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A Family of Convolution Operators for Multivalent Analytic Functions

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Abstract. In this paper, we consider a family of multiplier transformations and several subclasses of multivalent functions which are defined by means of convolution. Several interesting results are derived. Some (known or new) special cases of the multivalent function classes, which are investigated here, are also discussed.

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1. Introduction and Definitions

Let $\mathscr{A}(p)$ denote the class of functions of the following form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

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Let $f, g \in \mathscr{A}(p)$, f be given by (1) and

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}. \tag{2}$$

Then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n} =: (g * f)(z) \qquad (p \in \mathbb{N}). \tag{3}$$

Also, if f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g in $\mathbb{U}$, and we write

$$f \prec g \qquad (z \in \mathbb{U}), \tag{4}$$

if there exists a Schwarz function w such that

$$f(z) = g\big(w(z)\big) \quad \text{and} \quad |w(z)| \leq |z| \qquad (z \in \mathbb{U}).$$

We now define a linear operator $L_k^c : \mathscr{A}(p) \to \mathscr{A}(p)$ as follows: let the linear operator L_0

$$L_0 : \mathscr{A}(p) \to \mathscr{A}(p) \qquad (k \in \mathbb{N};\ c \in \mathbb{C} \setminus \{0\}) \tag{5}$$

be given and

$$cL_k^c f(z) = z\big(L_{k-1}^c f(z)\big)' + (c-p) L_{k-1}^c f(z) \tag{6}$$

with

$$L_0^c := L_0. \tag{7}$$

It can easily be seen from (6) that the operator L_k^c is linear and it satisfies the following property:

$$L_0^c f(z) = z^p + \sum_{n=1}^{\infty} A_{p+n} z^{p+n}, \tag{8}$$

which implies that

$$L_k^c f(z) = z^p + \sum_{n=1}^{\infty} (1 + n/c)^k A_{p+n} z^{p+n}. \tag{9}$$

We also have

$$cL_1^c f = z(L_0^c f)' + (c-p) L_0^c f, \tag{10}$$

$$cL_k^c f = z^{p+1} (z^{-p} L_{k-1}^c f)' + cL_{k-1}^c f \tag{11}$$

and

$$\frac{L_k^c f}{z^p} = \frac{z}{c} \left(\frac{L_{k-1}^c f}{z^p} \right)' + \frac{L_{k-1}^c f}{z^p}. \tag{12}$$

By appropriately choosing L_k^c given by (8), we obtain several applications studied by various earlier authors (see, for example, [2, 3, 4, 5, 6, 8, 14, 9, 10, 11, 15, 20]; see also [13, 17, 18, 21]). We now define the following analytic function class.

Definition 1. *Let* $\mathfrak{q}$ *and* h *be analytic in* $\mathbb{U}$. *Also let the function* h *be convex univalent in* $\mathbb{U}$ *with* $h(0) = \mathfrak{q}(0) = 1$. *Then* $\mathfrak{q} \in \mathscr{P}(h)$ *if and only if*

$$\mathfrak{q}(z) \prec h(z) \qquad (z \in \mathbb{U}). \tag{13}$$

Some well-known examples of the convex function h are listed below.

(i) If

$$h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad \text{and} \quad 0 \leqq \alpha < 1,$$

then

$$\Re\big(h(z)\big) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1).$$

(ii) If

$$h(0) = 1 \quad \text{and} \quad h(z) = \left(\frac{1+z}{1-z}\right)^{\beta} \qquad (0 < \beta < 1),$$

then

$$\Big|\arg\big(h(z)\big)\Big| < \frac{\beta\pi}{2} \qquad (z \in \mathbb{U}).$$

(iii) Let

$$h(z) = \frac{M(1+z)}{M + (1-M)z} \qquad \left(M > \frac{1}{2}\right).$$

Also

$$h(\mathbb{U}) = \{w : |w - M| < M\}.$$

(iv) If

$$h(z) = \sqrt{z+1} \quad \text{and} \quad \Re\left(\sqrt{z+1}\right) \geqq 0 \qquad (z \in \mathbb{U}),$$

then $h(\mathbb{U})$ is the interior of the right part of the Bernoulli lemniscate [see 1].

