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Weakly Special Radical Class and Special Radical Class of Ternary Semirings

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Abstract. In this paper, we consider the weakly special radical classes and special radical classes of ternary semirings. Some of our results are similar to those in rings theory as well as in semiring theory. In particular, the upper radicals of the above two classes are determined.

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1. Introduction

In this paper, we consider "The Singular Ideal of a ternary Semiring" mentioned in [3]. The notion of a ternary semiring was first introduced by T. K. Dutta and S. Kar in [3]. Subsequently, many related notions of semiring and ring have been generalized to ternary semirings. Some earlier works of ternary semiring may be found in [4]-[12] and [13, 14, 15, 16]. The Partitioning and subtractive ideals of ternary semirings were considered by J. N. Chaudhari and K. J. Ingale in [2]. For the general radical theory of rings,the reader is referred to the classical monograph of N. J. Divinsky [17]. For definitions and properties of ideals, homomorphism, quotient for ternary semirings,singular ideals, singular ternary semirings, non-singular ternary semiring, the reader is referred to [2]. The concepts of radical class for hemirings were given by D. M. Olson and T. L. Jenkins in 1983, see [27]. Moreover,the general theory to upper radicals was extended by A. C. Nance in [27] and the special radical classes and properties of special radicals were investigated by M. D. Olson, G.A.P. Heyman and H. J. L. Roux in [26]. The properties of the weakly special radical class of hemirings were also studied by the above authors.

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In this paper, we first extend the above notions to ternary semirings. Then, we define again the weakly special radical class and special radical class for ternary semirings. We will show that the class of all semiprime and non singular ternary semirings forms a weakly special radical class whereas the class of all prime non-singular ternary semirings forms a special radical class. As a consequence of this result, the upper radicals will be determined by the above two classes which are called the singular radical and special singular radical of ternary semirings, respectively. For the radical properties,the reader is referred to the well known monograph of N. J. Divinsky.

Throughout this paper, S will be used to denote a ternary semiring with zero and $S^* = S \setminus \{0\}$. Also we use M to denote a right ternary S-semimodule with zero.

2. The Radical Class, Weakly Special Radical Class and Special Radical Class

Let S be a ternary semiring and M a right ternary S-semimodule. We define $Z_S(M)$ [2] by $\{m \in M : r_S(m)$ be an essential right ideal of $S\}$.

We first give the following crucial definition.

Definition 1 ([2])**.** *A ternary subsemimodule $Z_S(M)$ of M is called a singular ternary subsemimodule of the right ternary S-semimodule M.*

We call a singular ternary subsemimodule $Z_S(S)$ a right ideal of a ternary semiring S and call this kind of right ideal the (right) singular ideal [2] of a ternary semiring S which is denoted by $Z(S)$, that is, $Z(S) = \{t \in S : r_S(t)$ is an essential right ideal of $S\}$.

A ternary semiring S is said to satisfy the condition α[2] if for any nonzero element a in S, $r_S(a) \neq S$ or equivalently $aSS = 0$ implies that $a = 0$.

The properties of singular modules of a ternary modules are given in the following propositions, see [2].

Proposition 1. *$Z_S(M)$ is a ternary subsemimodule of M.*

Proposition 2. *Let S be a ternary semiring with condition α. Then the singular ideal $Z(S)$ is a k-ideal of S.*

Proposition 3. *If I is an ideal of a ternary semiring S and as a ternary semiring, I is semiprime, then $Z(I) = I \cap Z(S)$.*

We give the following example to show that there exists a prime(semiprime) ternary semiring satisfying the condition α.

Example 1 ([2])**.** *Let S be a prime(semiprime) ternary semiring. Then S satisfies the condition α.*

The following theorem of ternary semirings can be found in [2].

Theorem 1. *Let S and S' be two semi-isomorphic ternary semirings. Then S is singular(nonsingular) if and only if S' is singular(resp. nonsingular).*

We state the following known definition.

Definition 2 ([5]). *A non-empty subset A of a ternary semiring S is called a p-system if for each $a \in A$ there exist elements x_1, x_2, x_3, x_4 of S such that $ax_1ax_2a \in A$ or $ax_1x_2ax_3x_4a \in A$ or $ax_1x_2ax_3ax_4 \in A$ or $x_1ax_2ax_3x_4a \in A$.*

In the following theorem, we characterize the semiprime ideal of a ternary semiring.

