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On Decompositions of Continuity and Complete Continuity in Ideal Topological Spaces

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Abstract. We define new classes of sets called $\delta\beta_I$-open set, $\delta\alpha - I$-open set, $\delta_\beta - I-$set, semi$^* - I$-open set, $s\delta_I - g$-closed set in ideal topological spaces. Using these sets, we obtain decompositions of continuity and complete continuity in ideal topological spaces. Also, we investigate some properties of these sets and relationship other generalized sets.

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1. Introduction and Preliminaries

Recently, Ekici and Noiri [3] have introduced $pre^* - I$-open sets to obtain a decomposition of continuity and defined $\alpha_* - I$-open sets and showed that the family of $\alpha_* - I$-open sets is a topology in ideal topological space. In [14], the authors have studied some new classes of functions in ideal topological spaces. In this paper, we define new classes of sets called $\delta\beta_I$-open set, $\delta\alpha - I$-open set, $\delta_\beta - I$-set, semi$^* - I$-open set, $s\delta_I - g$-closed set in ideal topological spaces. Using these sets, we obtain decompositions of continuity and complete continuity in ideal topological spaces. Also, we investigate some properties of these sets and relationship other generalized sets.

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y), always mean topological spaces on which no separation axiom is assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively.

A subset of a space (X, τ) is said to be regular open (resp. regular closed) [12] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A is called δ-open [12] if for each $x \in A$, there exists a regular open set G such that $x \in G \subset A$. The complement of a $delta$-open set is called δ-closed. A point $x \in X$ is called a δ-cluster point of A if $Int(Cl(U)) \cap A \neq \varnothing$ for each open set U containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$. The δ-interior of A is the union of all regular open sets of X contained in A and it is denoted by $Int_\delta(A)$.

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An ideal I on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies $i)$ $A \in I$ and $B \subset A$ implies $B \in I$, $ii)$ $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space is a topological space (X,τ) with an ideal I on X and if $P(X)$ is the set of all subsets of X, a set operator $(\cdot)^* : P(X) \to P(X)$ called a local function [8] of A with respect to τ and I is defined as follows: for $A \subset X$, $A^*(I) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. We simply write A^* instead of $A^*(I,\tau)$. X^* is often a proper subset of X. The hypothesis $X = X^*$ [6] is equivalent to the hypothesis $\tau \cap I = \varnothing$. For every ideal topological space, there exists a topology $\tau^*(I)$ or briefly τ^*, finer than τ, generated by $\beta(I,\tau) = \{U \setminus I : U \in \tau \text{ and } I \in I\}$, but in general $\beta(I,\tau)$ is not always a topology [7]. Additionally, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$. If I is an ideal on X, then (X,τ,I) is called an ideal topological space or simply an ideal space. A subset A of an ideal space (X,τ,I) is said to be $R-I$-open [14] if $A = Int(Cl^*(A))$. A point x in an ideal space (X,τ,I) is called a $\delta_I - cluster$ point of A if $Int(Cl^*(U)) \cap A \neq \varnothing$ for each neighborhood U of x. The set of all δ_I-cluster points of A is called the δ_I-closure of A and is denoted by $\delta Cl_I(A)$. A is said to be δ_I-closed [14] if $\delta Cl_I(A) = A$. The complement of δ_I-closed set is called δ_I-open set.

Lemma 1 ([7]). *Let (X,τ,I) be an ideal topological space and A, B be subsets of X.*

1. *If $A \subset B$, then $A^* \subset B^*$*
2. *If $G \in \tau$, then $G \cap A^* \subset (G \cap A)^*$*
3. *$A^* = Cl(A^*) \subset Cl(A)$.*

