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A Note on Nearly Quasi-Einstein Manifolds

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Abstract. The object of the present paper is to study nearly quasi-Einstein manifold. Also we have studied decomposable Riemannian manifold and it is shown that a decomposable Riemannian manifold is nearly quasi-Einstein if and only if both the decompositions are Einstein.

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1. Introduction

It is well known that a Riemannian manifold $(M^n, g)(n > 2)$ is Einstein if its Ricci tensor S of type (0,2) is of the form $S = \alpha g$, where α is a constant, which reduces to $S = \frac{r}{n} g$, r being the scalar curvature (constant) of the manifold.

The notion of quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is said to be quasi-Einstein manifold [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following:

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1}$$

where α, β are scalars of which $\beta \neq 0$ and A is a nowhere vanishing 1-form defined by $g(X, \rho) = A(X)$ for all X; ρ being a unit vector field, called the generator of the manifold. Such an n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$. The scalars α, β are known as the associated scalars of the manifold. Also the 1-form A is called the associated 1-form of the manifold. From the above definition it follows that every Einstein manifold is quasi-Einstein. In particular, every Ricci-flat (e.g. Schwarzschild spacetime) manifold is quasi-Einstein.

Recently the notion of quasi-Einstein manifold have been weakened by De and Gaji [2, 14] and they introduced the notion of nearly quasi-Einstein manifold with the existence of such

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notion. A Riemannian manifold $(M^n, g)(n > 2)$ is called *nearly quasi-Einstein* if its Ricci tensor S is not identically zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta D(X,Y), \tag{2}$$

where α, β are non-zero scalars and D is a symmetric non-zero (0, 2) tensor. The scalars α, β are known as associated scalars and D is called the associated tensor of the manifold. Such an n-dimensional manifold is denoted by $N(QE)_n$.

The present paper deals with a study of $N(QE)_n(n > 2)$. The paper is organized as follows. Section 2 is concerned with Ricci-pseudosymmetric $N(QE)_n$ and we obtain a $N(QE)_n$ is Ricci-pseudosymmetric if and only if it is D-pseudosymmetric. Section 4 deals with decomposable Riemannian manifold. It is proved that a decomposable Riemannian manifold is nearly quasi-Einstein if and only if both the decompositions are Einstein. Section 5 deals with some global properties of $N(QE)_n$ and it is proved that under certain condition such a manifold does not admit non-zero Killing vector field, non-zero projective Killing vector field and non-zero conformal Killing vector field. Finally the last section deals with an interesting example of nearly quasi-Einstein manifold with non-vanishing scalar curvature which is not quasi-Einstein.

2. Ricci-pseudosymmetry $N(QE)_n$

An n-dimensional Riemannian manifold (M^n, g) is called Ricci-pseudosymmetric [4] if the tensor $R \cdot S$ and $Q(g,S)$ are linearly dependent, where

$$(R(X,Y) \cdot S)(Z,U) = -S(R(X,Y)Z,U) - S(Z,R(X,Y)U), \tag{3}$$

$$Q(g,S)(Z,U;X,Y) = -S((X \wedge_g Y)Z,U) - S(Z,(X \wedge_g Y)U). \tag{4}$$

Thus the condition of Ricci-pseudosymmetry is

$$(R(X,Y) \cdot S)(Z,U) = L_S Q(g,S)(Z,U;X,Y) \tag{5}$$

holding on the set $U_S = \{x \in M : S \neq \frac{r}{n} g \text{ at } x\}$, where L_S is some function on U_S. If $R \cdot S = 0$ then M is called Ricci-semisymmetric. Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true [4]. In [2] De and Gaji studied Ricci-semisymmetric $N(QE)_n$.