Theorem 2 ([5]). *A proper ideal Q of a ternary semiring S is semiprime if and only if its complement P^c is a p-system.*

Proposition 4. *Let S be a ternary semiring. If Q is a semiprime ideal of S and I is an ideal of S then $Q \cap I$ is a semiprime ideal of I.*

Proof. Let J be an ideal of I such that $J^3 \subseteq I \cap Q$. Then $J^3 \subseteq Q$. If possible, let $J \nsubseteq I \cap Q$. Then $J \nsubseteq Q$. Hence, there exists an element $a \in J$ but $a \notin Q$. Now by theorem 2, Q is a p-system. Then $a \in Q^c$ implies that there exist elements x_1, x_2, x_3, x_4 of S such that $ax_1ax_2a \in Q^c$ or $ax_1x_2ax_3x_4a \in Q^c$ or $ax_1x_2ax_3ax_4 \in Q^c$ or $x_1ax_2ax_3x_4a \in Q^c$. We consider the following situation:

If $ax_1ax_2a \in Q^c$, then there exist elements s_1, s_2, s_3, s_4 of S such that $ax_1ax_2as_1ax_1ax_2as_2ax_1ax_2a \in Q^c$ or $ax_1ax_2as_1s_2ax_1ax_2as_3s_4ax_1ax_2a \in Q^c$ or $ax_1ax_2as_1s_2ax_1ax_2as_3ax_1ax_2as_4 \in Q^c$ or $s_1ax_1ax_2as_2ax_1ax_2as_3s_4ax_1ax_2a \in Q^c$.

Now consider the following cases:

(i)

$$\begin{aligned} ax_1ax_2as_1ax_1ax_2as_2ax_1ax_2a &= a(x_1ax_2as_1ax_1ax_2as_2)a(x_1ax_2)a \\ &= ai_1ai_2a \in J^3 \subseteq Q \end{aligned}$$

as $i_1, i_2 \in I$ where $i_1 = x_1ax_2as_1ax_1ax_2as_2$ and $i_2 = x_1ax_2$.

(ii)

$$\begin{aligned} ax_1ax_2as_1s_2ax_1ax_2as_3s_4ax_1ax_2a &= a(x_1ax_2as_1s_2ax_1ax_2as_3s_4)a(x_1ax_2)a \\ &= ai_1ai_2a \in J^3 \subseteq Q \end{aligned}$$

as $i_1, i_2 \in I$ where $i_1 = x_1ax_2as_1s_2ax_1ax_2as_3s_4$ and $i_2 = x_1ax_2$.

(iii)

$$\begin{aligned} ax_1ax_2as_1s_2ax_1ax_2as_3ax_1ax_2as_4 &= a(x_1ax_2)a(s_1s_2a)(x_1ax_2as_3ax_1)a(x_2as_4) \\ &= ai_1ai_2i_3ai_4 \in J^3 \subseteq Q \end{aligned}$$

as $i_1, i_2, i_3, i_4 \in I$ where $i_1 = x_1ax_2, i_2 = s_1s_2a, i_3 = x_1ax_2as_3ax_1$ and $i_4 = x_2as_4$.

(iv)

$$s_1ax_1ax_2as_2ax_1ax_2as_3s_4ax_1ax_2a = (s_1ax_1)a(x_2as_2)a(x_1ax_2)(as_3s_4ax_1ax_2)a$$
$$= i_1ai_2ai_3i_4a \in J^3 \subseteq Q$$

as $i_1, i_2, i_3, i_4 \in I$ where $i_1 = s_1ax_1, i_2 = x_2as_2, i_3 = x_1ax_2$ and $i_4 = as_3s_4ax_1ax_2$.

Now, from (i) , (ii), (iii) and (iv), we can easily see that $ax_1ax_2a \notin Q^c$.

Hence, $J \subseteq Q$, and whence, $J \subseteq I \cap Q$. This proves that $I \cap Q$ is a semiprime ideal of S.

We state below the following definition of m-system of a ternary semiring.

Definition 3 ([4]). *A non-empty subset A of a ternary semiring S is called an m-system if for each $a, b, c \in A$ there exist elements x_1, x_2, x_3, x_4 of S such that $ax_1bx_2c \in A$ or $ax_1x_2bx_3x_4c \in A$ or $ax_1x_2bx_3cx_4 \in A$ or $x_1ax_2bx_3x_4c \in A$.*

We now characterize the prime ideals of a ternary semiring.

Theorem 3 ([4]). *A proper ideal P of a ternary semiring S is prime if and only if its complement P^c is an m-system.*

Proposition 5. *Let S be a ternary semiring and P a prime ideal of S. If I is an ideal of S. Then $P \cap I$ is a prime ideal of I.*

Proof. Proceeding as in Proposition 4, the proposition follows immediately.

In below, we let $\mathscr{S} = \{S : S$ be a ternary semiring such that every nonzero homomorphic image S' of S contains a nonzero ideal which is singular as a ternary semiring$\}$.