Definition 1. *A subset A of an ideal topological space (X,τ,I) is called*

a) α-open [10] if $A \subset Int(Cl(Int(A)))$

b) $preopen$ [9] if $A \subset Int(Cl(A))$

c) $Pre-I$-open [2] if $A \subset Int(Cl^*(A))$

d) $\alpha - I$-open [4] if $A \subset Int(Cl^*(Int(A)))$

e) δ-preopen [11] if $A \subset Int(Cl_\delta(A))$

f) $pre^* - I$-open [3] if $A \subset Int(\delta Cl_I(A))$

g) $\alpha_* - I$-open [3] if $A \subset Int(Cl^*(Int_\delta(A)))$

h) strongly $\alpha - I$-open [3] if $A \subset Int(Cl^*(\delta Int_I(A)))$

i) β_I^*-open [3] if $A \subset Cl^*(Int(Cl_\delta(A)))$

j) $t - I$-set [5] if $Int(Cl^*(A)) = Int(A)$

k) $\delta\beta_I^*$-open [13] if $A \subset Cl^*(Int(\delta Cl_I(A)))$

2. $\delta\beta_I$-Open Sets

Definition 2. *A subset A of an ideal space(X,τ,I) is said to be $\delta\beta_I$-open if $A \subset Cl(Int(\delta Cl_I(A)))$.*

Remark 1. *The following diagram holds for a subset A of an ideal space (X,τ,I).*

$$\begin{array}{ccccccccc}
\text{open} & & & & & & & & \\
\downarrow & & & & & & & & \\
\alpha - I\text{-open} & \rightarrow & \text{pre}-I\text{-open} & \rightarrow & \text{pre}^* - I\text{-open} & \rightarrow & \delta\beta_I^*\text{-open} & \rightarrow & \delta\beta_I\text{-open} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\alpha\text{-open} & \rightarrow & \text{preopen} & \rightarrow & \delta\text{-preopen} & \rightarrow & \beta_I^*\text{-open} & \rightarrow & \delta\beta\text{-open}
\end{array}$$

Figure 1: Diagram

None of these implications is reversible, as shown in the following example and in [3]

Example 1. *Let $X = \{a,b,c,d\}$, $\tau = \{X,\varnothing,\{a\},\{b\},\{a,b\},\{a,c\},\{a,b,c\}\}$ and $I = P(X)$. Then the set $\{c,d\}$ is $\delta\beta-$open, but it is not $\delta\beta_I$-open.*

The set $\{b,d\}$ is $\delta\beta_I$-open set, but it is not both β_I^*-open and $\delta\beta_I^*$-open. If we take $I = \{\varnothing\}$, then the set $\{c,d\}$ is $\delta\beta_I$-open, but it is not pre$^* - I$-open. The family of all $\delta\beta_I$-open (resp. $\delta\beta_I$-closed) sets of X is denoted by $\delta\beta IO(X)$ (resp. $\delta\beta IC(X)$).

Definition 3. *Let (X,τ,I) be an ideal space.*

a) The union of all $\delta\beta_I$-open sets contained in A is called the $\delta\beta_I$-interior of A and is denoted by $\delta_\beta Int_I(A)$

b) The intersection of all $\delta\beta_I$-closed sets containing A is called the $\delta\beta_I$-closure of A and is denoted by $\delta_\beta Cl_I(A)$.

Theorem 1. *The following properties hold for the $\delta\beta_I$-closure of a set A in a space (X,τ,I).*

a) A is $\delta\beta_I$-closed in X if and only if $A = \delta_\beta Cl_I(A)$,

b) $\delta_\beta Cl_I(A) \subset \delta_\beta Cl_I(B)$ whenever $A \subset B \subset X$,

c) $\delta_\beta Cl_I(A)$ is $\delta\beta_I$-closed,

d) $\delta_\beta Cl_I(\delta_\beta Cl_I(A)) = \delta_\beta Cl_I(A)$,

e) $x \in \delta_\beta Cl_I(A)$ if $A \cap U \neq \varnothing$ for every $\delta\beta_I$-open set containing x.

Proof. Straightforward.

We give the following Lemma using in the sequel.

Lemma 2. *Let A be a subset of a space (X,τ,I). Then*

a) $\delta Cl_I(A) \cap U \subset \delta Cl_I(A \cap U)$, for any δ_I-open set U in X, *b)* $\delta Int_I(A \cup F) \subset \delta Int_I(A) \cup F$*, for any* δ_I*-closed set* F *in* X*.*

Proposition 1. *Let* (X, τ, I) *be an ideal space. If* $A \subset B \subset \delta Cl_I(A)$ *and* B *be a* $\delta\beta_I$*-open, then* A *is* $\delta\beta_I$*-open.*

Proof. Let $A \subset B \subset \delta Cl_I(A)$ and B be a $\delta\beta_I$-open. Then we have $\delta Cl_I(A) = \delta Cl_I(B)$. Thus, $A \subset B \subset Cl(Int(\delta Cl_I(B))) = Cl(Int(\delta Cl_I(A)))$ and hence A is $\delta\beta_I$-open set.