(v) If

$$h(z) = 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2 \quad \text{and} \quad \Im\left(\sqrt{z}\right) > 0 \qquad (z \in \mathbb{U}),$$

then $h(\mathbb{U})$ is the interior of the parabola given by

$$\left\{w : [\Im(w)]^2 = 2\Re(w) - 1\right\}.$$

Definition 2. *Let* L_0 *be a linear operator on* $\mathscr{A}(p)$ *and let* L_k^c *be given by* (6). *Then, for* $\lambda \geqq 0$, *a function* $f \in \mathscr{A}(p)$ *is said to be in the class* $\mathscr{S}_k^c(p, \lambda; h)$ *if and only if*

$$\left((1-\lambda)\frac{L_k^c f(z)}{z^p} + \lambda\frac{L_{k+1}^c f(z)}{z^p}\right) \in \mathscr{P}(h). \tag{14}$$

2. Preliminary Results

We need each of the following lemmas in our present investigation.

Lemma 1 (see [7] and [12]). *Let h be an analytic and convex univalent function in $\mathbb{U}$. Let the function f be analytic in $\mathbb{U}$ with $h(0) = f(0) = 1$. If*

$$f(z) + \frac{zf'(z)}{\gamma} \prec h(z) \qquad \big(z \in \mathbb{U};\ \Re(\gamma) \geqq 0;\ \gamma \neq 0\big), \tag{15}$$

then

$$f(z) \prec g(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t)\, \mathrm{d}t \prec h(z) \qquad (z \in \mathbb{U}).$$

Moreover, the function g is convex univalent in $\mathbb{U}$ and it is the best dominant of the subordination (15) *in the sense that in the sense that $f \prec g$ for all f satisfying* (15)*, and if there exists q such that $f \prec q$ for all f satisfying* (15)*, then $g \prec q$.*

Lemma 2 (see [19]). *Let the functions $\mathfrak{q}$ and h be analytic in $\mathbb{U}$ with $\mathfrak{q}(0) = 1$. Suppose also that*

$$\Re\big(\mathfrak{q}(z)\big) > \frac{1}{2} \qquad (z \in \mathbb{U}).$$

Then

$$(q * h)(\mathbb{U}) \subset \text{co}\,\{h(\mathbb{U})\},$$

where co $\{h(\mathbb{U})\}$ *is the convex hull of $h(\mathbb{U})$.*

Lemma 3 (see [16]). *Let*

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \qquad \text{and} \qquad g(z) \prec G(z) \quad (z \in \mathbb{U}).$$

If the functions F and G are convex in $\mathbb{U}$, then

$$(f * g)(z) \prec (F * G)(z) \qquad (z \in \mathbb{U}).$$

Unless otherwise stated, we shall assume throughout this paper that

$$\lambda \geqq 0, \;\; c \in \mathbb{C} \setminus \{0\}, \;\; \Re(c) > 0, \;\; k, p \in \mathbb{N}, \quad \text{and} \quad z \in \mathbb{U}.$$

3. Main Results

Our first main result in this paper is contained in Theorem 1 below.

Theorem 1. *If the function f belongs to the class $\mathscr{S}_k^c(p, \lambda; h)$, then*

$$\frac{L_k^c f(z)}{z^p} \in \mathscr{P}(h).$$

Moreover, if $\lambda > 0$, then

$$\frac{L_k^c f(z)}{z^p} \in \mathscr{P}(g), \tag{16}$$

where

$$g(z) = \frac{c}{\lambda} z^{-\frac{c}{\lambda}} \int_0^z t^{\frac{c}{\lambda}-1} h(t)\, \mathrm{d}t \prec h(z) \qquad (z \in \mathbb{U}), \tag{17}$$

the function g is convex univalent in $\mathbb{U}$ and g is the best dominant of the subordination

$$\frac{L_k^c f(z)}{z^p} \prec g \qquad (z \in \mathbb{U}).$$

Proof. The proof for the case when $\lambda = 0$ is trivial. We, therefore, suppose that $\lambda > 0$. Let

$$f \in \mathscr{S}_k^c(p, \lambda; h). \tag{18}$$

Then, by (12), we have

$$(1-\lambda)\frac{L_k^c f(z)}{z^p} + \lambda \frac{L_{k+1}^c f(z)}{z^p} = \frac{L_k^c f(z)}{z^p} + \frac{\lambda z}{c}\left(\frac{L_k^c f(z)}{z^p}\right)' \in \mathscr{P}(h). \tag{19}$$