Proposition 6. *$\mathscr{S} = \{S : S$ is a ternary semiring such that for every nonzero homomorphic image S' of S, $\beta(S') \neq 0$ or $Z(S') \neq 0\}$.*

Proof. Let $\mathscr{S}' = \{S : S$ is a ternary semiring such that for every nonzero homomorphic image S' of S,$\beta(S') \neq 0$ or $Z(S') \neq 0\}$. Let $S \in \mathscr{S}$ and S' be a nonzero homomorphic image of S. So S' contains a nonzero ideal I (say) which is singular as a ternary semiring i.e. $Z(I) = I$. Suppose $\beta(S') = 0$. Then S' is a semiprime ternary semiring; so I is a semiprime ternary semiring by Proposition 4. Hence,by Proposition 3, $Z(I) = I \cap Z(S')$ i.e. $I \cap Z(S') = I$ which implies that $I \subseteq Z(S')$. Consequently $Z(S') \neq 0$. Hence $\beta(S') \neq 0$ or $Z(S') \neq 0$. Therefore, $S \in \mathscr{S}'$.

Conversely, let $S \in \mathscr{S}'$ and S' be a nonzero homomorphic image of S such that $\beta(S') \neq 0$ or $Z(S') \neq 0$. Suppose that $\beta(S') \neq 0$. Then S' contains a nonzero ideal I such that $I^3 = 0$. We now prove that $Z(I) = I$. For this purpose,let $x \in I$ and H a nonzero right ideal of I. Then $xHI \subseteq xII \subseteq III = 0$. This leads $H \subseteq r_I(x)$,that is, $r_I(x) \cap H = H \neq 0$. Therefore, $x \in Z(I)$. Hence, $Z(I) = I$. If $\beta(S') = 0$, then $Z(S') \neq 0$. Since $\beta(S') = 0$, S' is a semiprime ternary semiring. Let $Z(S') = I$. Then, by Example 1 and Proposition 2, I is a nonzero ideal of the semiprime ternary semiring S'. Hence. by Proposition 4, I is a semiprime ternary semiring. Thus, by Prop 3, $Z(I) = I \cap Z(S') = I$ that is,I is singular as a ternary semiring. In both cases $S \in \mathscr{S}$. Thus, $\mathscr{S} = \mathscr{S}'$.

Recall the following definition of hereditary class given in N. J. Divinsky [3].

Definition 4. *A class ρ of ternary semirings is called hereditary if I is an ideal of a ternary semiring S and $S \in \rho$ then $I \in \rho$.*

Now, by D. M. Olson and A. C. Nance [27], we define the regular class in a ternary semiring as follows:

Definition 5. *A class $\mathscr{M}$ of ternary semirings is called regular if $S \in \mathscr{M}$ and I is a nonzero ideal of the ternary semiring S, then there is a nonzero homomorphic image of I in $\mathscr{M}$.*

Following D. M. Olson and A. C. Nance [27], we define the radical class in ternary semiring.

Definition 6. *A nonempty class γ of ternary semirings is called a radical class if the following conditions hold:*

(R1) γ is homomorphically closed.

(R2) If $S \notin \gamma$, then S contains a proper k-ideal K such that S/K has no nonzero γ-ideals (ideals which are as ternary semirings are in the class γ).

As an example of the radical class in ternary semiring, we have the following Lemma.

Lemma 1. *If I is a nil ideal of a ternary semiring S, then $\overline{I}$ is also a nil ideal of S.*

Proof. Let I be a nil ideal of S and $x \in \overline{I}$. Then $x + i \in I$ for some $i \in I$ Since I is nil, for each t in S there exists a positive integer n(depending on t) such that $[(x+i)t]^n(x+i) = 0$. Since I is an ideal of S, so $[(x+i)t]^n(x+i)$ can be written as $(xt)^n x+h$ for a particular $h \in I$. Thus, $(xt)^n x+h = 0$, and as a result, we have $(xt)^{n+1}h+hth = 0$. Now, $(xt)^{2n+1}x = (xt)^{n+1}[(xt)^n x + h] + hth = hth$. But, $hth \in I$, so $(xt)^{2n+1}x \in I$. $(xt)^{2n+1}x$ is nilpotent. Hence, for each $t \in S$, $[(xt)^{2n+1}xt]^k[(xt)^{2n+1}x] = 0$ for some positive integer k $\Rightarrow (xt)^{(2n+2)k+2n+1}x = 0$. Thus, x is nilpotent and thus $\overline{I}$ is a nil ideal of S.

Theorem 4. *The class $\mathscr{N}$ of all nil ternary semirings is a radical class.*

Proof. Clearly, the class $\mathscr{N}$ is homomorphically closed. Let S be a ternary semiring such that $S \notin \mathscr{N}$. Now, using Zorn's Lemma, we choose an ideal M of S which is maximal with respect to being a nil ideal. Since $S \notin \mathscr{N}$, M is a proper ideal of S. By Lemma 1 and the maximality of M, we see that $\overline{M} = M$. This means that M is a k-ideal of S. If I/M is any $\mathscr{N}$-ideal of S/M then for any $x \in I$, x/M is nilpotent. Now, for each $t/M \in S/M$, there exists a positive integer n such that $[(x/M)(t/M)]^n x/M = ((xt)^n x)/M = 0/M$. But then $(xt)^n x \in M$ which makes $(xt)^n x$ and therefore, x is nilpotent. Thus, I is a nil ideal of S, and so $I \subseteq M$ since M is maximal. Hence, $I/M = (0)$, and so S/M has a no non-zero $\mathscr{N}$-ideal. This shows that $\mathscr{N}$ is a radical class.