Proposition 2. *Let* (X, τ, I) *be an ideal space. If* $A \subset B \subset Cl(A)$ *and* A *be a* $\delta\beta_I$*-open, then* B *is* $\delta\beta_I$*-open.*

Proof. Let $A \subset B \subset Cl(A)$ and A be $\delta\beta_I$-open. Then $A \subset Cl(Int(\delta Cl_I(A)))$. Since $B \subset Cl(A)$, then $B \subset Cl(Cl(Int(\delta Cl_I(A)))) = Cl(Int(\delta Cl_I(A))) \subset Cl(Int(\delta Cl_I(B)))$.

Thus B is $\delta\beta_I$-open set.

Corollary 1. *Let* (X, τ, I) *be an ideal space. If* A *is* $\delta\beta_I$*-open, then* $Cl(A)$ *is* $\delta\beta_I$*-open.*

Proposition 3. *Let* (X, τ, I) *be an ideal space and* $A \subset X$ *is* $\delta\beta_I$*-closed if and only if* $Cl(Int(\delta Int_I(A))) \subset A$*.*

Proof. Let $A \in \delta\beta IC(X) \iff X - A \in \delta\beta IO(X)$.

$$\begin{aligned}
&\iff X - A \subset Cl(Int(\delta Cl_I(X - A)))\\
&\quad = Cl(Int(X - \delta Int_I(A))) = Cl(X - Cl(\delta Int_I(A)))\\
&\quad = X - Int(Cl(\delta Int_I(A))) \iff Int(Cl(\delta Int_I(A))) \subset A.
\end{aligned}$$

Remark 2. *The intersection of any two* $\delta\beta_I$*-open sets need not be* $\delta\beta_I$*-open set as shown example below.*

Example 2. *Let* $X = \{a, b, c, d\}$*,* $\tau = \{X, \varnothing, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ *and* $I = P(X)$*. Hence* $\{b, d\}, \{a, c, d\}$ *are* $\delta\beta_I$*-open sets, but the set* $\{d\}$ *is not* $\delta\beta_I$*-open.*

Definition 4. *A subset A of an ideal space* (X, τ, I) *is said to be* $\delta\alpha{-}I$*-open if* $A \subset Int(Cl(\delta Int_I(A)))$*.*

The family of all $\delta\alpha - I$-open (resp. $\delta\alpha - I$-closed) sets of X is denoted by $\delta\alpha IO(X)$ (resp. $\delta\alpha IC(X)$).

It is obvious that every δ_I-open set is $\delta\alpha - I$-open.

Proposition 4. *Let* (X, τ, I) *be an ideal topological space. Then, the family of* $\delta\alpha - I$*-open sets is a topology for* X*.*

Proof. It is obvious that $\varnothing$ and X are $\delta\alpha - I$-open sets.

Let *A*, B be $\delta\alpha - I$-open sets. Then

$$\begin{aligned}
A \cap B &\subset Int(Cl(\delta Cl_I(A))) \cap Int(Cl(\delta Cl_I(B)))\\
&= Int(Cl(\delta Int_I(A)) \cap Int(Cl(\delta Int_I(B))))
\end{aligned}$$

$$\begin{aligned}
&\subset Int(Cl(\delta Int_I(A) \cap Cl(\delta Int_I(B)))\\
&= Int(Cl(Int(\delta Int_I(A)) \cap Cl(\delta Int_I(B))))\\
&\subset Int(Cl(Cl(\delta Int_I(A) \cap \delta Int_I(B))))\\
&= Int(Cl(\delta Int_I(A \cap B)))
\end{aligned}$$

Hence, $A \cap B$ is a $\delta\alpha - I$-open set.

For the last axiom of topology, let A_i be $\delta\alpha - I$-open sets for $i \in I$. Then

$$A_i \subset Int(Cl(\delta Int_I(A_i))) \subset Int(Cl(\delta Int_I(\cup_{i\in I} A_i))).$$

Thus, $\cup_{i\in I} A_i \subset Int(Cl(\delta Int_I(\cup_{i\in I} A_i)))$. This implies that $\cup_{i\in I} A_i$ is a $\delta\alpha - I$-open set.