Now we prove the following:

Theorem 1. *A nearly quasi-Einstein manifold is Ricci-pseudosymmetric if and only if it is D-pseudosymmetric.*

Proof. We now consider a Ricci-pseudosymmetric $N(QE)_n$. Then from (3)–(5), we can write

$$\begin{aligned} S(R(X,Y)Z,U) + S(Z,R(X,Y)U) = L_S\{S(X,U)g(Y,Z) \\ -S(Y,U)g(X,Z) + S(X,Z)g(Y,U) - S(Y,Z)g(X,U)\}. \end{aligned} \tag{6}$$

Using (2) in (6), we get

$$D(R(X,Y)Z,U)+D(Z,R(X,Y)U)=L_S\{D(X,U)g(Y,Z)$$
$$-D(Y,U)g(X,Z)+D(X,Z)g(Y,U)-D(Y,Z)g(X,U)\}, \tag{7}$$

which implies that the manifold is D-pseudosymmetric.

Conversely, if the manifold is D-pseudosymmetric, then (7) holds. By virue of (2), it follows from (7), we get the relation(6) and consequently, the manifold is Ricci-pseudosymmetric.

Corollary 1. *A nearly quasi-Einstein manifold is Ricci-semisymmetric if and only if it is D-semisymmetric [2].*

3. Decomposable Riemannian manifold

A non-flat Riemannian manifold (M^n,g) is said to be decomposable [19] if it can be expressed as $M_1^p\times M_2^{n-p}$ for $2\le p\le n-2$, that is, in some coordinate neighbourhood of the Riemannian manifold (M^n,g), the metric can be expressed as

$$ds^2=g_{ij}dx^idx^j=\tilde{g}_{ab}dx^adx^b+\overset{*}{g}_{\alpha\beta}\,dx^\alpha dx^\beta, \tag{8}$$

where $\tilde{g}_{ab}$ are functions of $x^1,x^2,\cdots,x^p(p<n)$ denoted by $\tilde{x}$ and $\overset{*}{g}_{\alpha\beta}$ are functions of $x^{p+1},x^{p+2},\cdots,x^n$ denoted by $\overset{*}{x}$; $a,b,c,\cdots$ run from 1 to p and $\alpha,\beta,\gamma,\cdots$ run from $p+1$ to n. The two parts of (8) are the metrics of $M_1^p(p\ge 2)$ and $M_2^{n-p}(n-p\ge 2)$ which are called the decomposition of the manifold $M^n=M_1^p\times M_2^{n-p}(2\le p\le n-2)$.

Let (M^n,g) be a Riemannian manifold such that $M_1^p\times M_2^{n-p}$ for $2\le p\le n-2$. Here throughout this section each object denoted by a "tilde" is assumed to be from M_1 and each object denoted by a "star" is assumed to be from M_2.

Let $\tilde{X},\ \tilde{Y},\ \tilde{Z},\ \tilde{U},\ \tilde{V}\in\chi(M_1)$ and $\overset{*}{X},\ \overset{*}{Y},\ \overset{*}{Z},\ \overset{*}{U},\ \overset{*}{V}\in\chi(M_2)$, then we have the following relations:

$$R(\overset{*}{X},\tilde{Y},\tilde{Z},\tilde{U})=0=R(\tilde{X},\overset{*}{Y},\tilde{Z},\overset{*}{U})=R(\tilde{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U}),$$

$$(\nabla_{\overset{*}{X}}R)(\tilde{Y},\tilde{Z},\tilde{U},\tilde{V})=0=(\nabla_{\tilde{X}}R)(\tilde{Y},\overset{*}{Z},\tilde{U},\overset{*}{V})=(\nabla_{\overset{*}{X}}R)(\tilde{Y},\overset{*}{Z},\tilde{U},\overset{*}{V}),$$

$$R(\tilde{X},\tilde{Y},\tilde{Z},\tilde{U})=\tilde{R}(\tilde{X},\tilde{Y},\tilde{Z},\tilde{U});R(\overset{*}{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U})=\overset{*}{R}(\overset{*}{X},\overset{*}{Y},\overset{*}{Z},\overset{*}{U}),$$

$$S(\tilde{X},\tilde{Y})=\tilde{S}(\tilde{X},\tilde{Y});S(\overset{*}{X},\overset{*}{Y})=\overset{*}{S}(\overset{*}{X},\overset{*}{Y}),$$

$$(\nabla_{\tilde{X}}S)(\tilde{Y},\tilde{Z})=(\tilde{\nabla}_{\tilde{X}}S)(\tilde{Y},\tilde{Z});(\nabla_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{Z})=(\overset{*}{\nabla}_{\overset{*}{X}}S)(\overset{*}{Y},\overset{*}{Z}),$$

$$\text{and } r=\tilde{r}+\overset{*}{r},$$

where r, $\tilde{r}$, and $\overset{*}{r}$ are the scalar curvature of M, M_1, M_2 respectively.

