{b}I & \frac{1}{b}Q''_{33} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover, through direct calculations, we have

$$PQ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Q'_{23} & Q''_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad QP = \begin{pmatrix} 0 & Q_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$QPQ = \begin{pmatrix} 0 & 0 & Q_{12}Q'_{23} & Q_{12}Q''_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$\begin{aligned}(aP+bQ+cPQ+dQP)^D =& \frac{1}{a}P+\frac{1}{b}Q-(\frac{1}{a}+\frac{1}{b}+\frac{c}{ab})PQ-(\frac{1}{a}+\frac{1}{b}+\frac{d}{ab})QP \\ &+(\frac{1}{a}+\frac{2}{b}+\frac{c}{ab}+\frac{d}{ab}+\frac{cd}{ab^2})QPQ.\end{aligned}$$

Now we can derive some special cases from Theorem 1. These results are also the cases of Theorem 2.1 in [4].

Corollary 1. *Let P and Q be two idempotents in $B(\mathscr{H})$. Assume that $PQP = 0$, then the following statements hold.*

(i) $(P+Q)^D = P+Q-2(PQ+QP)+3QPQ$.

(ii) $(P-Q)^D = P-Q-QPQ$.

If either of the stronger condition $PQ = 0$ or $QP = 0$ is satisfied, then by Theorem 1, we obtain the following results.

Corollary 2. *Let P and Q be two idempotents in $B(\mathscr{H})$ and $a, b \in \mathbb{C}, ab \neq 0$. Then the following statements hold.*

(i) If $QP = 0$, *then* $(aP+bQ)^D = \frac{1}{a}P+\frac{1}{b}Q-(\frac{1}{a}+\frac{1}{b})PQ$.

(ii) If $PQ = 0$, *then* $(aP+bQ)^D = \frac{1}{a}P+\frac{1}{b}Q-(\frac{1}{a}+\frac{1}{b})QP$.

Next we discuss the Drazin inverse of $aP+bQ+cPQ+dQP$ under the assumption that $PQP = P$.

Theorem 2. *Let P and Q be two idempotents in $B(\mathscr{H})$, then for any $a,b,c,d \in \mathbb{C}, ab \neq 0$, the combinations $aP+bQ+cPQ+dQP$ are Drazin invertible under the condition $PQP = P$. The Drazin inverses of $aP+bQ+cPQ+dQP$ can be represented as following:*

(i) If $a+b+c+d \neq 0$, then

$$\begin{aligned}(aP+bQ+cPQ+dQP)^D =& \frac{(a+c)(a+d)}{(a+b+c+d)^3}P + \frac{1}{b}Q + \frac{(b+c)(a+c)}{(a+b+c+d)^3}PQ \\ &+ \frac{(a+d)(b+d)}{(a+b+c+d)^3}QP + [\frac{(b+c)(b+d)}{(a+b+c+d)^3} - \frac{1}{b}]QPQ.\end{aligned}$$

(ii) If $a+b+c+d = 0$, then

$$(aP+bQ+cPQ+dQP)^D = \frac{1}{b}(Q-QPQ).$$

Proof. If $PQP = P$, then P and Q can be written as

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & Q_1 \\ Q_2 & Q_3 \end{pmatrix}$$

under the space decomposition of $\mathscr{H} = \mathscr{R}(P) \oplus \mathscr{R}(P)^\perp$. The idempotency of Q yields that $Q_1Q_2 = 0$, $Q_1Q_3 = 0$, $Q_3Q_2 = 0$ and $Q_2Q_1 + Q_3^2 = Q_3$. It follows that $\mathscr{R}(Q_2) \subseteq \mathscr{N}(Q_1)$, $\mathscr{R}(Q_2) \subseteq \mathscr{N}(Q_3)$, $\mathscr{R}(Q_3) \subseteq \mathscr{N}(Q_1)$. With respect to the space decomposition $\mathscr{H} = \overline{\mathscr{R}(Q_1)} \oplus \mathscr{R}(Q_1)^\perp \oplus \overline{\mathscr{R}(Q_2)} \oplus \mathscr{R}(Q_2)^\perp$, P and Q can be represented as

$$P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 & 0 & Q_{11}' \\ 0 & I & 0 & 0 \\ Q_{21} & Q_{22} & 0 & Q_{31} \\ 0 & 0 & 0 & Q_{32} \end{pmatrix},$$

where $Q_{11}Q_{32} = 0, Q_{32}^2 = Q_{32}$ and $Q_{21}Q_{11} + Q_{31}Q_{32} = Q_{31}$. So, under the space decomposition of $\mathscr{H} = \overline{\mathscr{R}(Q_1)} \oplus \mathscr{R}(Q_1)^\perp \oplus \overline{\mathscr{R}(Q_2)} \oplus \mathscr{R}(Q_{32}) \oplus \mathscr{R}(Q_{32})^\perp$, the operators P and Q can then be further written as

$$P = \begin{pmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 & 0 & 0 & Q_{11}'' \\ 0 & I & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & Q_{31}' & Q_{31}'' \\ 0 & 0 & 0 & I & Q_{32}'' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $Q_{21}Q_{11}'' + Q_{31}'Q_{32}'' = Q_{31}''$.

