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COORDINATED SEARCH FOR A RANDOMLY LOCATED TARGET ON THE PLANE

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Abstract. This paper presents a coordinated search technique that allows two searchers who start together at the intersection point of two roads (the vertical road acts x-axis and the horizintal road acts y-axis) in known region, we consider this point is the center of this region and it is $(0,0)$. The two searchers wanted to detect the lost target which is randomly located on the region. This lost target has symmetric distribution. We will find the expected value of detecting the target and the optimal seach plan which minimizes this expected value in the case of the target has a circular normal distribution, numerical resluts show the effectiveness of this technique and demonstrates the applicablity of it to real world search scenarios.

AMS subject classifications: 60K30, 90B40.

Key words: coordinated search, optimal search plan, symmetric distribution.

1. INTRODUCTION

The study of search plans for any lost target either located or moved and having symmetric or unsymmetric distribution is important and has recently various applications, such as searching for a faulty unit in large linear system, such as electrical power lines, this kind of search is called linear search problem, see [1], [2], [3] and [4]. The coordinated search technique is one of a set of techniques, which studied on the line when the target has symmetric

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or unsymmetric distribution, see [5], [6] and [7]. If the target located on a known region, like petrol or gas supply under ground, it would study, see [8], but if it moved, like missing boats, submarines and missing system, a Bayesian approach would formulate for a target whose prior distribution and probabilistic motion model are known and generalized the approach for multi-vehicle search, see [9] and [10]. Similary, the tracking study commenced with a simple feedback motion tracking algorithm, and has evolved with the developements of a number of recursive filtering techniques, see [11].

The primary concern of the paper thus lies in the coordinated search technique which allows two searchers s_1 and s_2 start together and looking for the target from the point $(0,0)$ (center of the known region), where the region is devided by two roads and they intersect in the center of this region, one of these roads is horizintal (y-axis) and the other is vertical (x-axis). We will devide the region to many circles as in fig.1.

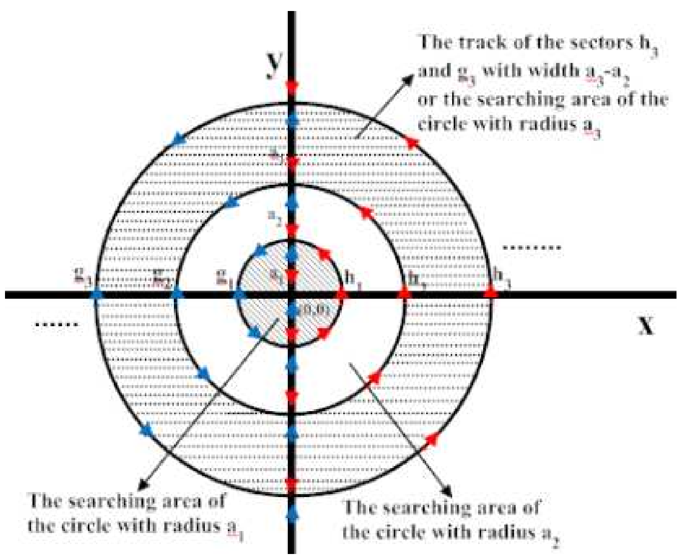

Figure 1

The searcher s_1 searchs on the right hand side of the horizintal road and the searcher s_2 searchs on the left hand side of the horizintal road. They are start together from the point $(0,0)$, the searcher s_1 goes through a $-ve$ part on y-axis with adistance a_1 (redius of the first circle) and then he starts searching on the sector h_1, after finishing the search on h_1 he arrives to y-axis again, then he returns through the same distance a_1 on $+ve$ part on y-axis to the starting point $(0,0)$,also the searcher s_2 goes through a $+ve$ part on y-axis with adistance a_1 (redius of the circles) and then he starts searching on the sector g_1, after finishing the search on g_1 he arrives to y-axis again, then he returns through the same distance a_1 on $-ve$ part on y-axis to the starting point $(0,0)$. The searchers arrive to the starting point at the same moment to tell each other if one of them has detected the target or not. If one of them detect it then he will tell the other or they do the previous path in the second circle with another redius a_2 and so on. The searchers wish to minimize the expected time to detect the target. In this problem we consider the searchers search only on the sectors and it's tracks as in fig. 1, any one of these tracks has width $a_i - a_{i-1}$ to cover all area and not neglect any part, but they go through y-axis ($+ve$ and $-ve$) only witout searching.