Proposition 5. *Let (X, τ, I) be an ideal space. If A is $\delta\beta_I$-open and B is $\delta\alpha - I$-open, then $A \cap B$ is $\delta\beta_I$-open.*

Proof. Let $A \in \delta\beta IO(X)$ and $B \in \delta\alpha IO(X)$. Then, we have $A \subset Cl(Int(\delta Cl_I(A)))$ and $B \subset Int(Cl(\delta Int_I(B)))$, respectively. This implies that

$$\begin{aligned}
A \cap B &\subset Cl(Int(\delta Cl_I(A))) \cap Int(Cl(\delta Int_I(B)))\\
&\subset Cl(Int((Int(\delta Cl_I(A)))) \cap (Cl(\delta Int_I(B)))))\\
&\subset Cl(Int(Cl(\delta Cl_I(A) \cap \delta Int_I(B)))) \subset Cl(Int(Cl(\delta Cl_I(A \cap B))))\\
&\subset Cl(Int(\delta Cl_I(\delta Cl_I(A \cap B)))) = Cl(Int(\delta Cl_I(A \cap B))).
\end{aligned}$$

Corollary 2. *A set A in (X, τ, I) is a $\delta\beta_I$-open if and only if $U \cap A \in \delta\beta IO(X)$, for every δ_I-open set U of (X, τ, I).*

Proof. Let A be a $\delta\beta_I$-open set. Then we have

$$\begin{aligned}
U \cap A &\subset U \cap Cl(Int(\delta Cl_I(A)))\\
&= Int(U) \cap Cl(Int(\delta Cl_I(A))) \subset Cl(Int(U) \cap Int(\delta Cl_I(A)))\\
&= Cl(Int(U \cap \delta Cl_I(A))) \subset Cl(Int(\delta Cl_I(U \cap A)))
\end{aligned}$$

by Lemma 2. Hence $U \cap A \in \delta\beta IO(X)$.

Definition 5. *A subset A of an ideal space (X, τ, I) is said to be*

a) strongly$-t-I$-set [3] if $Int(\delta Cl_I(A)) = Int(A)$

b) $\delta_\beta - t$-set [5] if $Cl(Int(Cl_\delta(A))) = Int(A)$

c) $\delta_\beta - t - I$-set if $Cl(Int(\delta Cl_I(A))) = Int(A)$

d) $\delta\alpha^* - I$-set if $Int(Cl(\delta Int_I(A))) = \delta Int_I(A)$

Proposition 6. *a)* $\delta_\beta - t$*-set is a* $\delta_\beta - t - I$*-set.*

b) A $\delta_\beta - t - I$*-set is a strongly*$-t - I$*-set.*

Proof. Obvious.

Proposition 7. *Let A and B be subsets of an ideal space* (X, τ, I)*. If A and B are* $\delta_\beta - t - I$*-sets, then* $A \cap B$ *is a* $\delta_\beta - t - I$*-set.*

Proof. Let A and B be $\delta_\beta - t - I$-sets. Then

$$\begin{aligned} Int(A \cap B) &\subset Cl(Int(\delta Cl_I(A \cap B))) \\ &\subset Cl(Int(\delta Cl_I(A) \cap \delta Cl_I(B))) \\ &= Cl(Int(\delta Cl_I(A)) \cap Int(\delta Cl_I(B))) \\ &\subset Cl(Int(\delta Cl_I(A))) \cap Cl(Int(\delta Cl_I(B))) \\ &= Int(A) \cap Int(B) = Int(A \cap B) \end{aligned}$$

This implies that $A \cap B$ is a $\delta_\beta - t - I$-set.