Let the function $H(z)$ be given by

$$H(z) := \frac{L_k^c f(z)}{z^p} \qquad (z \in \mathbb{U}). \tag{20}$$

Then, by (19), it follows that

$$\left(H(z) + \frac{\lambda}{c} z H'(z)\right) \in \mathscr{P}(h)$$

and

$$\left(H(z) + \frac{\lambda}{c} z H'(z)\right) \prec h(z) \qquad (z \in \mathbb{U}). \tag{21}$$

Now, using Lemma 1 in (21) with

$$\gamma = \frac{c}{\lambda} \qquad \text{and} \qquad \lambda > 0, \tag{22}$$

we obtain (17). This shows that $H \in \mathscr{P}(g)$, where the function g is given by (17). Consequently, the proof of Theorem 1 is complete.

We take

$$L_0 f(z) = f(z) * \phi(a, c, z), \tag{23}$$

where

$$\phi(a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{p+n} \qquad (c \neq 0, -1, -2, -3, \ldots;\ z \in \mathbb{U})$$

and $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the familiar Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0;\ \lambda \neq 0), \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$. We also let

$$h(z) = \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1). \tag{24}$$

Then, by applying Theorem 1, we obtain the subordination (17) with

$$g(z) = \begin{cases} \dfrac{A}{B} + \left(1 - \dfrac{A}{B}\right)(1+Bz)^{-1}_2 F_1\left(1,1;\dfrac{c-1}{\lambda}+1;\dfrac{Bz}{Bz+1}\right) & (B \neq 0)' \\ 1 - \left(\dfrac{c-1}{c-1+\lambda}\right) Az & (B=0), \end{cases}$$

where ${}_2F_1$ is the Gauss hypergeometric function defined by

$${}_2F_1(\alpha,\beta;\gamma;z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \qquad (z \in \mathbb{U};\ \gamma \neq 0,-1,-2,-3,\ldots). \tag{25}$$

Theorem 2. *Let* $0 \leqq \lambda_1 \leqq \lambda_2$. *Then*

$$\mathscr{S}_k^c(p,\lambda_2;h) \subset \mathscr{S}_k^c(p,\lambda_1;h). \tag{26}$$

Proof. Suppose that $f \in \mathscr{S}_k^c(p,\lambda_2;h)$. A simple computation will then yield

$$\begin{aligned}(1-\lambda_1)&\frac{L_k^c f(z)}{z^p} + \lambda_1 \frac{L_{k+1}^c f(z)}{z^p} \\ &= \left(1 - \frac{\lambda_1}{\lambda_2}\right)\frac{L_k^c f(z)}{z^p} + \frac{\lambda_1}{\lambda_2}\left((1-\lambda_2)\frac{L_k^c f(z)}{z^p} + \lambda_2 \frac{L_{k+1}^c f(z)}{z^p}\right).\end{aligned} \tag{27}$$

It can now be easily shown that the class $\mathscr{P}(h)$ is a convex set. We can write (27) as follows:

$$(1-\lambda_1)\frac{L_k^c f(z)}{z^p} + \lambda_1 \frac{L_{k+1}^c f(z)}{z^p} = \left(1 - \frac{\lambda_1}{\lambda_2}\right) h_1(z) + \frac{\lambda_1}{\lambda_2} h_2(z) = \psi(z), \tag{28}$$

where $h_1 \in \mathscr{P}(h)$, by Theorem 1, and $h_2 \in \mathscr{P}(h)$, since $f \in \mathscr{S}_k^c(p,\lambda_2;h)$. We thus find that $\psi \in \mathscr{P}(h)$. Consequently, $f \in \mathscr{S}_k^c(p,\lambda_1;h)$. This proves Theorem 2.