In [19] Yano and Kon find a necessary and sufficient condition that both the decompositions of a decomposable Riemannian manifold are Einstein and they obtained that

Theorem 2. *In a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$, a necessary and sufficient condition that the two decompositions are both Einstein is that the Ricci tensor of the manifold has the form*

$$S(X,Y) = ag(X,Y) + bF(X,Y), \tag{9}$$

a and b being necessarily constant and F is a (0, 2) type metric tensor such that

$$F(X,Y) = \tilde{g}(\tilde{X},\tilde{Y}) + \overset{*}{g}(\overset{*}{X},\overset{*}{Y}). \tag{10}$$

By virtue of Theorem 2, we can state the following:

Theorem 3. *A decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$ is nearly quasi-Einstein if and only if both the decompositions are Einstein.*

4. Some global properties of $N(QE)_n$

This section is concerned with a compact, orientable $N(QE)_n (n > 2)$ without boundary with α, β as associated scalars and D as the structure tensor. Then we prove the following:

Theorem 4. *If in a compact, orientable $N(QE)_n (n > 2)$ without boundary, the associated scalars and the structure tensor are such that $\alpha < 0$ and $\beta D(X,X) < 0$, then there exists no non-zero Killing vector field in this manifold.*

Proof. It is known that [17] for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \left[S(X,X) - |\nabla X|^2 - (divX)^2 \right] dv \leq 0, \tag{11}$$

where "dv" denotes the volume element of M. If X is a Killing vector field, then $divX = 0$ [18]. Hence (11) takes the following form

$$\int_M \left[S(X,X) - |\nabla X|^2 \right] dv = 0. \tag{12}$$

Let us consider $\alpha < 0$ and $\beta D(X,X) < 0$. Hence by virtue of (2) we have

$$\int_{M=N(QE)_n} \left[\alpha |X|^2 + \beta D(X,X) - |\nabla X|^2 \right] dv$$
$$\geq \int_M \left[S(X,X) - |\nabla X|^2 \right] dv,$$

which yields by virtue of (12) that

$$\int_M \left[\alpha |X|^2 + \beta D(X,X) - |\nabla X|^2 \right] dv \geq 0.$$

If $\alpha < 0$ and $\beta D(X,X) < 0$, then the last relation reduces to

$$\int_M \left[\alpha|X|^2 + \beta D(X,X) - |\nabla X|^2\right] dv = 0.$$

Hence $X = 0$. This proves the theorem.

Definition 1. *[18] A vector field X in a Riemannian manifold (M^n, g) $(n > 2)$ is said to be projective Killing vector field if it satisfies*

$$(\$_X g)(Y,Z) = \omega(Y)Z + \omega(Z)Y$$

for any vector fields Y and Z, ω being a certain 1-form and \$ is the operator of Lie differentiation.

Theorem 5. *If in a compact, orientable $N(QE)_n (n > 2)$ without boundary, the associated scalars and the structure tensor are such that $\alpha \leq 0$ and $\beta D(X,X) \leq 0$, then a projective Killing vector field has vanishing covariant derivative, and if $\alpha < 0$ and $\beta D(X,X) < 0$, then there exists no non-zero projective Killing vector field in this manifold.*

Proof. We know that [17] for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \left[S(X,X) - \frac{1}{4}|d\xi|^2 - \frac{n-1}{2(n+1)}(divX)^2\right] dv = 0, \tag{13}$$

where ξ is an 1-form corresponding to the vector field X. We now assume $\alpha \leq 0$ and $\beta D(X,X) \leq 0$. Therefore (13) yields $S(X,X) \leq 0$ and hence from (13) we obtain $d\xi = 0$ and $divX = 0$. This implies that X is harmonic as well as a Killing vector field. Consequently its covariant derivative vanishes. This proves the theorem.