Definition 6. *Let* (X, τ, I) *be an ideal space.*

a) A subset A in X is said to be $\delta_\beta - B - I$*-set (resp. strongly*$B - I$*-set [3],* $\delta_\beta - B$*-set [5]) if there is a* $U \in \tau$ *and a* $\delta_\beta - t - I$*-set (resp. strongly*$-t - I$*-set,* $\delta_\beta - t$*-set) V in X such that* $A = U \cap V$*.*

b) A subset A in X is said to be $\delta - C$*-set if there is a* δ_I*-open set U in X and a* $\delta\alpha^* - I$*-set V in X such that* $A = U \cap V$*.*

Proposition 8. *a) A* $\delta_\beta - t - I$*-set A is a* $\delta_\beta - B - I$*-set.*

b) An open set is a $\delta_\beta - B - I$*-set.*

c) A δ_I*-open set is a* $\delta - C$*-set.*

Proposition 9. *a) A* $\delta_\beta - B$*-set is a* $\delta_\beta - B - I$*-set.*

b) A $\delta_\beta - B - I$*-set is a strongly*$B - I$*-set.*

Remark 3. *The converses of the statements in Proposition 6 and Proposition 9 are false as in the following example.*

Example 3. *Let* $X = \{a, b, c, d\}$*,* $\tau = \{X, \emptyset, \{a\}, \{a, c\}, \{a, b, c\}, \{c, d\}, \{c\}, \{a, c, d\}\}$ *and* $I = \{\emptyset, \{c\}\}$*. Hence* $\{c, d\}$ *is strongly*$-t - I$*-set (resp. strongly* $B - I$*-set), but it is not* $\delta_\beta - t - I$*-set (resp.* δ_β*-B-I-set).* $\{d\}$ *is* $\delta_\beta - t - I$*-set (resp.* $\delta_\beta - B - I$*-set), but it is not* $\delta_\beta - t$*-set (resp.* $\delta_\beta - B$*-set).*

Lemma 3. *Let* (X, τ, I) *be an ideal space and A be a subset of X.*

a) If A is open, then $\delta Cl_I(A) = Cl(A)$*,*

b) If A is closed, then $\delta Int_I(A) = Int(A)$*.*

Proof.

a) Since every δ_I-open set is open, we have $Cl(A) \subset \delta Cl_I(A)$ [14]. Conversely, let $x \notin Cl(A)$. Then there exists an open set U containing x such that $U \cap A = \emptyset$.

Since A is an open set, $Int(Cl(U)) \cap A = \emptyset$ and we know that $Int(Cl^*(U)) \subset Int(Cl(U))$, i.e. $Int(Cl^*(U)) \cap A = \emptyset$. This means that $x \notin \delta Cl_I(A)$. So, we get the result.

b) This follows from (a).

Theorem 2. *For a subset A of an ideal space* (X, τ, I)*, the following properties are equivalent;*

a) A is regular open,

b) $Int(\delta Cl_I(A)) = A$*,*

c) A is $pre^* - I$*-open and a strongly*$-t-I$*-set.*

Proof. $a) \Longrightarrow b)$. Let A be regular open. Then A is open and by Lemma 3, $\delta Cl_I(A) = Cl(A)$. Therefore, we have $Int(\delta Cl_I(A)) = Int(Cl(A)) = A$.

$b) \Longrightarrow c)$. Straightforward.

$c) \Longrightarrow a)$. Let A be pre$^* - I$-open and strongly$-t-I$-set. Then $A \subset Int(\delta Cl_I(A)) = Int(A) \subset A$ and A is open, $A = Int(\delta Cl_I(A)) = Int(Cl(A))$.

Theorem 3. *Let A be a subset of an ideal space* (X, τ, I)*. Then the following properties are equivalent;*

a) A is open,

b) A is $pre^* - I$*-open and a strongly* $B - I$*-set,*

c) A is $\delta\beta_I$*-open and a* $\delta_\beta - B - I$*-set.*

Proof. $a) \Longleftrightarrow b)$ It follows from [3, Theorem 33]

$a) \Longrightarrow c)$ Diagram 1 and Proposition 8

$c) \Longrightarrow a)$ Let A be a $\delta\beta_I$-open and a $\delta_\beta - B - I$-set. Then there exist an open set U and a $\delta_\beta - t - I$-set V in X such that $A = U \cap V$. Since V is $\delta_\beta - t - I$-set and A is $\delta\beta_I$-open, then

$$\begin{aligned} A \subset & Cl(Int(\delta Cl_I(A))) = Cl(Int(\delta Cl_I(U \cap V))) \\ \subset & Cl(Int(\delta Cl_I(U) \cap \delta Cl_I(V))) = Cl(Int(\delta Cl_I(U)) \cap Int(\delta Cl_I(V))) \\ \subset & Cl(Int(\delta Cl_I(U))) \cap Cl(Int(\delta Cl_I(V))) = Cl(Int(\delta Cl_I(U))) \cap Int(V). \end{aligned}$$

Thus,

$$A = U \cap V = (U \cap V) \cap U \subset Cl(Int(\delta Cl_I(U))) \cap Int(V) \cap U$$

$=U \cap Int(V)$ and $U \cap Int(V) \subset U \cap V = A$.