Let (X,Y) be two indpendent random variables which they are represent the position of the located target in the region . Any track has width $a_i - a_{i-1}$, such that, when any searcher moves on the sector of any circle they cover track with width $a_i - a_{i-1}$ (i.e. search on one direction (inside of the sector) of its position).

The searchers goes on y-axis ($+ve$ and $-ve$ parts) as in the above steps with equal speeds ($v_1$and v_2), and searches with "*regular speed*" v_3 on the sectors and it's tracks, they return to $(0,0)$ after searching successively through y-axis ($+ve$ and $-ve$ parts) until the target is detect. Our aim is to calculate the expected value of the time for detecting the target; also we wish to find the Optimal Search Plan (O.S.P.) to detect it.

2. THE SEARCH PATH

The searchers s_1 and s_2 follow search paths e and f, respectively to detect the target. The first search path e_1 of s_1 is defined as follows: The searcher s_1 goes a distance a_1, through a $-ve$ part in y-axis, after that he searchs on the sector h_1 and it's track, then he returns to the origin with the same distance a_1 through a $+ve$ part in y-axis, also the second search path e_2 of s_1 is defined as follows: The searcher s_1 goes a distance a_2, through a $-ve$ part in y-axis, after that he searchs on the sector h_2 and it's track, then he returns to the origin with the same distance a_2 through a $+ve$ part in y-axis, and so on then the search path e of s_1 is completely defined by a sequence $\{e_i, i \geq 0\}$, where i is a nonegative integer. Also, the first search path f_1 of s_2 is defined as follows: The searcher s_2 goes a distance a_1, through a $+ve$ part in y-axis, after that he searchs on the sector g_1 and it's track, then he returns to the origin with the same distance a_1 through a $-ve$ part in y-axis, and the second search path f_2 of s_2 is defined as follows: The searcher s_2 goes a distance a_2, through a $+ve$ part in y-axis, after that he searchs on the sector g_2 and it's track, then he returns to the origin with the same distance a_2 through a $-ve$ part in y-axis, and so on then the search path f of s_2 is completely defined by a sequence $\{f_i, i \geq 0\}$, where i is a nonegative integer.

Let the search plan be defined by :

$\phi = (e, f) \in \Phi$, where Φ is the set of all search plans.

The circle number j, $j = 1, 2, ...$ is devided into an equal two sectors h_j, g_j, $j = 1, 2, ...$ as in fig. 2, then the distances which made by the searchers in y-axis are equal and the searching areas (set of tracks) in the two parts are equal also so that, the target has symmetric distribution . We consider the searchers goes on y-axis with equal speeds $(v_1 = v_2 = 1)$, and search on the sectors and it's tracks with regular speed v_3, where the searching process done only on the sectors and the areas produced by the tracks which made by the seachers inside the sectors but we added the time which the searchers taked it through going on y-axis to the time of the searching process.

Let (X, Y) are independent, random variables and they represent the target position on the region with probability density function $f(x, y)$ and distribution function $F(x, y)$, where we consider the surface of the region be a "*Standrad Eculidan 2-space E*" , with points designated by ordered pairs (x, y). The circles devide into sectors as in fig. 2, and these sectors are also devided into an equal small sectors $l_i, i = 1, 2, ..., n$, each sector of them make small search area of the track which the search done on it by which we mean for the moment that the searcher sees every thing of his position, and nothing beyond that.