Theorem 3. *The following inclusion relationship holds true:*

$$\mathscr{S}_k^c(p,\lambda;h) \subset \mathscr{S}_{k-1}^c(p,\lambda;h). \tag{29}$$

Proof. Let $f \in \mathscr{S}_k^c(p,\lambda;h)$ and suppose that

$$\left((1-\lambda)\frac{L_{k-1}^c f(z)}{z^p} + \lambda f rac L_k^c f(z) z^p\right) = H(z).$$

Then, from (12), we have

$$\begin{aligned}
&\left((1-\lambda)\frac{L_{k-1}^c f(z)}{z^p} + \lambda\ \frac{L_k^c f(z)}{z^p}\right) + \frac{z}{c}\left((1-\lambda)\frac{L_{k-1}^c f(z)}{z^p} + \lambda\ \frac{L_k^c f(z)}{z^p}\right)'\\
&\quad = H(z) + \frac{1}{c}\ zH'(z)\\
&\quad = (1-\lambda)\left[\frac{L_{k-1}^c f(z)}{z^p} + \frac{z}{c}\left(\frac{L_{k-1}^c f(z)}{z^p}\right)'\right]\\
&\qquad + \lambda\left[\frac{L_k^c f(z)}{z^p} + \frac{z}{c}\left(\frac{L_k^c f(z)}{z^p}\right)'\right]\\
&\quad = \left((1-\lambda)\frac{L_k^c f(z)}{z^p} + \lambda\ \frac{L_{k+1}^c f(z)}{z^p}\right) \in \mathscr{P}(h).
\end{aligned}$$

We thus find that

$$\left(H(z) + \frac{1}{c}\ zH'(z)\right) \prec h(z) \qquad (z \in \mathbb{U}). \tag{30}$$

By applying Lemma 1, it follows that

$$H(z) \prec \frac{c}{z^c}\int_0^z t^{c-1}h(t)\ \mathrm{d}t \prec h(z) \qquad (z \in \mathbb{U}),$$

which shows that $H \in \mathscr{P}(h)$. Consequently, we have

$$\left((1-\lambda)\frac{L_{k-1}^c f(z)}{z^p} + \lambda\ \frac{L_k^c f(z)}{z^p}\right) \in \mathscr{P}(h). \tag{31}$$

This evidently proves that $f \in \mathscr{S}_{k-1}^c(p,\lambda;h)$.

Corollary 1. *For $\Re(c) > 0$, let $f \in \mathscr{S}_k^c(p,\lambda;h)$. Then*

$$\frac{L_s^c f(z)}{z^p} \in \mathscr{P}(h) \qquad (s \in \{0,1,2,\ldots,k\}). \tag{32}$$

Proof. We can readily deduce the assertion (32) of the above Corollary from the assertion (17) of Theorem 1. The details involved are being omitted here.

In order to get the convolution results of the multivalent analytic function class $\mathscr{S}_k^c(p,\lambda;h)$, it is necessary to put the following restrictions on the operator L_k^c:

$$L_k^c(f * g) = (L_k^c f) * g = f * (L_k^c g), \tag{33}$$

where $f, g \in \mathscr{S}_k^c(p,\lambda;h)$ $(k \in \mathbb{N})$. We now prove our next result contained in Theorem 4 below.

Theorem 4. *Let the operator L_k^c satisfy the condition* (33). *If $f_j \in \mathscr{S}_k^c(p,\lambda;h_j)$ $(j=1,2)$, then each of the following inclusion relationships holds true:*

$$G(z) = (1-\lambda)L_k^c(f_1 * f_2)(z) + \lambda L_{k+1}^c(f_1 * f_2)(z) \in \mathscr{S}_k^c(p,\lambda,h_1 * h_2), \tag{34}$$

$$L_k^c(f_1 * f_2)(z) \in \mathscr{S}_k^c(p,\lambda;h_1 * h_2) \tag{35}$$

and

$$\frac{L_k^c\left[L_k^c(f_1 * f_2)(z)\right]}{z^p} \in \mathscr{P}(h_1 * h_2). \tag{36}$$

Proof. Since

$$f_1 \in \mathscr{S}_k^c(p,\lambda;h_1) \qquad \text{and} \qquad f_2 \in \mathscr{S}_k^c(p,\lambda;h_2), \tag{37}$$

it follows that

$$\left((1-\lambda)\frac{L_k^c f_1(z)}{z^p} + \lambda\frac{L_{k+1}^c f_1(z)}{z^p}\right) \in \mathscr{P}(h_1) \tag{38}$$

and

$$\left((1-\lambda)\frac{L_k^c f_2(z)}{z^p} + \lambda\frac{L_{k+1}^c f_2(z)}{z^p}\right) \in \mathscr{P}(h_2). \tag{39}$$

