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Some Remarks on Finitely Quasi-injective Modules

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Abstract. Let R be a ring. A right R-module M is called finitely quasi-injective if each R-homomorphism from a finitely generated submodule of M to M can be extended to an endomorphism of M. Some conditions under which finitely generated finitely quasi-injective modules are of finite Goldie Dimensions are given, and finitely generated finitely quasi-injective Kasch modules are studied.

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1. Introduction

Throughout the paper, R is an associative ring with identity and all modules are unitary. If M_R is a right R-module with $S = End(M_R)$, and $A \subseteq S, X \subseteq M, B \subseteq R$, then we denote the Jacobson radical of S by $J(S)$, and we write $l_S(X) = \{s \in S \mid sx = 0, \forall x \in X\}$, $r_M(A) = \{m \in M \mid am = 0, \forall a \in A\}$, $l_M(B) = \{m \in M \mid mb = 0, \forall b \in B\}$. Following [5], we write $W(S) = \{s \in S \mid Ker(s) \subseteq^{ess} M\}$.

At first let we recall some concepts. A module M_R is called finitely quasi-injective (or FQ-injective for short) [7] if each R-homomorphism from a finitely generated submodule of M to M can be extended to an endomorphism of M; a ring R is said to be right F-injective if R_R is finitely quasi-injective. F-injective rings have been studied by many authors such as [2, 3, 8]. A module M_R is called a C_1 module if every submodule of M is essential in a direct summand of M, C_1 modules are also called CS modules. A module M_R is called a C_2 module if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. A module M_R is called a C_3 module if, whenever N and K are submodules of M with $N \subseteq^{\oplus} M, K \subseteq^{\oplus} M$, and $N \cap K = 0$, then $N \oplus K \subseteq^{\oplus} M$. A module M_R is called continuous if it is both C_1 and C_2. A module M_R is called quasi-continuous if it is both C_1 and C_3. It is well-known that C_2 modules are C_3 modules, and so continuous modules are quasi-continuous.

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A module M_R is said to be Kasch [1] provided that every simple module in $\sigma[M]$ embeds in M, where $\sigma[M]$ is the category consisting of all M-subgenerated right R-modules. In this note we shall mainly study finitely generated finitely quasi-injective modules with finite Goldie Dimensions, and finitely generated finitely quasi-injective Kasch modules, respectively.

2. Main Results

We begin with some Lemmas.

Lemma 1 ([10, Theorem 1.2]). *For a module M_R with $S = End(M_R)$, the following statements are equivalent:*

(1) M_R is FQ-injective;

(2) (a) $l_S(A \bigcap B) = l_S(A) + l_S(B)$ for any finitely generated submodules A, B of M, and

(b) $l_M r_R(m) = Sm$ for any $m \in M$. where $l_M r_R(m)$ consists of all elements $z \in M$ such that $mx = 0$ implies $zx = 0$ for any $x \in R$.

Lemma 2 ([10, Theorem 2.1, Theorem 2.2]). *Let M_R be a finitely generated FQ-injective module with $S = End(M_R)$. Then*

(1) $l_S(Ker\alpha) = S\alpha$ for any $\alpha \in S$.

(2) $W(S) = J(S)$.

Lemma 3 ([10, Theorem 2.3]). *Let M_R be a finitely generated finite dimensional FQ-injective module with $S = End(M_R)$. Then S is semilocal.*

Lemma 4. *Let M_R be a finitely generated FQ-injective module. Then it is a C_2 module.*

Proof. Write $S = End(M_R)$. Let N be a submodule of M and $N \cong eM$ for some $e^2 = e \in S$. Then there exists some $s \in S$ such that $N = seM$ and $Ker(se) = Ker(e)$. By Lemma 2(1), we have $Sse = Se$, and hence $e = tse$ for some $t \in S$ with $t = et$. Thus $(set)^2 = set$ and $N = (se)M = (set)M$. Therefore N is a direct summand of M. □

Lemma 5. *Let M_R be a quasi-continuous module with $S = End(M_R)$. Then idempotents of $S/W(S)$ can be lifted.*