Lemma 2. *If ϕ is a semi-isomorphism from a ternary semiring S onto a ternary semiring T and I is a nonzero ideal of S, then $\phi(I)$ is a nonzero ideal of T.*

Proof. Clearly, $\phi(I)$ is an ideal of T. If $\phi(I) = (0)$, then, $I \subseteq Ker\phi = (0)$, as ϕ is semi-isomorphism. Thus $I = (0)$, a contradiction. This shows that $\phi(I)$ is a nonzero ideal of T.

The following Theorem is a theorem for the regular radical class of the ternary semirings.

Theorem 5. *If $\mathscr{M}$ is a regular class of ternary semirings, then $\mathscr{U}\mathscr{M} = \{$ternary semirings S : no nonzero homomorphic image of S is in $\mathscr{M}\}$ is a radical class.*

Proof. Suppose that $S \in \mathscr{U}\mathscr{M}$ and $\phi(S)$ is a nonzero homomorphic image of S. Let $\psi(\phi(S))$ be a nonzero homomorphic image of $\phi(S)$. Then $\psi(\phi(S)) = (\psi\phi)(S)$ is a nonzero homomorphic image of S. Since $S \in \mathscr{U}\mathscr{M}$, $(\psi\phi)(S) \notin \mathscr{M}$. Hence $\phi(S) \in \mathscr{U}\mathscr{M}$. Thus $\mathscr{U}\mathscr{M}$ is homomorphically closed.

Next, we suppose that $S \notin \mathscr{U}\mathscr{M}$. Then there exists a nonzero homomorphic image $\phi(S) \in \mathscr{M}$. Now as ϕ is nonzero, $Ker\phi$ is a proper k-ideal of S and $S/Ker\phi \simeq \phi(S)$, and let the semi-isomorphism be ψ. If I is a nonzero $\mathscr{U}\mathscr{M}$-ideal of $S/Ker\phi$, then by Lemma 2, $\psi(I)$ is a nonzero ideal of $\phi(S)$. Since $I \in \mathscr{U}\mathscr{M}$ and $\mathscr{U}\mathscr{M}$ is homomorphically closed, $\psi(I) \in \mathscr{U}\mathscr{M}$. Since $\phi(S) \in \mathscr{M}$ and $\mathscr{M}$ is regular, $\psi(I)$ has a nonzero homomorphic image in $\mathscr{M}$. This, however, contradicts to $\psi(I) \in \mathscr{U}\mathscr{M}$, As a result, we have shown that $S/Ker\phi$ has no nonzero $\mathscr{U}\mathscr{M}$-ideals.Thus, $\mathscr{U}\mathscr{M}$ is indeed a radical class.

Definition 7. *Let S be a ternary semiring and A be a nonempty subset of S. Then, the annihilator of A in S, denoted by $ann_S(A)$, is defined by $\{x \in S : Axs = 0$ and $Asx = 0$ for all $s \in S\}$.*

Proposition 7. *Let S be a ternary semiring and A be a right ideal of S. Then $ann_S(A)$ is a k-ideal of S.*

Proof. Obviously, $ann_S(A)$ is nonempty since $0 \in ann_S(A)$. Also, if $a, b \in ann_S(A)$, then $a + b \in ann_S(A)$. Let $x \in ann_S(A)$ and $s_1, s_2 \in S$. Then, $Axs = 0$ and $Asx = 0$ for all $s \in S$ and so $Axs_1s_2s = 0$ and $Asxs_1s_2 = 0$ for all $s \in S$. This leads to $xs_1s_2 \in ann_S(A)$. Hence, $ann_S(A)$ is a right ideal of S. Also $As_1s_2xs \subseteq Axs = 0$ and $Ass_1s_2x \subseteq ASx = 0$ for all $s \in S$ as A is a right ideal of S. So $s_1s_2x \in ann_S(A)$. Hence, $ann_S(A)$ is a left ideal of S. Again, we have $As_1xs_2s = (As_1x)s_2s = 0$ and $Ass_1xs_2 \subseteq AxS = 0$ for all $s \in S$. So $s_1xs_2 \in ann_S(A)$. Hence, $ann_S(A)$ is a lateral ideal of S. Thus, $ann_S(A)$ is an ideal of S. Now let $a, a + b \in ann_S(A)$. Then $Aas = 0 = A(a + b)s$ and $Asa = 0 = As(a + b)$ for all $s \in S$. This implies $Abs = 0$ and $Asb = 0$ for all $s \in S$. So, $b \in ann_S(A)$. This proves that $ann_S(A)$ is a k-ideal of S.