(i) If $a+b+c+d \neq 0$, then

$$aP+bQ+cPQ+dQP = \begin{pmatrix} (a+b+c+d)I & 0 & 0 & 0 & (b+c)Q_{11}'' \\ 0 & (a+b+c+d)I & 0 & 0 & 0 \\ (b+d)Q_{21} & (b+d)Q_{22} & 0 & bQ_{31}' & bQ_{31}'' \\ 0 & 0 & 0 & bI & bQ_{32}'' \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $b \neq 0$, let $a' = \frac{a}{b}$, $c' = \frac{c}{b}$, $d' = \frac{d}{b}$, then we consider the following combination

$$a'P+Q+c'PQ+d'QP = \begin{pmatrix} (a'+1+c'+d')I & 0 & 0 & 0 & (1+c')Q''_{11} \\ 0 & (a'+1+c'+d')I & 0 & 0 & 0 \\ (1+d')Q_{21} & (1+d')Q_{22} & 0 & Q'_{31} & Q''_{31} \\ 0 & 0 & 0 & I & Q''_{32} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let

$$S = \begin{pmatrix} I & 0 & 0 & 0 & \frac{1+c'}{(a'+1+c'+d')}Q''_{11} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & Q''_{32} \\ 0 & 0 & I & 0 & Q''_{32} \\ 0 & 0 & 0 & 0 & I \end{pmatrix},$$

then

$$S^{-1} = \begin{pmatrix} I & 0 & 0 & 0 & -\frac{1+c'}{(a'+1+c'+d')}Q''_{11} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & -Q''_{32} \\ 0 & 0 & I & 0 & -Q''_{32} \\ 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

Direct calculation shows that

$$S(a'P+Q+c'PQ+d'QP)S^{-1} =$$
$$\begin{pmatrix} (a'+1+c'+d')I & 0 & 0 & 0 & 0 \\ 0 & (a'+1+c'+d')I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ (1+d')Q_{21} & (1+d')Q_{22} & Q'_{31} & 0 & \frac{a'-c'd'}{(a'+1+c'+d')}Q_{21}Q''_{11} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that

$$(a'P+Q+c'PQ)^D = S^{-1}(S(a'P+Q+c'PQ)S^{-1})^D S$$
$$= \begin{pmatrix} \frac{1}{(a'+1+c'+d')}I & 0 & 0 & 0 & \frac{1+c'}{(a'+1+c'+d')^2}Q''_{11} \\ 0 & \frac{1}{(a'+1+c'+d')}I & 0 & 0 & 0 \\ \frac{1+d'}{(a'+1+c'+d')^2}Q_{21} & \frac{1+d'}{(a'+1+c'+d')^2}Q_{22} & 0 & Q'_{31} & \frac{(1+c')(1+d')}{(a'+1+c'+d')^3}Q_{21}Q''_{11} + Q'_{31}Q''_{32} \\ 0 & 0 & 0 & I & Q''_{32} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{(a'+c')(a'+d')}{(a'+1+c'+d')^3}P + Q + \frac{(1+c')(a'+c')}{(a'+1+c'+d')^3}PQ + \frac{(1+d')(a'+d')}{(a'+1+c'+d')^3}QP$$
$$+[\frac{(1+c')(1+d')}{(a'+1+c'+d')^3} - 1]QPQ.$$

Moreover, since $(cT)^D = \frac{1}{c}T^D$ holds for any $c \neq 0$ and any Drazin invertible operator $T \in \mathbf{B}(\mathscr{H})$. Hence

$$\begin{aligned}(aP+bQ+cPQ+dQP)^D &= [b(a'P+Q+c'PQ+d'QP)]^D \\ &= \frac{1}{b}(a'P+Q+c'PQ+d'QP)^D \\ &= \frac{(a+c)(a+d)}{(a+b+c+d)^3}P + \frac{1}{b}Q + \frac{(b+c)(a+c)}{(a+b+c+d)^3}PQ \\ &\quad + \frac{(a+d)(b+d)}{(a+b+c+d)^3}QP + [\frac{(b+c)(b+d)}{(a+b+c+d)^3} - \frac{1}{b}]QPQ.\end{aligned}$$

(ii) If $a+b+c+d = 0$, then

$$\begin{aligned}(aP+bQ+cPQ+dQP)^D &= \begin{pmatrix} 0 & 0 & 0 & 0 & (b+c)Q''_{11} \\ 0 & 0 & 0 & 0 & 0 \\ (b+d)Q_{21} & (b+d)Q_{22} & 0 & bQ'_{31} & bQ''_{31} \\ 0 & 0 & 0 & bI & bQ''_{32} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^D \\ &= \frac{1}{b}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q'_{31} & Q'_{31}Q''_{32} \\ 0 & 0 & 0 & I & Q''_{32} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{b}(Q - QPQ).\end{aligned}$$

Now we can derive some special cases from Theorem 2. These results are the special cases of Theorem 2.3 in [4].