Proof. Let $s^2 - s \in W(S)$, then $Ker(s^2 - s) \lhd M$. If $x \in Ker(s^2 - s)$, then $(1 - s)x \in Ker(s)$, $sx \in Ker(1 - s)$, and hence $x = (1 - s)x + sx \in Ker(s) \oplus Ker(1 - s)$. It shows that $Ker(s^2 - s) \subseteq Ker(s) \oplus Ker(1 - s)$, and thus $Ker(s) \oplus Ker(1 - s) \lhd M$. Now let N_1 and N_2 be maximal essential extensions of $Ker(s)$ and $Ker(1 - s)$ in M, respectively. Then it is clear that $N_1 \cap N_2 = 0$ and $N_1 \oplus N_2 \lhd M$. Since M is a C_1 module and N_1 and N_2 are closed submodules of M, N_1 and N_2 are direct summand of M. But M is a C_3 module, $N_1 \oplus N_2$ is a direct summand of M, so that $N_1 \oplus N_2 = M$. This implies that there exists an $e^2 = e \in S$ such that $N_1 = (1 - e)M$ and $N_2 = eM$. Let $y \in Ker(s), z \in Ker(1 - s)$, then noting that $y \in (1 - e)M$ and $z \in eM$, we have $(e - s)(y + z) = z - sz = (1 - s)z = 0$, so that $Ker(s) \oplus Ker(1 - s) \subseteq Ker(e - s)$. And hence $e - s \in W(S)$, that is, idempotents modulo $W(S)$ lift. □

Corollary 1. *Let M_R be a finitely generated FQ-injective C_1 module with $S = End(M_R)$. Then S is semiperfect if and only if S is semilocal.*

Proof. Since M_R is a finitely generated FQ-injective module , by Lemma 4, it is a C_2 module and hence a C_3 module. Thus M_R is quasi-continuous by the condition that M_R is a C_1 module. And so the result follows from Lemma 5 and Lemma 2(2). □

Recall that a ring R is called right MP-injective [11] if every monomorphism from a principal right ideal of R to R extends to an endomorphism of R; a ring R is called right MGP-injective [11] if, for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any R-monomorphism from a^nR to R extends to an endomorphism of R; a ring R is said to be right AP-injective [6] if, for any $a \in R$, there exists a left ideal X_a such that $lr(a) = Ra \oplus X_a$; a ring R is called right AGP-injective if, for any $0 \neq a \in R$, there exists a positive integer n and a left ideal X_{a^n} such that $a^n \neq 0$ and $lr(a^n) = Ra^n \oplus X_{a^n}$. Clearly, right MP-injective rings are right MGP-injective, and right AP-injective rings are right AGP-injective. If R is a right MP-injective rings, then R is right C_2 by [11, Theorem 2.7] and $J(R) = Z(R_R)$ by [11, Theorem 3.4]. If R is a right AP-injective rings, then R is right C_2 by [9, Corollary 3.4] and $J(R) = Z(R_R)$ by [6, Corollary 2.3]. So by Lemma 5, we have immediately the following corollary.

Corollary 2. *Let R be a right CS ring. If R is right MP-injective or right AP-injective, then R is semiperfect if and only if R is semilocal.*

Let M be a right R-module. A finite set $A_1,\ldots,A_n$ of proper submodules of M is said to be coindependent if for each $i, 1 \leq i \leq n, A_i + \cap_{j\neq i} A_j = M$, and a family of submodules of M is said to be coindependent if each of its finite subfamily is coindependent. The module M is said to have finite dual Goldie dimension if every coindependent family of submodules of M is finite. Refer to [4] for details concerning the dual Goldie dimension.

Lemma 6 ([4, Propositions 2.43]). *A ring R is semilocal if and only if R_R has finite dual Goldie dimension, if and only if $_RR$ has finite dual Goldie dimension.*

Theorem 1. *Let M_R be a finitely generated FQ-injective module with $S = End(M_R)$. Then the following conditions are equivalent:*

(1) S is semilocal.

(2) M_R is finite dimensional.

Furthermore, if M is a C_1 module, then these conditions are equivalent to:

(3) S is semiperfect.

Proof. (1) ⇒ (2). If M_R is not finite dimensional, then there exists $0 \neq x_i \in M, i = 1, 2, 3, \cdots$, such that $\sum_{i=1}^{\infty} x_iR$ is a direct sum. Since M_R is FQ-injective, by Lemma 1, for any positive integer m and any finite subset

$$I \subset \mathbb{N} \setminus m, S = l_S(0) = l_S(x_mR \cap \sum_{i\in I} x_iR) = l_S(x_m) + l_S(\sum_{i\in I} x_iR) = l_S(x_m) + \cap_{i\in I} l_S(x_i).$$

Thus $l_S(x_i), i = 1,2,3,\cdots$ is an infinite coindependent family of submodules of ${}_SS$. By Lemma 6, S is not semilocal, a contradiction.

(2) $\Rightarrow$ (1). By Lemma 3.

Furthermore, if M is a C_1 module, then since it is a C_2 module by Lemma 4 and $W(S) = J(S)$ by Lemma 2(2), we have (1) $\Longleftrightarrow$ (3) by Lemma 5. □

The equivalence of (1) and (2) in the next Corollary 3 appeared in [8, Corollary 4.5].