Hence $A = U \cap Int(V)$ and A is an open set.

Theorem 4. *Let A be a subset of an ideal space (X, τ, I). Then the following properties are equivalent;*

a) A is δ_I-open,

b) A is $\delta\alpha - I$-open and a $\delta - C$-set.

Proof. The proof is similar with Theorem 3

3. Decompositions of Continuity and δ_I-Continuity

Definition 7. *a) Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function. If for each $V \in \sigma$, $f^{-1}(V)$ is a $\delta\beta_I$-open (resp. $\delta_\beta - B - I$-set), then f is said to be $\delta - \beta - I$-continuous (resp. $\delta_\beta - B - I$-continuous).*

b) Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function. If for each δ_I-open set V in Y, $f^{-1}(V)$ is a δ_I-open, then f is said to be $\delta - I$-continuous [10].

c) Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function. If for each δ_I-open set V in Y, $f^{-1}(V)$ is a $\delta\alpha - I$-open (resp. $\delta - C$-set), then f is said to be $\delta\alpha - I$-continuous (resp. $\delta - C$-continuous).

By Theorem 3, we obtain the following Theorem.

Theorem 5. *For a function $f : (X, \tau, I) \to (Y, \sigma)$, the following properties are equivalent;*

a) f is continuous,

b) f is $\delta - \beta - I$-continuous and $\delta_\beta - B - I$-continuous.

Remark 4. *$\delta - \beta - I$-continuity and $\delta_\beta - B - I$-continuity are independent notions of each other.*

Example 4. *Let $X = Y = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, c\}, \{a, b, c\}, \{c, d\}, \{c\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{c\}\}$ and $\sigma = \{\emptyset, Y, \{a, c\}\}$. Define a function $f : (X, \tau, I) \to (Y, \sigma)$ such that $f(x) = x$. Then f is $\delta - \beta - I$-continuous, but it is not $\delta_\beta - B - I$-continuous. If we change the topology on Y as $\sigma_1 = \{\emptyset, Y, \{d\}\}$ in the function $f : (X, \tau, I) \to (Y, \sigma_1)$ defined as $f(x) = x$, then f is $\delta_\beta - B - I$-continuous, but it is not $\delta - \beta - I$-continuous.*

Theorem 6. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following properties are equivalent;*

a) f is $\delta - I$-continuous,

b) f is $\delta\alpha - I$-continuous and $\delta - C$-continuous.

Remark 5. *$\delta\alpha - I$-continuity and $\delta - C$-continuity are independent notions of each other.*

Example 5. *Let $X = Y = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, c\}, \{a, b, c\}, \{c, d\}, \{c\}, \{a, c, d\}\}$ and $I = \{\emptyset, \{c\}\}$. Also let $\sigma = \{\emptyset, Y, \{b, c\}\}$ and $J = P(X)$. Define a function $f : (X, \tau, I) \to (Y, \sigma, J)$ such that $f(x) = x$.Then f is $\delta - C$-continuous, but it is not $\delta\alpha - I$-continuous. If we change the topology on Y as $\sigma_1 = \{\emptyset, Y, \{a, c\}\}$ and $J = P(X)$ in the function $f : (X, \tau, I) \to (Y, \sigma_1, J)$ such that $f(x) = x$, then f is $\delta\alpha - I$-continuous, but it is not $\delta - C$-continuous.*

4. Decomposition of Complete Continuity

Definition 8. *A subset A of an ideal space (X, τ, I) is said to be semi$^* - I$-open (resp. semi$^* - I$-closed) set if $A \subset Cl(\delta Int_I(A))$ (resp. $Int(\delta Cl_I(A)) \subset A$)*

The intersection of all semi$^* - I$-closed sets containing A is called semi$^* - \delta - I$-closure of A and denoted by $s\delta Cl_I(A)$.