Following D. M. Olson, G.A.P. Heyman and H. J. L. Roux [26], we define he weakly special radical class in ternary semirings as follows:

Definition 8. *A class $\mathscr{M}$ of ternary semirings is called a weakly special radical class if $\mathscr{M}$ is a hereditary class of semiprime ternary semirings satisfying the following conditions:*

(x) If S is a ternary semiring and S is semi-isomorphic to T with $T \in \mathscr{M}$, then $S \in \mathscr{M}$.

(z) If $I \in \mathscr{M}$ and I is an ideal of a ternary semiring S, then $S/ann_S(I) \in \mathscr{M}$.

We give the following crucial Lemma.

It is noted that the h-prime and h-semiprime ideals in semirings and Γ-semirings have been recently investigated by S. Sardar and others in [29]. For semiprime ternary semirings, we have the following main theorem.

Theorem 6. *If $\mathscr{M}$ is a hereditary class of semiprime ternary semirings which satisfies properties: "if $S \in \mathscr{M}$ and S is semi-isomorphic to T then $T \in \mathscr{M}$" then the following conditions are equivalent:*

(1) If $A \in \mathscr{M}$ and A is an ideal of S, then $S/ann_S(A) \in \mathscr{M}$;

(2) If $A \in \mathscr{M}$ with A an ideal of S and $ann_S(A) = 0$, then $S \in \mathscr{M}$.

(3) If $A \in \mathscr{M}$ and A is an essential ideal of S, then $S \in \mathscr{M}$.

Proof. For an hereditary radical class of semiprime ternary semirings, it is clear that $(1) \Rightarrow (2)$. For $(2) \Rightarrow (3)$, suppose that A is an essential ideal of S and $A \in \mathscr{M}$. Now let $a \in A \cap ann_S(A)$, then $AaS = 0$ and $ASa = 0$. This implies that $aAa = 0$ and $AAa = 0$ since $a \in A$ and $A \subseteq S$. Thus $aAaAa = 0$, $aAAaAAa = 0$, $aAAaAaA = 0$ and $AaAaAAa = 0$.Thus, $a = 0$ since A is semiprime. Because A is essential, $ann_S(A) = 0$.

Now by (2), $S \in \mathscr{M}$. Assume that (3) holds and consider a ternary semiring $A \in \mathscr{M}$ with A an ideal of S. Then, by using the arguments as in above, we see that $A \cap ann_S(A) = (0)$ since A is semiprime. Now $(A + ann_S(A))/ann_S(A)$ is a nonzero ideal of $S/ann_S(A)$. Also, we have $A = A/(A \cap ann_S(A)) \simeq (A + ann_S(A))/ann_S(A)$, and so by the given condition,we deduce that $(A + ann_S(A))/ann_S(A) \in \mathscr{M}$. However,$(A + ann_S(A))/ann_S(A)$ is an essential ideal in $S/ann_S(A)$,for if $H/ann_S(A)$ is a nonzero ideal of $S/ann_S(A)$, then $H \cap A \neq 0$, otherwise, $AHS, ASH \subseteq H \cap A$ would be zero which implies that $H \subseteq ann_S(A)$ which is impossible. Also, $(H \cap A) \cap ann_S(A) = (0)$. Thus, there exits an $a(\neq 0) \in H \cap A$ such that $a/ann_S(A) \neq 0/ann_S(A)$. Also, $a/ann_S(A) \in [A + ann_S(A)/ann_S(A)]$. Hence,
$[A + ann_S(A)/ann_S(A)] \bigcap (H/ann_S(A)) \neq 0/ann_S(A)$. But then, by the condition (3), we have $S/ann_S(A) \in \mathscr{M}$ and so, condition (1) follows, as desired.

Lemma 3. *If the ternary semirings $S \simeq T$ and T are semiprime, then S is semiprime.*

Proof. Suppose that ϕ is a semi-isomorphism and A is an ideal of S such that $A^3 = (0)$. Then, $\phi(A)$ is an ideal of T and $\phi(A^3) = (0) \Rightarrow [\phi(A)]^3 = (0)$. Thus, $\phi(A) = (0)$ since T is semiprime $\Rightarrow A \subseteq Ker\,\phi = (0) \Rightarrow A = (0)$. This shows that S is a semiprime ternary semiring.

The semiprime ternary semirings is described in the following Lemma.

Lemma 4. *If the ternary semirings $S \simeq T$ and S are semiprime, then T is semiprime.*

Proof. Suppose that $\phi : S \to T$ is a semi-isomorphism and A' is an ideal of T such that $A'^3 = (0)$. Then, there exists an ideal A of S such that $\phi(A) = A'$, as ϕ is surjective. Now $\phi(A^3) = [\phi(A)]^3 = A'^3 = (0)$. Thus $A^3 = (0)$, as $ker\phi = 0$. Since S is semiprime, $A = (0)$, we have $A' = (0)$. Thus, T is a semiprime ternary semiring.