Definition 2. *[18] A vector field X in a Riemannian manifold (M^n, g) $(n > 2)$ is said to be conformal Killing vector field if it satisfies*

$$\$_X g = 2\rho g$$

for any vector field X, where ρ is given by $\rho = -\frac{1}{n}(divX)$ and \$ is the operator of Lie differentiation.

Theorem 6. *If in a compact, orientable $N(QE)_n (n > 2)$ without boundary, the associated scalars and the structure tensor are such that $\alpha < 0$ and $\beta D(X,X) < 0$, then there exists no non-zero conformal Killing vector field in this manifold.*

Proof. It is known from [17] that for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \left[S(X,X) - |\nabla X|^2 - \frac{n-2}{n}(divX)^2\right] dv = 0, \tag{14}$$

where dv denotes the volume element of M. Now we assume that the associated scalars and the structure tensor are such that $\alpha < 0$ and $\beta D(X,X) < 0$. Then proceeding similarly as before we obtain

$$\nabla X = 0, \quad divX = 0.$$

This proves the theorem.

5. Example of $N(QE)_n$

We define a Riemannian metric g on the n-dimensional real number space $\mathbb{R}^n$ by the formula

$$ds^2 = e^{kx^1}[(dx^1)^2 + \sin^2 x^3 (dx^2)^2 + (dx^3)^2] + f(x^4)(dx^4)^2 + \sum_{l=5}^{n}(dx^l)^2, \tag{15}$$

where x^1 is non-zero finite, $0 < x^3 < \frac{\pi}{2}$, k is a non-zero finite real number excepting ± 2 and f is a positive smooth function of x^4 only. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensors are given by

$$\Gamma^1_{11} = \frac{k}{2} = \Gamma^2_{12} = \Gamma^3_{13} = -\Gamma^1_{33}, \Gamma^1_{22} = -\frac{k}{2}\sin^2 x^3,$$
$$\Gamma^3_{22} = -\sin x^3 \cos x^3, \Gamma^2_{23} = \cot x^3, \Gamma^4_{44} = \frac{1}{2}\frac{f'(x^4)}{f(x^4)},$$
$$R_{2332} = \Big(\frac{k^2}{4} - 1\Big)e^{kx^1}\sin^2 x^3, S_{22} = \Big(\frac{k^2}{4} - 1\Big)\sin^2 x^3, S_{33} = \Big(\frac{k^2}{4} - 1\Big).$$

Here the scalar curvature of the manifold is $r = 2(\frac{k^2}{4} - 1)e^{-kx^1} \neq 0$. Therefore $\mathbb{R}^n$ with the considered metric is a Rimennian manifold (M^n, g) of non-vanishing scalar curvature. We shall now show that this M^n is a nearly quasi-Einstein manifold, i.e., it satisfies (2).

Let us now consider the associated scalars and the components of the structure tensor of D as follows:

$$\alpha = \frac{1}{2}\Big(\frac{k^2}{4} - 1\Big)e^{-kx^1}, \ \beta = \frac{1}{2}\Big(\frac{k^2}{4} - 1\Big), \tag{16}$$

and

$$D_{ij}(x) = \begin{cases} \sin^2 x^3 & \text{for} \quad i,j = 2,2, \\ 1 & \text{for} \quad i,j = 3,3, \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

at any point $x \in M$.
Then it can be easily shown that the manifold under consideration is nearly quasi-Einstein manifold. Hence we can state the following:

Theorem 7. *Let* (M^n, g) *be a Riemannian manifold endowed with the metric given in* (15). *Then* (M^n, g) *is a nearly quasi-Einstein manifold with non-vanishing scalar curvature, which is not quasi-Einstein.*

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