Corollary 3. *Let R be a right F-injective ring. Then the following conditions are equivalent:*

(1) R is semilocal.

(2) R is right finite dimensional.

Furthermore, if R is a right CS ring, then these conditions are equivalent to:

(3) R is semiperfect.

Theorem 2. *Let M_R be a FQ-injective Kasch module with $S = End(M_R)$, then*

(1) $r_M l_S(K) = K$ for every finitely generarated submodule K_R of M_R.

(2) Sm is simple if and only if mR is simple. In particular, $Soc(M_R) = Soc({}_SM)$.

(3) $l_M(J(R)) \lhd_S M$.

Moreover, if M_R is finitely generated, then

(4) $l_S(T)$ is a minimal left ideal of S for any maximal submodule T of M.

(5) $l_S(Rad(M)) \lhd_S S$.

Proof. (1). Always $K \subseteq r_M l_S(K)$. If $m \in r_M l_S(K) - K$, let $K \subseteq T \subseteq^{max} (mR + K)$. By the Kasch hypothesis, let $\sigma : (mR + K)/T \to M$ be monic, and define $\gamma : mR + K \to M$ by $\gamma(x) = \sigma(x + T)$. Since M_R is FQ-injective, $\gamma = s\cdot$ for some $s \in S$, so $sK = \gamma(K) = 0$. This gives $sm = 0$ as $m \in r_M l_S(K)$. But $sm = \sigma(m + T) \neq 0$ because $m \notin T$, a contradiction. Therefore, $r_M l_S(K) = K$.

(2). If mR is simple. Then if $0 \neq sm \in Sm$, define $\gamma : mR \to smR$ by $\gamma(x) = sx$. Then γ is a right R-isomorphism, and hence γ^{-1} extends to an endomorphism of M. Thus, $m = \gamma^{-1}(sm) = \alpha(sm)$ for some $\alpha \in S$, and so Sm is simple. Conversely, If Sm is simple. By (1), $r_M l_S(m) = mR$ for each $m \in M$, which implies that for any $m \in M$, every S-homomorphism from Sm to M is right multiplication by an element of R. Now for any $0 \neq ma \in mR$, the right multiplication $\cdot a : Sm \to Sma$ is a left S-isomorphism. So let $\theta : Sma \to Sm$ be its inverse, then θ is a right multiplication by an element b of R. Thus, $m = \theta(ma) = mab \in (ma)R$. Hence mR is simple.

(3). Let $0 \neq m \in M$. Suppose that T is a maximal submodule of mR. By the Kasch hypothesis, let $\sigma : mR/T \to M$ be monic, and define $f : mR \to M$ by $f(x) = \sigma(x + T)$. Since M_R is FQ-injective, $f = s\cdot$ for some $s \in S$, and then $sm = f(m) = \sigma(m + T) \neq 0$. But

$smJ(R) = f(m)J(R) = \sigma(m+T)J(R) = 0$, so $0 \neq sm \in Sm \cap l_M(J(R))$. Therefore, $l_M(J(R)) \lhd {}_SM$.

(4). Let T be any maximal submodule of M. Since M_R is Kasch, there exists a monomorphism $\varphi : M/T \to M$. Define $\alpha : M \to M$ by $x \mapsto \varphi(x+T)$. Then $0 \neq \alpha \in S, \alpha T = \varphi(0) = 0$, and so $l_S(T) \neq 0$. For any $0 \neq s \in l_S(T)$, we have $T \subseteq Ker(s) \neq M$, and so $Ker(s) = T$ by the maximality of T. It follows that $l_S(T) = l_S(ker(s)) = Ss$ by Lemma 2(1). Therefore, $l_S(T)$ is a minimal left ideal of S.

(5). If $0 \neq a \in S$, choose a maximal submodule T of the right R-module aM. Since M is Kasch, there exists a monomorphism $f : aM/T \to M$. Define $g : aM \to M$ by $g(x) = f(x+T)$. Since M is FQ-injective and finitely generated, $g = s\cdot$ for some $s \in S$. Take $y \in M$ such that $ay \notin T$, then $say = g(ay) = f(ay+T) \neq 0$, and hence $sa \neq 0$. If $a(Rad(M)) \nsubseteq T$, then $a(Rad(M)) + T = aM$. But $a(Rad(M)) << aM$ because M is finitely generated, so $T = aM$, a contradiction. Thus $a(Rad(M)) \subseteq T$, and then $(sa)(Rad(M)) = g(a(Rad(M))) = f(0) = 0$, whence $0 \neq sa \in Sa \cap l_S(Rad(M))$. This shows that $l_S(Rad(M)) \lhd_S S$. □