Theorem 7. *Let A be a subset of an ideal space (X, τ, I). Then $s\delta Cl_I(A) = A \cup Int(\delta Cl_I(A))$.*

Proof. Since

$$\begin{aligned} Int(\delta Cl_I(A \cup Int(\delta Cl_I(A)))) \subset & Int(\delta Cl_I(A) \cup \delta Cl_I(Int(\delta Cl_I(A)))) \\ =& Int(\delta Cl_I(A)) \subset A \cup Int(\delta Cl_I(A)), \end{aligned}$$

$A \cup Int(\delta Cl_I(A))$ is semi$^* - I$-closed containing A and hence $s\delta Cl_I(A) \subset A \cup Int(\delta Cl_I(A))$. On the other hand, since $s\delta Cl_I(A)$ is semi$^* - I$-closed,

$$Int(\delta Cl_I(A)) \subset Int(\delta Cl_I(s\delta Cl_I(A))) \subset s\delta Cl_I(A).$$

Thus $A \cup Int(\delta Cl_I(A)) \subset s\delta Cl_I(A)$.

Definition 9. *A subset A of an ideal space (X, τ, I) is said to be semi $- \delta_I$-generalized-closed (briefly, $s\delta_I - g$-closed) if $s\delta Cl_I(A) \subset U$, whenever $A \subset U$ and U is pre$^* - I$-open.*

Theorem 8. *For a subset A of an ideal space (X, τ, I), the following properties are equivalent;*

a) A is regular open,

b) A is pre$^* - I$-open and semi$-\delta_I$-generalized-closed.

Proof. $(a) \Longrightarrow (b)$. Let A be a regular open. Since every regular open set is pre$^* - I$-open, A is pre$^* - I$-open. By $s\delta Cl_I(A) = A \cup Int(\delta Cl_I(A)) = Int(\delta Cl_I(A)) = Int(Cl(A)) = A$ and Theorem 2, A is $s\delta_I - g$-closed.

$(b) \Longrightarrow (a)$. Let A be pre$^* - I$-open and a $s\delta_I - g$-closed set. Then $s\delta Cl_I(A) \subset A$ and hence A is strong semi$-I$-closed. Therefore, $Int(\delta Cl_I(A)) \subset A$. Since A is pre$^* - I$-open, $A \subset Int(\delta Cl_I(A))$ and $Int(\delta Cl_I(A) = A$. Thus, by Theorem 2, A is regular open.

To obtain decomposition of complete continuity, we introduce the following new functions.

Definition 10. *A function $f : (X,\tau) \to (Y,\sigma)$ is said to be completely continuous [1] if for each $V \in \sigma$, $f^{-1}(V)$ is regular open in (X,τ).*

Definition 11. *A function $f : (X,\tau,I) \to (Y,\sigma)$ is said to be pre$^* - I$-continuous [3] (resp. contra $s\delta_I - g$-continuous) if for each $V \in \sigma$, $f^{-1}(V)$ is pre$^* - I$-open (resp. $s\delta_I - g$-closed) in (X,τ,I).*

By Theorem 8, we obtain the following decomposition of complete continuity.

Theorem 9. *For a function $f : (X,\tau,I) \to (Y,\sigma)$, the following properties are equivalent;*

a) f is completely continuous,

b) f is pre$^* - I$-continuous and contra $s\delta_I - g$-continuous.

Remark 6. *By the following example, pre$^* - I$-continuity and contra $s\delta_I - g$-continuity are independent concepts.*

Example 6. *Let $X = Y = \{a,b,c,d\}$, $\tau = \{X,\varnothing,\{a\},\{a,c\},\{a,b,c\},\{c,d\},\{c\},\{a,c,d\}\}$ and $I = \{\varnothing,\{c\}\}$ and $\sigma = \{\varnothing,Y,\{a,c\}\}$ as in Example 4. Define a function $f : (X,\tau,I) \to (Y,\sigma)$ such that $f(x) = x$. Then f is pre$^* - I$-continuous, but it is not contra $s\delta_I - g$-continuous. If we change the topology on Y as $\sigma_1 = \{\varnothing,Y,\{b\}\}$ in the function $f : (X,\tau,I) \to (Y,\sigma_1)$ defined as $f(x) = x$, then f is contra $s\delta_I - g$-continuous, but it is not pre$^* - I$-continuous.*

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