In the following theorem, we consider the semiprime non-singular ternary semirings.

Theorem 7. *The class $\wp$ of semiprime non-singular ternary semirings forms a weakly special radical class.*

Proof. Let $S \in \wp$ and I be a nonzero ideal of S. Then by Lemma 4, I is a semiprime ternary semiring. Also, from Proposition 3, $Z(I) = I \cap Z(S) = (0)$, since Z(S)=0 and S is nonsingular, $I \in \wp$. Thus, the class $\wp$ is a hereditary class of semiprime nonsingular ternary semirings. By Theorem 1 and Lemma 3, the class $\wp$ satisfies the property (x) of the definition of weakly special radical class. In order to prove the condition (z) of the weakly special radical class, in view of Theorem 1, Lemma 4 and Lemma 6, it suffices to prove that if $I \in \wp$ and I is an essential ideal of a ternary semiring S then $S \in \wp$. To prove that S is a semiprime ternary semiring, we let $K^3 = 0$, where K is an ideal of S. Let $K' = K \cap I$. Then K' is an ideal of I. Now $K'^3 \subseteq K^3 = 0$ implies $K' = 0$ since I is a semiprime ternary semiring. Since I is an essential ideal and so $K = 0$. This shows that S is a semiprime ternary semiring. Again if S is not a nonsingular ternary semiring, then by Example 1 and Proposition 2, we can easily see that $Z(S)$ is a nonzero ideal of S and thus $I \cap Z(S) \neq 0$, since I is an essential ideal of S. Hence, by Proposition 3, $Z(I) \neq 0$ which is a contradiction, since $I \in \wp$. Therefore, S is nonsingular and hence, $S \in \wp$. This shows that $\wp$ is a weakly special radical class.

3. Supernilpotent Radical Class and Weakly Special Radical Class of Ternary Semirings

We first give the following useful Definition.

Definition 9. *Let S be a ternary semiring. Then, we call an ideal I of S nilpotent if there exists a positive integer n such that $I^{2n+1} = 0$. The semiring S is said to be a nilpotent ternary semiring if S is nilpotent as an ideal of itself.*

Following D. M. Olson and A. C. Nance [27], we define the supernilpotent radical class in ternary semiring as follows:

Definition 10. *A radical class of ternary semirings is called a supernilpotent radical class if it is hereditary and contains all the nilpotent ternary semirings.*

The following Lemma is a crucial lemma.

Lemma 5. *Let S be a ternary semiring. If I is a semiprime k-ideal J and J is an ideal of S, then I is an ideal of S.*

Proof. Since I is a semiprime k-ideal of J, it is easy to see that J/I is a semiprime ternary semiring. Now, $(I+SIS+JSISJ)$ is an ideal of J and we have $(I+SIS+JSISJ/I)^5 \subseteq I/I = 0/I$. Thus, $I + SIS + JSISJ/I = 0/I$ as J/I is semiprime ternary semiring. Since I is k-ideal of J,$SIS \subseteq I$. Again, $(I + ISS)$ is an ideal of J and $(I + ISS/I)^3/I \subseteq I/I = (0)$. Thus, $(I + ISS)/I = 0/I$ as J/I is a semiprime ternary semiring. Now as I is a k-ideal of J, $ISS \subseteq I$. Also, $(I+SSI)$ is an ideal of J and $(I+SSI/I)^3 \subseteq I/I = (0)$. Hence,we have $(I+SSI)/I = (0)$ as J/I is a semiprime ternary semiring. Now as I is a k-ideal of J, we have $SSI \subseteq I$. This proves that I is an ideal of S.

Corollary 1. *Let S be a ternary semiring. If I is a prime k-ideal of J and J is an ideal of S,then I is an ideal of S.*

We now formulate a theorem of weakly special radical class of ternary semirings.

Theorem 8. *If $\mathscr{M}$ is a weakly special radical class of ternary semirings, then $\mathscr{U}\mathscr{M} = \{$ ternary semirings S : no nonzero homomorphic image of S is in $\mathscr{M}\}$ is a supernilpotent radical class.*

Proof. Since $\mathscr{M}$ is a weakly special radical class of ternary semirings, $\mathscr{M}$ is a hereditary class. Let $S \in \mathscr{M}$ and I be a nonzero ideal of S. Then $I \in \mathscr{M}$. Now I is a homomorphic image of itself. This means that $\mathscr{M}$ is regular. Hence, by Theorem 5, $\mathscr{U}\mathscr{M}$ is a radical class. In order to show that $\mathscr{U}\mathscr{M}$ is a hereditary class , let $S \in \mathscr{U}\mathscr{M}$ and J be a nonzero ideal of S. If $J \notin \mathscr{U}\mathscr{M}$, then there is a nonzero homomorphic image $\phi(J)$ of J in $\mathscr{M}$. Let $K = ker\phi$. Then K is a k-ideal of J and $J/K \simeq \phi(J)$. Since $\phi(J) \in \mathscr{M}$, $\phi(J)$ is semiprime, and so by Lemma 3, J/K is semiprime. Now $\phi(J)$ is nonzero, and hence, $J/K \neq (0)$. Since K is a k-ideal, K is a semiprime ideal of J and, whence K is an ideal of S, by Proposition 5.