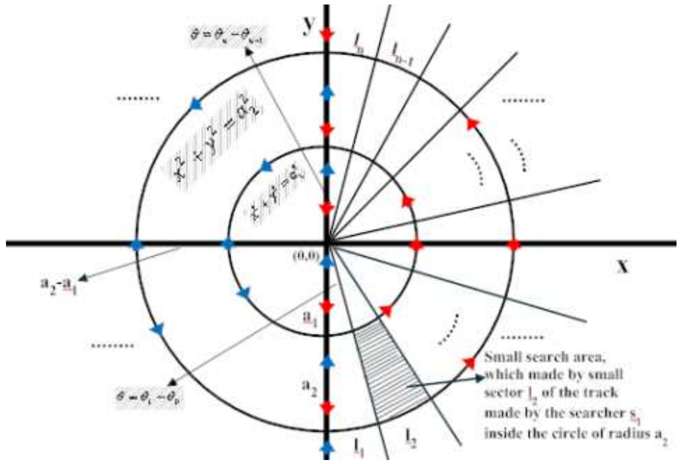

Figure 2

Let t_1 be the time which the searcher s_1 takes it in the search path $\{e_i, i \geq 0\}$, where i is a nonegative integer in the first part to return to $(0,0)$, and t_2 be the time which the searcher s_2 takes it in the search path $\{f_i, i \geq 0\}$, where i is a nonegative intege in the second part to return to $(0,0)$, (where they go on y-axis from the origin before searching on the sectors and return after finishing on the sectors to the origin with equal speed $(v_1 = v_2 = 1)$, then in this case the time of going througth y-axis is equal to the distances which done, and searching on the sectors $h_i, g_i, i = 1, 2, ...$ and it's tracks (searching areas of the sectors) with "*regular speed*" v_3, then we consider the searching time on the sectors and inside them (on the tracks which made by the searchers) is equal to $\tau_i = \frac{2\pi}{\omega_i}$, where τ_i is the"*time league*", ω_i is called "*angular velocity* " the searching time τ_i depends on ω_i which depends on the redius a_i) and $t(\phi)$ be the time for the searchers to detect the target.

Theorem 2.1. *The expected value of the time for detecting the target is given by :*

$$E(t(\phi)) = \left(4a_1 + \frac{2\pi}{\omega_1}\right)\left[\sum_{k=1}^{n}\int_{0}^{a_1}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta\right]$$

$$+\sum_{i=2}^{\infty}\left[\begin{array}{c}\left(4a_1+\frac{2\pi}{\omega_1}\right)\sum_{k=1}^{n}\int_{a_i-a_{i-1}}^{a_i}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)rdrd\theta \\ +\left(4a_i+\frac{2\pi}{\omega_i}\right)\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_s-a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)rdrd\theta\end{array}\right] \quad (1)$$

Proof. IF the target lies in any point of the track of g_1, then $t_1 = a_1 + \frac{1}{2}.\frac{2\pi}{\omega_1} + a_1 = 2a_1 + \frac{\pi}{\omega_1}$.

IF the target lies in any point of the track of g_2, then $t_1 = 2(a_1 + a_2) + \pi(\frac{1}{\omega_1} + \frac{1}{\omega_2})$.

IF the target lies in any point of the track of g_3, then $t_1 = 2(a_1+a_2+a_3)+\pi(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{1}{\omega_3})$, and so on.

IF the target lies in any point of the track of h_1, then $t_2 = 2a_1 + \frac{\pi}{\omega_1}$.

IF the target lies in any point of the track of h_2, then $t_2 = 2(a_1 + a_2) + \pi(\frac{1}{\omega_1} + \frac{1}{\omega_2})$.

IF the target lies in any point of the track of h_3, then $t_2 = 2(a_1+a_2+a_3)+\pi(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{1}{\omega_3})$, and so on.

But since each sector are devided into an equal small sectors $l_i, i = 1, 2, ..., n$, where these sctors make a set of an equal cones have the same vertex $(0, 0)$ as in figure 2, but the searcher can cover a track with width $a_i - a_{i-1}$, so that he cover an area from each cone and these areas are equal with width $a_i - a_{i-1}$ and the cones is determined by a set of lines with equations $x = m_k y = \tan\theta\ y$, where $\theta = \theta_k - \theta_{k-1}$, $k = 1, 2, .., n$, where this set of equations make a set of an equal small areas, by which we mean for the moment that the searcher sees every thing of his position, and nothing beyond that, so that to evaluate the expected value of the time for the searchers to detect the target, we use the polar coordinates with $x = r\ \cos\theta$ and $y = r\ \sin\theta$, $r : a_{i-1} \longrightarrow a_i$, $i = 1, 2, 3, ...$ and $\theta : \theta_{k-1} \longrightarrow \theta_k$, $k = 1, 2, 3, ..., n$, where $r_0 = 0$, $\theta_0 = 0$. The searchers search on the sectors and it's tracks in anti clockwise. According to our assumptions that the target has symmetric distribution, hence