Corollary 3. *Let P and Q be two idempotents in $B(\mathscr{H})$. Assume that $PQP = P$, then the following statements hold.*

(i) $(P+Q)^D = \frac{1}{8}P + Q + \frac{1}{8}(PQ+QP) - \frac{7}{8}QPQ.$

(ii) $(P-Q)^D = QPQ - Q.$

If the stronger condition $QP = P$ is satisfied, then by Theorem 2, we can also derive the formulaes of Drazin inverses of linear combinations of P and Q.

Corollary 4. *Let P and Q be two idempotents in $B(\mathscr{H})$, and $a, b \in \mathbb{C}, ab \neq 0$. If $QP = P$, then*

$$(aP+bQ)^D = \begin{cases} \frac{a}{(a+b)^2}P + \frac{1}{b}Q + [\frac{b}{(a+b)^2} - \frac{1}{b}]PQ, & \text{when } a+b \neq 0 \\ \frac{1}{b}(Q-PQ), & \text{when } a+b = 0. \end{cases}$$

Next we discuss the Drazin inverse of $aP+bQ+cPQ+dQP$ under the assumption that $PQP=PQ$.

Theorem 3. *Let P and Q be two idempotents in $B(\mathscr{H})$, then for any $a,b,c,d\in\mathbb{C}, ab\neq 0$, the combinations $aP+bQ+cPQ+dQP$ are Drazin invertible under the condition $PQP=PQ$. The Drazin inverses of $aP+bQ+cPQ+dQP$ can be represented as following:*

(i) If $a+b+c+d\neq 0$, then

$$\begin{aligned}(aP+bQ+cPQ+dQP)^D =&\frac{1}{a}P+\frac{1}{b}Q+[\frac{1}{(a+b+c+d)}-\frac{b+d}{(a+b+c+d)^2}-\frac{1}{a}]PQ\\ &-(\frac{a+b+d}{ab})QP+[\frac{b+d}{(a+b+c+d)^2}+\frac{b+d}{ab}]QPQ.\end{aligned}$$

(ii) If $a+b+c+d=0$, then

$$\begin{aligned}(aP+bQ+cPQ+dQP)^D =&\frac{1}{a}P+\frac{1}{b}Q-\frac{1}{a}PQ\\ &-(\frac{1}{a}+\frac{1}{b}+\frac{d}{ab})QP+\frac{b+d}{ab}QPQ.\end{aligned}$$

Proof. If $PQP=PQ$, then P and Q can be written as

$$P=\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix},\quad Q=\begin{pmatrix} Q_1 & 0\\ Q_2 & Q_3\end{pmatrix}$$

under the space decomposition of $\mathscr{H}=\mathscr{R}(P)\oplus\mathscr{R}(P)^{\perp}$. The idempotency of Q yields that $Q_1^2=Q_1$, $Q_3^2=Q_3$, $Q_3Q_2=0$ and $Q_2Q_1+Q_3^2=Q_2$. With respect to the space decomposition $\mathscr{H}=\mathscr{R}(Q_1)^{\perp}\oplus\mathscr{R}(Q_1)\oplus\mathscr{R}(Q_3^*)\oplus\mathscr{R}(Q_3^*)$, P and Q can be further represented as

$$P=\begin{pmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},\quad Q=\begin{pmatrix} 0 & 0 & 0 & 0\\ Q_{11} & I & 0 & 0\\ Q_{21} & 0 & I & 0\\ Q_{23} & Q_{24} & Q_{31} & 0\end{pmatrix},$$

where $Q_{24}Q_{11}+Q_{31}Q_{21}=Q_{23}$.