Now J/K is a nonzero ideal of S/K, and so by the property (z), $(S/K)/ann_S(J/K) \in \mathscr{M}$, since by the property (x), $J/K \in \mathscr{M}$. But, if $S/K \subseteq ann_S(J/K)$, then we have $(J/K)^3 = (0)$ which cannot happen since J/K is a semiprime ternary semiring. This leads to $ann_S(J/K) \neq S/K$. Thus, $(S/K)/ann_S(J/K)$ is a homomorphic image of S which is in $\mathscr{M}$ and $(S/K)/ann_S(J/K) \neq (0)$, since any annihilator ideal is necessarily a k-ideal. However,this is impossible because $S \in \mathscr{U}\mathscr{M}$. Hence, we deduce that $J \in \mathscr{U}\mathscr{M}$ and hence, we have proved that $\mathscr{U}\mathscr{M}$ is hereditary.

Finally, if S is any nilpotent ternary semiring, then $\phi(S)$ is nilpotent for any nonzero homomorphism ϕ, and hence $\phi(S) \notin \mathscr{M}$. Thus, $S \in \mathscr{U}\mathscr{M}$ and hence, $\mathscr{U}\mathscr{M}$ is supernilpotent.

We here call $\mathscr{U}\mathscr{M}$ the upper radical class determined by the class $\mathscr{M}$.

Proposition 8. *$\mathscr{S} = \{S : S$ is a ternary semiring such that every nonzero homomorphic image S' of S contains a nonzero ideal which is singular as a ternary semiring$\}$ is an upper radical class determined by the class $\wp$ of semiprime non-singular ternary semirings.*

Proof. Now $U\wp = \{S :$ no nonzero homomorphic image of S is in $\wp\} = \{S :$ for every nonzero homomorphic image S_1 of S either $\beta(S_1) \neq 0$ or $Z(S_1) \neq 0\} = \mathscr{S}$ (by Proposition 6, since $S_1 \in \wp$ implies $\beta(S_1) = 0$ and $Z(S_1) = 0$).

We now simply call $\mathscr{S}$ the singular radical.

Following D. M. Olson and A. C. Nance [27], we define the special radical class of a ternary semiring as follows:

Definition 11. *A class $\mathscr{M}$ of ternary semirings is called a special radical class if $\mathscr{M}$ is a hereditary class of prime ternary semiring satisfying the following conditions:*

(1) If S is ternary semi-isomorphic to T and $T \in \mathscr{M}$, then $S \in \mathscr{M}$.

(2) If $I \in \mathscr{M}$ and I is an ideal of a ternary semiring S, then $S/ann_S(I) \in \mathscr{M}$.

For prime ternary semirings, we have he following Lemmas.

Lemma 6. *If the ternary semirings $S \simeq T$ and T is prime, then S is prime.*

Proof. Suppose that ϕ is a semi-isomorphism and A, B, C are ideals of S such that $ABC = (0)$. Then $\phi(A), \phi(B), \phi(C)$ are ideals of T and $\phi(ABC) = (0) \Rightarrow \phi(A)\phi(B)\phi(C) = (0)$. Since T is prime, $\phi(A) = (0)$ or, $\phi(B) = (0)$ or, $\phi(C) = (0)$ which implies that $A \subseteq Ker\phi = (0) \Rightarrow A = (0)$ or, $B \subseteq Ker\phi = (0) \Rightarrow B = (0)$, or, $C \subseteq Ker\phi = (0) \Rightarrow A = (0)$. Thus, we have either A=(0) or B=(0) or C=(0). Hence, S is prime ternary semiring.

In the following lemma, we study the hereditary radical class of ternary semirings.

Lemma 7. *Let $\mathscr{M}$ be a hereditary radical class of prime ternary semirings which satisfies the following properties: "if $S \in \mathscr{M}$ and S is semi-isomorphic to T then $T \in \mathscr{M}$".*

Then the following conditions are equivalent:

(1) If $A \in \mathscr{M}$ and A is an ideal of S, then $S/ann_S(A) \in \mathscr{M}$;

(2) If $A \in \mathscr{M}$ with A an ideal of S and $ann_S(A) = 0$, then $S \in \mathscr{M}$.

(3) If $A \in \mathscr{M}$ and A is an essential ideal of S, then $S \in \mathscr{M}$.

Proof. The proof follows from Lemma 6.