$$E(t(\phi)) = \left(2a_1 + \frac{\pi}{\omega_1}\right)\left[\begin{array}{c}\int_0^{a_1}\int_0^{\theta_1} g(r\cos\theta, r\sin\theta)rdrd\theta + ... \\ +\int_0^{a_1}\int_{\theta_{n-1}}^{\theta_n} g(r\cos\theta, r\sin\theta)rdrd\theta\end{array}\right]$$

$$+\left(2(a_1+a_2)+\pi(\frac{1}{\omega_1}+\frac{1}{\omega_2})\right)\left[\begin{array}{l}\int\limits_{a_2-a_1}^{a_2}\int\limits_{0}^{\theta_1} g(r\cos\theta, r\sin\theta)rdrd\theta+...\\ +\int\limits_{a_2-a_1}^{a_2}\int\limits_{\theta_{n-1}}^{\theta_n} g(r\cos\theta, r\sin\theta)rdrd\theta\end{array}\right]$$

$$+\left(2(a_1+a_2+a_3)+\pi(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{1}{\omega_3})\right)\left[\begin{array}{l}\int\limits_{a_3-a_2}^{a_3}\int\limits_{0}^{\theta_1} g(r\cos\theta, r\sin\theta)rdrd\theta+...\\ +\int\limits_{a_3-a_2}^{a_3}\int\limits_{\theta_{n-1}}^{\theta_n} g(r\cos\theta, r\sin\theta)rdrd\theta\end{array}\right]$$

$$+...$$

$$+\left(2a_1+\frac{\pi}{\omega_1}\right)\left[\begin{array}{l}\int\limits_{0}^{a_1}\int\limits_{0}^{\theta_1} g(r\cos\theta, r\sin\theta)rdrd\theta+...\\ +\int\limits_{0}^{a_1}\int\limits_{\theta_{n-1}}^{\theta_n} g(r\cos\theta, r\sin\theta)rdrd\theta\end{array}\right]$$

$$+\left(2(a_1+a_2)+\pi(\frac{1}{\omega_1}+\frac{1}{\omega_2})\right)\left[\begin{array}{l}\int\limits_{a_2-a_1}^{a_2}\int\limits_{0}^{\theta_1} g(r\cos\theta, r\sin\theta)rdrd\theta+...\\ +\int\limits_{a_2-a_1}^{a_2}\int\limits_{\theta_{n-1}}^{\theta_n} g(r\cos\theta, r\sin\theta)rdrd\theta\end{array}\right]$$

$$+\left(2(a_1+a_2+a_3)+\pi(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{1}{\omega_3})\right)\left[\begin{array}{l}\int\limits_{a_3-a_2}^{a_3}\int\limits_{0}^{\theta_1} g(r\cos\theta, r\sin\theta)rdrd\theta+...\\ +\int\limits_{a_3-a_2}^{a_3}\int\limits_{\theta_{n-1}}^{\theta_n} g(r\cos\theta, r\sin\theta)rdrd\theta\end{array}\right]$$

$$+...$$

and so on, then

$$E(t(\phi))=2\left[2a_1+\frac{\pi}{\omega_1}\right]\left[\sum_{k=1}^{n}\int\limits_{0}^{a_1}\int\limits_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)rdrd\theta\right]$$

$$+2\left[2(a_1+a_2)+\pi\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)\right]\left[\sum_{k=1}^{n}\int_{a_2-a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta\right]$$

$$+2\left[2(a_1+a_2+a_3)+\pi\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}+\frac{1}{\omega_3}\right)\right]\left[\sum_{k=1}^{n}\int_{a_3-a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta\right]$$

$$+\ldots$$

$$=\left(4a_1+\frac{2\pi}{\omega_1}\right)\left[\begin{array}{c}\sum_{k=1}^{n}\int_{0}^{a_1}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta \\ +\sum_{k=1}^{n}\int_{a_2-a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta \\ +\sum_{k=1}^{n}\int_{a_3-a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta+\ldots\end{array}\right]$$