Lemma 7. *Let M_R be a finitely generated Kasch module with $S = end(M_R)$. If S is left finite dimensional, then $M/RadM$ is semisimple.*

Proof. Let T be any maximal submodule of M. Since M_R is Kasch, there exists a monomorphism $\varphi : M/T \to M$. Define $\alpha : M \to M$ by $x \mapsto \varphi(x+T)$. Then $0 \neq \alpha \in S, \alpha T = \varphi(0) = 0$, and so $l_S(T) \neq 0$. Let $\Omega = \{K \mid 0 \neq K = l_S(X)$ for some $X \subseteq M\}$, then $l_S(T)$ is minimal in Ω for any maximal submodule T of M. In fact, if $l_S(T) \supseteq l_S(X) \neq 0$, where $X \subseteq M$, then $T \subseteq r_M l_S(X) \neq M$. So $T = r_M l_S(X)$, and hence $l_S(T) = l_S(X)$. Since S is left finite dimensional, there exist some minimal members $I_1, I_2, \cdots, I_n$ in Ω such that $I = \oplus_{i=1}^n I_i$ is a maximal direct sum of minimal members in Ω. Now we establish the following claims:

Claim 1. $r_M(I_i)$ is a maximal submodule of M for each i.

Since M is finitely generated and Kasch, $r_M(I_i) \subseteq T_i = r_M l_S(T_i)$ for some maximal submodule T_i. Thus $I_i \supseteq l_S r_M l_S(T_i) = l_S(T_i) \neq 0$, and so $I_i = l_S(T_i)$ by the minimality of I_i in Ω. Now we choose $0 \neq a_i \in l_S(T_i)$. Then $T_i = r_M(a_i)$, and hence $r_M(I_i) = r_M l_S(T_i) = r_M l_S r_M(a_i) = r_M(a_i) = T_i$.

Claim 2. $RadM = \cap_{i=1}^n r_M(I_i)$.

Clearly, $RadM \subseteq \cap_{i=1}^n r_M(I_i)$. If T is a maximal submodule of M, then $l_S(T) \cap I \neq 0$. Taking some $0 \neq b \in l_S(T) \cap I$, we have $T = r_M(b) \supseteq \cap_{i=1}^n r_M(I_i)$. This gives that $\cap_{i=1}^n r_M(I_i) \subseteq RadM$, and the claim follows.

Finally, observing that each $M/r_M(I_i)$ is simple by Claim 1, and the mapping

$$f : M/RadM \to \oplus_{i=1}^n M/r_M(I_i); m + RadM \mapsto (m + r_M(I_1), \cdots, m + r_M(I_n))$$

is a monomorphism by Claim 2, we have that $M/RadM$ is semisimple. □

Theorem 3. *Let M_R be a finitely generated and FQ-injective Kasch module with $S = End(M_R)$. Then the following conditions are equivalent:*

(1) $M/Rad(M)$ *is semisimple.*

(2) S *is left finitely cogenerated.*

(3) S *is left finite dimensional.*

In this case, $Soc({}_SS) = l_S(Rad(M))$*, and* $G({}_SS) = c({}_SSoc({}_SS)) = c(M/Rad(M))$.

Proof. (1) ⇒ (2). It is trival in case $M = 0$. If $M \neq 0$, then $M/RadM \neq 0$ because M is finitely generated. As $M/RadM$ is semisimple, there exist maximal submodules $T_1, T_2, \cdots, T_n$ such that $M/RadM \cong \oplus_{i=1}^n M/T_i$. Hence, by Theorem 2(4),

$$l_S(RadM) \cong {}_SHom_R(M/RadM,\ {}_SM_R) \cong {}_SHom_R(\oplus_{i=1}^n M/T_i,\ {}_SM_R) \cong \oplus_{i=1}^n l_S(T_i)$$

is an n-generated semisimple left ideal of S. This implies that $l_S(RadM) = Soc({}_SS) \lhd_S S$ by Theorem 2(5), and therefore S is left finitely cogenerated, and $G({}_SS) = n = c({}_SSoc({}_SS))$.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). See Lemma 7. □

Corollary 4. *Let R be a right F-injective right Kasch ring. Then the following conditions are equivalent:*

(1) R *is semilocal.*

(2) R *is left finitely cogenerated.*

(3) R *is left finite dimensional.*

In this case, $Soc({}_RR) = l_R(J(R))$*, and* $G({}_RR) = c({}_RSoc({}_RR)) = c(R/J(R))$.

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