Proposition 9. *The class $\wp'$ of prime nonsingular ternary semirings is a special radical class.*

Proof. Let $S \in \wp'$ and I be a nonzero ideal of S. Then by Proposition 5, I is a prime non-singular ternary semiring. Since every prime ternary semiring is semiprime, by Proposition 3, $Z(I) = I \cap Z(S) = (0)$ since $Z(S) = 0$ as S is nonsingular. Hence, we have $I \in \wp'$. Thus,the class $\wp'$ is a hereditary class of prime nonsingular ternary semirings. By Theorem 1 and by Lemma 6, the class $\wp'$ satisfies the property (1) of the definition of special radical class. We now proceed to prove that the class $\wp'$ is a special radical class. In view of Theorem 1 and Lemma 6, it suffices to prove that if $I \in \wp'$ and I is an essential ideal of a ternary semiring S,then $S \in \wp'$.In order to prove that S is a prime ternary semiring, we let $ABC = 0$, where A , B and C are three ideals of S. Suppose that $A \neq 0$, $B \neq 0$ and $C \neq 0$. Let $A' = A \cap I$,$B' = B \cap I$

and $C' = C \cap I$. Then, A' , B' and C' are nonzero ideals of I, as I is an essential ideal of S. Now $A'B'C' \subseteq ABC = 0$. Since I is a prime ternary semiring, $A'B'C' = 0$ implies either $A' = 0$ or $B' = 0$ or $C' = 0$, a contradiction. This shows that S is a prime ternary semiring. Again, Suppose that S is not a nonsingular ternary semiring. Then, by Example 1 and Proposition 2, $Z(S)$ is a nonzero ideal of S and $I \cap Z(S) \neq 0$, since I is an essential ideal of S. By Prop 3, $Z(I) \neq 0$, which is a contradiction. Therefore, S is nonsingular. Thus, $S \in \wp'$. This proves that $\wp'$ is a special radical class.

Finally,we state a theorem of the special radical classes of a ternary ring S.

Theorem 9. *Let S be a ternary ring. If $\mathscr{M}$ is a special radical class of S, then $\mathscr{U}\mathscr{M}$ is a supernilpotent radical class of S.*

Proof. In order to show that $\mathscr{U}\mathscr{M}$ is hereditary, let $S \in \mathscr{U}\mathscr{M}$ and J be a nonzero ideal of S. If $J \notin \mathscr{U}\mathscr{M}$ then there is a nonzero homomorphic image $\phi(J)$ of J in $\mathscr{M}$. Let $K = ker\phi$. Then K is a k-ideal of J and we have $J/K \simeq \phi(J)$. But $\phi(J)$ is prime because it is in $\mathscr{M}$, and by Lemma 6, J/K is prime. Now $\phi(J)$ is nonzero, hence $J/K \neq (0)$. Since K is a k-ideal, K is a prime ideal of J and, hence, K is an ideal of S by Lemma 1.

Now J/K is a nonzero ideal of S/K, and hence by the property (z), we have $(S/K)/\ ann_S(J/K) \in \mathscr{M}$, since by the property (x), $J/K \in \mathscr{M}$. If $S/K \subseteq ann_S(J/K)$, then we have $(J/K)^3 = (0)$ which cannot happen because J/K is a prime ternary semiring. Hence, $ann_S(J/K) \neq S/K$. Thus, $(S/K)/ann_S(J/K)$ is a homomorphic image of S which is in $\mathscr{M}$ and $(S/K)/ann_S(J/K) \neq (0)$ as an annihilator ideal is necessarily a k-ideal. However, this is impossible as $S \in \mathscr{U}\mathscr{M}$. Hence, $J \in \mathscr{U}\mathscr{M}$ and we have shown that $\mathscr{U}\mathscr{M}$ is a hereditary radical class.

Finally, if S is an nilpotent ternary semiring, then $\phi(S)$ is nilpotent for any nonzero homomorphism ϕ, and hence $\phi(S) \notin \mathscr{M}$ has no nonzero nilpotent ternary semiring can be prime.Thus, $S \in \mathscr{U}\mathscr{M}$ and hence, $\mathscr{U}\mathscr{M}$ is supernilpotent.

Finally, we state a Theorem for the upper radical class of ternary semirings.

Theorem 10. *The upper radical class determined by the class $\wp'$ of prime nonsingular ternary semirings is a supernilpotent radical class.*

Proof. The proof of the above theorem follows immediately from Proposition 9 and by Theorem 9.

Remark 1. *We call the upper radical class determined by the class $\wp'$ the special singular radical class. It is clear that above upper radical class is contained in $\mathscr{S}$.*

In closing this paper, we notice that in the 1983 paper of D. M. Olson and T. L Jenkins [28] on radical theorems for hemirings, they asked an open problem. Is the class of nil hemirings a radical class?

It seems that this open problem of Olson-Jenkins has not yet been answered in the literature. Naturally, we ask a new open problem: Is the class of all nil singular ternary semirings also a radical class ?

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