$$+\left(4a_2+\frac{2\pi}{\omega_2}\right)\left[\begin{array}{c}\sum_{k=1}^{n}\int_{a_2-a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta \\ +\sum_{k=1}^{n}\int_{a_3-a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta \\ +\sum_{k=1}^{n}\int_{a_4-a_3}^{a_4}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta+\ldots\end{array}\right]$$

$$+\left(4a_3+\frac{2\pi}{\omega_3}\right)\left[\begin{array}{c}\sum_{k=1}^{n}\int_{a_3-a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta \\ +\sum_{k=1}^{n}\int_{a_4-a_3}^{a_4}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta+\ldots\end{array}\right]$$

$$+\ldots$$

$$=\left(4a_1+\frac{2\pi}{\omega_1}\right)\left[\sum_{k=1}^{n}\int_{0}^{a_1}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta\right]$$

$$+\sum_{i=2}^{\infty}\left[\begin{array}{c}\left(4a_1+\frac{2\pi}{\omega_1}\right)\sum_{k=1}^{n}\int_{a_i-a_{i-1}}^{a_i}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta \\ +\left(4a_i+\frac{2\pi}{\omega_i}\right)\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_s-a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta)r\,dr\,d\theta\end{array}\right]. \blacksquare$$

3. OPTIMAL SEARCH PLAN

Definition 3.1. *Let $\phi^* \in \Phi$ be a search plan, then ϕ^* is an optimal search plan, if $E(t(\phi^*)) = \inf\{E(t(\phi)), \phi \in \Phi\}$.*

The goal of the searching strategy could be minimize the expected time to detect the target of *circular normal* distributon by minimizing the mean time to detection with respect to determining the optimal redius a_i, $i = 1, 2, ...$ which make us to cover all area.

If the target has a bivariate normal distribution with parameters σ_1 and σ_2. And since we consider the surface of the region be a standard *Eculidean* 2−space E, with points designated by ordered pairs (x, y). This is a reasonable assumption for small areas about the target's reported position. In this coordinate system, the target's reported position is $(0,0)$. Let (X, Y) give the target's actual position. Then X is normally distributed with mean 0 and standard deviation σ_1. In addition, X is independent of Y, which is normally distributed with mean 0 and standard deviation σ_2. Let

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2}\exp\left[-\frac{1}{2}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}\right)\right] \qquad \text{for} \quad (X,Y)\in E. \tag{2}$$

The function f is the probability density function of the *bivariate normal* distribution. Thus the distribution of error in the navigation system yields f as given in (2) for the density of the target distribution. If $\sigma_1 = \sigma_2 = \sigma$ then (2) becomes

$$f(x,y) = \frac{1}{2\pi\sigma^2}\exp[-(x^2+y^2)/2\sigma^2] \qquad \text{for} \quad (X,Y)\in E. \tag{3}$$

and the target distribution is called *circular normal.*

Theorem 3.1. *Let (X, Y) be two independent random variables have circular normal distribution with joint continuous distribution function $F(x, y)$ and joint density function $f(x, y)$ as in* (3)*. If ϕ is an optimal search plan, and with knowing the value of $a_1$we can get:*

$$\left\{\begin{array}{ll} a_2 \text{ from} & a_1 \exp\left(\frac{-a_1^2}{2\sigma^2}\right) = (a_2 - a_1)\exp\left(\frac{-(a_2-a_1)^2}{2\sigma^2}\right) \\ a_i \text{ from} & a_i \exp\left(\frac{-a_i^2}{2\sigma^2}\right) = (a_i - a_{i-1})\exp\left(\frac{-(a_i-a_{i-1})^2}{2\sigma^2}\right) \\ & +(a_{i+1} - a_i)\exp\left(\frac{-(a_{i+1}-a_i)^2}{2\sigma^2}\right), i = 3, 4, \dots \end{array}\right. \tag{4}$$