(i) If $a+b+c+d\neq 0$, then

$$aP+bQ+cPQ+dQP=\begin{pmatrix} aI & 0 & 0 & 0\\ (b+c+d)Q_{11} & (a+b+c+d)I & 0 & 0\\ (b+d)Q_{21} & 0 & bI & 0\\ (b+d)Q_{23} & (b+d)Q_{24} & bQ_{31} & 0\end{pmatrix}.$$

Since $ab\neq 0$ and $a+b+c+d\neq 0$ then the submatrix

$$\begin{pmatrix} aI & 0 & 0\\ (b+c+d)Q_{11} & (a+b+c+d)I & 0\\ (b+d)Q_{21} & 0 & bI\end{pmatrix}$$

of $aP+bQ+cPQ+dQP$ is invertible and it's inverse is

$$\begin{pmatrix} \frac{1}{a}I & 0 & 0 \\ -\frac{b+c+d}{a(a+b+c+d)}Q_{11} & \frac{1}{a+b+c+d}I & 0 \\ -\frac{b+d}{ab}Q_{21} & 0 & \frac{1}{b}I \end{pmatrix}$$

By using the results of Lemma 2 we have

$$(aP+bQ+cPQ+dQP)^D = \begin{pmatrix} \frac{1}{a}I & 0 & 0 & 0 \\ -\frac{b+c+d}{a(a+b+c+d)}Q_{11} & \frac{1}{a+b+c+d}I & 0 & 0 \\ -\frac{b+d}{ab}Q_{21} & 0 & \frac{1}{b}I & 0 \\ X & \frac{b+d}{(a+b+c+d)^2}Q_{24} & \frac{1}{b}Q_{31} & 0 \end{pmatrix},$$

where $X=-\frac{b+d}{ab}Q_{23}+[\frac{b+d}{ab}+\frac{b+d}{a^2}-\frac{(b+d)(b+c+d)}{(a+b+c+d)}(\frac{1}{a^2}+1)]Q_{24}Q_{11}$. The coefficients of P, Q, PQ, QP, QPQ in the expression of $(aP+bQ+cPQ+dQP)^D$ can be obtained by solving some linear equations. Then we have

$$(aP+bQ+cPQ+dQP)^D = \frac{1}{a}P+\frac{1}{b}Q+[\frac{1}{(a+b+c+d)}-\frac{b+d}{(a+b+c+d)^2}-\frac{1}{a}]PQ$$
$$-(\frac{a+b+d}{ab})QP+[\frac{b+d}{(a+b+c+d)^2}+\frac{b+d}{ab}]QPQ.$$

(ii) If $a+b+c+d=0$, then

$$aP+bQ+cPQ+dQP = \begin{pmatrix} aI & 0 & 0 & 0 \\ -aQ_{11} & 0 & 0 & 0 \\ (b+d)Q_{21} & 0 & bI & 0 \\ (b+d)Q_{23} & (b+d)Q_{24} & bQ_{31} & 0 \end{pmatrix}.$$

Let

$$S = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

then

$$S(aP+bQ+cPQ+dQP)S^{-1} = \begin{pmatrix} aI & 0 & 0 & 0 \\ (b+d)Q_{21} & bI & 0 & 0 \\ -aQ_{11} & 0 & 0 & 0 \\ (b+d)Q_{23} & bQ_{31} & (b+d)Q_{24} & 0 \end{pmatrix}.$$

$$(aP+bQ+cPQ+dQP)^D = S^{-1}[S(aP+bQ+cPQ)S^{-1}]^D S$$

$$
\begin{aligned}
&=\begin{pmatrix} \frac{1}{a}I & 0 & 0 & 0 \\ -\frac{1}{a}Q_{11} & 0 & 0 & 0 \\ -\frac{b+d}{ab}Q_{21} & 0 & \frac{1}{b}I & 0 \\ -\frac{b+d}{ab}Q_{31}Q_{21} & 0 & \frac{1}{b}Q_{31} & 0 \end{pmatrix} \\
&=\frac{1}{a}P+\frac{1}{b}Q-\frac{1}{a}PQ-(\frac{1}{a}+\frac{1}{b}+\frac{d}{ab})QP+\frac{b+d}{ab}QPQ.
\end{aligned}
$$

Now we can derive some special cases from Theorem 3. These results are also special cases of Theorem 2.6 in [4].

Corollary 5. *Let P and Q be two idempotents in $B(\mathscr{H})$. Assume that $PQP = PQ$, then the following statements hold.*

(i) $(P+Q)^D = P+Q-\frac{3}{4}PQ-2QP+\frac{5}{4}QPQ.$

(ii) $(P-Q)^D = P-Q-PQ+QPQ.$

We can also derive the formulaes of Drazin inverses of linear combinations of P and Q under the condition $PQP = PQ$.

Corollary 6. *Let P and Q be two idempotents in $B(\mathscr{H})$. Assume that $PQP = PQ$, then the following statements hold.*

$$
(aP+bQ)^D = \begin{cases} \frac{1}{a}P+\frac{1}{b}Q+[\frac{1}{a+b}-\frac{b}{(a+b)^2}-\frac{1}{a}]PQ & \\ -\frac{a+b}{ab}QP+[\frac{b}{(a+b)^2}+\frac{1}{a}]QPQ, & whe\ n\ \ a+b\neq 0 \\ \frac{1}{a}(P-Q-PQ+QPQ), & when\ \ a+b=0. \end{cases}
$$

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