Proof. from (1) we get:

$$E(t(\phi)) = \left(4a_1 + \frac{2\pi}{\omega_1}\right)\left[\sum_{k=1}^{n}\int_0^{a_1}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta\right]$$

$$+\sum_{i=2}^{\infty}\left[\begin{array}{c} \left(4a_1 + \frac{2\pi}{\omega_1}\right)\sum_{k=1}^{n}\int_{a_i - a_{i-1}}^{a_i}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta \\ +\left(4a_i + \frac{2\pi}{\omega_i}\right)\sum_{s=i}^{\infty}\sum_{k=1}^{n}\int_{a_s - a_{s-1}}^{a_s}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta \end{array}\right]$$

$$= \left(4a_1 + \frac{2\pi}{\omega_1}\right)\left[\begin{array}{c} \sum_{k=1}^{n}\int_0^{a_1}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta \\ \sum_{k=1}^{n}\int_{a_2 - a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta + \dots \end{array}\right]$$

$$+ \left(4a_2 + \frac{2\pi}{\omega_2}\right)\left[\begin{array}{c} \sum_{k=1}^{n}\int_{a_2 - a_1}^{a_2}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta \\ +\sum_{k=1}^{n}\int_{a_3 - a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta + \dots \end{array}\right]$$

$$+ \left(4a_3 + \frac{2\pi}{\omega_3}\right)\left[\begin{array}{c} \sum_{k=1}^{n}\int_{a_3 - a_2}^{a_3}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta \\ +\sum_{k=1}^{n}\int_{a_4 - a_3}^{a_4}\int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta + \dots \end{array}\right]$$

$$+\dots$$

$$= \left(4a_1 + \frac{2\pi}{\omega_1}\right)\left(\frac{n\theta(-\sigma^2)}{2\pi\sigma^2}\left[\begin{array}{c}\int_0^{a_1} \exp\left(\frac{-r^2}{2\sigma^2}\right) d\left(\frac{-r^2}{2\sigma^2}\right) \\ + \int_{a_2-a_1}^{a_2} \exp\left(\frac{-r^2}{2\sigma^2}\right) d\left(\frac{-r^2}{2\sigma^2}\right) + \ldots\end{array}\right]\right)$$

$$+ \left(4a_2 + \frac{2\pi}{\omega_2}\right)\left(\frac{n\theta(-\sigma^2)}{2\pi\sigma^2}\left[\begin{array}{c}\int_{a_2-a_1}^{a_2} \exp\left(\frac{-r^2}{2\sigma^2}\right) d\left(\frac{-r^2}{2\sigma^2}\right) \\ + \int_{a_3-a_2}^{a_3} \exp\left(\frac{-r^2}{2\sigma^2}\right) d\left(\frac{-r^2}{2\sigma^2}\right) + \ldots\end{array}\right]\right)$$

$$+ \left(4a_3 + \frac{2\pi}{\omega_3}\right)\left(\frac{n\theta(-\sigma^2)}{2\pi\sigma^2}\left[\begin{array}{c}\int_{a_3-a_2}^{a_3} \exp\left(\frac{-r^2}{2\sigma^2}\right) d\left(\frac{-r^2}{2\sigma^2}\right) \\ + \int_{a_4-a_3}^{a_4} \exp\left(\frac{-r^2}{2\sigma^2}\right) d\left(\frac{-r^2}{2\sigma^2}\right) + \ldots\end{array}\right]\right)$$

$$+\ldots$$

Then

$$\frac{\partial^2 E(t(\phi))}{\partial\omega_1\partial a_1} = \frac{n\theta}{\omega_1^2}\left[\frac{-a_1}{\sigma^2}\exp\left(\frac{-a_1^2}{2\sigma^2}\right) + \frac{(a_2-a_1)}{\sigma^2}\exp\left(\frac{-(a_2-a_1)^2}{2\sigma^2}\right)\right] = 0$$

which leads to

$$a_1\exp\left(\frac{-a_1^2}{2\sigma^2}\right) = (a_2-a_1)\exp\left(\frac{-(a_2-a_1)^2}{2\sigma^2}\right).$$

Also,

$$\frac{\partial^2 E(t(\phi))}{\partial\omega_2\partial a_2} = \frac{n\theta}{\omega_2^2}\left[\frac{-(a_2-a_1)}{\sigma^2}\exp\left(\frac{-(a_2-a_1)^2}{2\sigma^2}\right) + \frac{a_2}{\sigma^2}\exp\left(\frac{-a_2^2}{2\sigma^2}\right) + 0 + \frac{(a_3-a_2)}{\sigma^2}\exp\left(\frac{-(a_3-a_2)^2}{2\sigma^2}\right)\right]$$
$$= 0$$

due to

$$a_2\exp\left(\frac{-a_2^2}{2\sigma^2}\right) = (a_2-a_1)\exp\left(\frac{-(a_2-a_1)^2}{2\sigma^2}\right) + (a_3-a_2)\exp\left(\frac{-(a_3-a_2)^2}{2\sigma^2}\right).$$

similarly,

$$a_3\exp\left(\frac{-a_3^2}{2\sigma^2}\right) = (a_3-a_2)\exp\left(\frac{-(a_3-a_2)^2}{2\sigma^2}\right) + (a_4-a_3)\exp\left(\frac{-(a_4-a_3)^2}{2\sigma^2}\right).$$

And so on we can get:

$$a_i\exp\left(\frac{-a_i^2}{2\sigma^2}\right) = (a_i-a_{i-1})\exp(\frac{-(a_i-a_{i-1})^2}{2\sigma^2}) + (a_{i+1}-a_i)\exp(\frac{-(a_{i+1}-a_i)^2}{2\sigma^2}),\ i = 3,4,5,\ldots \blacksquare$$

Since the searcher searches inside the sector with width $a_i - a_{i-1}$, by choosing many values of $a_{1,}$ where the above theorem is true for all values of a_i, $i = 2, 3, 4, 5, ...$ we have two cases

case (1) If $a_i \geq a_{i-1,}$ this is the optimal case which we need to satisfy it along the search process.

case (2) If $a_i < a_{i-1}$, then in this case the value of a_1 is rejected.

If we take $0.5 \leq a_1 \leq 1$, $\sigma = 3$ we can choose the value of a_1 which can get a_i, $i = 2, 3, ...$ that minimize the expected value of the time to detect the target and satisfy the condition $a_i \geq a_{i-1}$ along the searching process. For example, if $a_1 = 0.5$, then by substuting in (4) as in the following:

$$(0.5)\exp\left(\frac{-(0.5)^2}{18}\right) = (a_2 - 0.5)\exp\left(\frac{-(a_2-0.5)^2}{18}\right), \text{ Solution is: } \{[a_2 = 7.3896]\},$$

then $a_2 > a_1$ and for calculating a_3

$$(7.3896)\exp\left(\frac{-(7.3896)^2}{18}\right) = (7.3896 - 0.5)\exp\left(\frac{-(7.3896-0.5)^2}{18}\right) + (a_3 - 7.3896)\exp\left(\frac{-(a_3-7.3896)^2}{18}\right),$$

Solution is: $\{[a_3 = -1.2435]\}$, then $a_3 < a_2$, so we stop the process and reject this value of a_1 and so on.

Some values of a_1 in the interval $(0.5, 1)$ are taken as in the table I to choose the best value of $a_1$between them, which satisfied the above optimal condition along the searching process.

a_1	$a_i, i = 2, 3, ...$	$decision$
0.5	$a_2 = 7.3896$	$a_2 > a_1$
	$a_3 = -1.2435$	$a_3 < a_2$ (stop, reject the value of $a_1 = 0.5$)
0.6	$a_2 = 7.1964$	$a_2 > a_1$
	$a_3 = -1.0879$	$a_3 < a_2$ (stop, reject the value of $a_1 = 0.6$)
0.7	$a_2 = 7.0358$	$a_2 > a_1$
	$a_3 = -0.94742$	$a_3 < a_2$ (stop, reject the value of $a_1 = 0.7$)
0.8	$a_2 = 6.8995$	$a_2 > a_1$
	$a_3 = -0.81871$	$a_3 < a_2$ (stop, reject the value of $a_1 = 0.8$)
0.9	$a_2 = 1.8$	$a_2 > a_1$
	$a_3 = 2.4588$	$a_3 > a_2$
	$a_4 = 7.7563$	$a_4 > a_3$
	$a_5 = 1.8247$	$a_5 < a_4$ (stop, reject the value of $a_1 = 0.9$)
1	$a_2 = 2.0$	$a_2 > a_1$
	$a_3 = 8.4055$	$a_3 > a_2$
	$a_4 = 1.5044$	$a_4 < a_3$ (stop, reject the value of $a_1 = 1$)

table I

Since $v_i = \omega_i r_i$ and we can obtain from (4) the optimal redius r_i and the speed v_i is "*regular speed*" on any circle so if we take $v_i = v_3$ =constant, we can obtain the optimal "*angular velocity*" in any circle from $\omega_i = \frac{v_3}{r_i}$, $i = 1, 2, 3, ...$

Special Case If the width is fixed (i.e. $a_i - a_{i-1} = a$), then $a_1 = a$, $a_2 = 2a$, $a_3 = 3a, ...,$ in (1) we get:

$$E(t(\phi)) = \sum_{i=1}^{\infty} \left[\left(4ia + \frac{2\pi}{\omega_i}\right) \sum_{s=i}^{\infty} \sum_{k=1}^{n} \int_{(s-1)a}^{sa} \int_{\theta_{k-1}}^{\theta_k} g(r\cos\theta, r\sin\theta) r\,dr\,d\theta \right], \tag{5}$$

then if the target has *circular normal distribution* with joint continuous distribution function $F(x, y)$ and joint density function $f(x, y)$ as in (3). If ϕ is an optimal search plan, then in (5) we get:

$$E(t(\phi)) = \sum_{i=1}^{\infty}\left[\left(4ia+\frac{2\pi}{\omega_i}\right)\left(\frac{n\theta(-\sigma^2)}{2\pi\sigma^2}\right)\sum_{s=i}^{\infty}\int_{(s-1)a}^{sa}\exp\left(\frac{-r^2}{2\sigma^2}\right)d\left(\frac{-r^2}{2\sigma^2}\right)\right]$$

$$= -\frac{n\theta}{2\pi}\left[\sum_{i=1}^{\infty}\left(4ia+\frac{2\pi}{\omega_i}\right).\sum_{s=i}^{\infty}\left(\exp\left(\frac{-s^2a^2}{2\sigma^2}\right)-\exp\left(\frac{-(s-1)^2a^2}{2\sigma^2}\right)\right)\right]$$

Then

$$\frac{\partial E(t(\phi))}{\partial a} = \left(-\frac{n\theta}{2\pi}\right)\left[\sum_{i=1}^{\infty}\frac{\partial}{\partial a}\left(4ia+\frac{2\pi}{\omega_i}\right)\right]\cdot\left[\sum_{s=i}^{\infty}\left(\exp\left(\frac{-s^2a^2}{2\sigma^2}\right)-\exp\left(\frac{-(s-1)^2a^2}{2\sigma^2}\right)\right)\right]$$

$$+\left(-\frac{n\theta}{2\pi}\right)\left[\sum_{i=1}^{\infty}\left(4ia+\frac{2\pi}{\omega_i}\right)\right]\cdot\left[\sum_{s=i}^{\infty}\frac{\partial}{\partial a}\left(\exp\left(\frac{-s^2a^2}{2\sigma^2}\right)-\exp\left(\frac{-(s-1)^2a^2}{2\sigma^2}\right)\right)\right]$$

and

$$\frac{\partial^2 E(t(\phi))}{\partial a\partial\omega_i} = \left(-\frac{n\theta}{2\pi}\right)\left[\sum_{i=1}^{\infty}\frac{-2\pi}{\omega_i^2}\right]\cdot\left[\sum_{s=i}^{\infty}\left(\frac{a(s-1)^2}{\sigma^2}\exp\left(\frac{-(s-1)^2a^2}{2\sigma^2}\right)-\frac{as^2}{\sigma^2}\exp\left(\frac{-s^2a^2}{2\sigma^2}\right)\right)\right]$$

$$=0$$

which leads to

$$\left[\sum_{i=1}^{\infty}\frac{1}{\omega_i^2}\right]\cdot\left[\sum_{s=i}^{\infty}\left(\frac{a(s-1)^2}{\sigma^2}\exp\left(\frac{-(s-1)^2a^2}{2\sigma^2}\right)-\frac{as^2}{\sigma^2}\exp\left(\frac{-s^2a^2}{2\sigma^2}\right)\right)\right]=0 \quad (6)$$

by solving (6) we can determine the optimal value of a, and substuting in (5), we can minimize the expected value of the time for detecting the target.

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