Also, from (38), (39) and Theorem 1, we have

$$\frac{L_k^c f_1(z)}{z^p} \in \mathscr{P}(h_1) \tag{40}$$

and

$$\frac{L_k^c f_2(z)}{z^p} \in \mathscr{P}(h_2). \tag{41}$$

Thus, by making use of (33), (38), (39) and Lemma 3, in conjunction with the technique used before, we have

$$\begin{aligned}
(1-\lambda)&\frac{L_k^c\left[(1-\lambda)L_k^c(f_1 * f_2)(z) + \lambda L_{k+1}^c(f_1 * f_2)(z)\right]}{z^p} \\
&+ \lambda\ \frac{L_{k+1}^c\left[(1-\lambda)L_k^c(f_1 * f_2)(z) + \lambda L_{k+1}^c(f_1 * f_2)(z)\right]}{z^p} \\
&= \left((1-\lambda)\frac{L_k^c g(z)}{z^p} + \lambda\frac{L_{k+1}^c g(z)}{z^p}\right) \in \mathscr{P}(h_1 * h_2),
\end{aligned}$$

that is, $G \in \mathscr{S}_k^c(p,\lambda;h_1 * h_2)$. This proves the first assertion (34) of Theorem 4. In order to demonstrate the second assertion (35) of Theorem 4, we again proceed in a similar manner and apply Lemma 3 to (38) and (41). We thus obtain

$$\left((1-\lambda)\frac{L_k^c\left[L_k^c(f_1 * f_2)(z)\right]}{z^p} + \lambda\frac{L_{k+1}^c\left[L_k^c(f_1 * f_2)(z)\right]}{z^p}\right) \in \mathscr{P}(h_1 * h_2), \tag{42}$$

which clearly implies (35). Finally, from (42) and Theorem 1, we obtain the third assertion (36) of Theorem 4.

As a special case of Theorem 4, we obtain a result proved in [11] (where $c = c_1$ and $k = 0$) for

$$h_j(z) = \frac{1 + A_j z}{1 + B_j z} \qquad (z \in \mathbb{U},\ j = 1, 2)$$

and

$$L_k^c f(z) = f(z) * {}_qF_r(z),$$

where ${}_qF_r$ is the generalized hypergeometric function defined by (see also [4] and [5])

$${}_qF_r(z) = {}_qF_r(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_r; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_q)_n}{(\beta_1)_n \ldots (\beta_r)_n} \frac{z^n}{n!}$$

$$(q, r \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ q \leqq r + 1) \tag{43}$$

for complex parameters

$$\alpha_1, \ldots, \alpha_q \qquad \text{and} \qquad \beta_1, \ldots, \beta_r \qquad (\beta_j \neq 0, -1, -2, \ldots;\ j = 1, \ldots, r). \tag{44}$$

Theorem 5. *Let the operator L_k^c satisfy the condition* (33). *If $f \in \mathscr{S}_k^c(p, \lambda; h)$ and $\mathfrak{q} \in \mathscr{A}(p)$ with*

$$\Re\left(\frac{\mathfrak{q}(z)}{z^p}\right) \geqq \frac{1}{2} \qquad (z \in \mathbb{U}), \tag{45}$$

then $f * \mathfrak{q} \in \mathscr{S}_k^c(p, \lambda; h)$.

Proof. By using the properties of convolution and (33), we have

$$\begin{aligned}
&(1-\lambda)\frac{L_k^c(f * \mathfrak{q})(z)}{z^p} + \lambda \frac{L_{k+1}^c(f * \mathfrak{q})(z)}{z^p} \\
&= \left((1-\lambda)\frac{L_k^c f(z)}{z^p} + \lambda \frac{L_{k+1}^c f(z)}{z^p}\right) * \frac{\mathfrak{q}(z)}{z^p} \\
&= H(z) * \frac{\mathfrak{q}(z)}{z^p} \qquad \big(H \in \mathscr{P}(h)\big).
\end{aligned}$$

Now, by using Lemma 2, we get

$$\left(H(z) * \frac{\mathfrak{q}(z)}{z^p}\right) \in \mathscr{P}(h),$$

which implies that

$$\left((1-\lambda)\frac{L_k^c(f * \mathfrak{q})(z)}{z^p} + \lambda \frac{L_{k+1}^c(f * \mathfrak{q})(z)}{z^p}\right) \in \mathscr{P}(h). \tag{46}$$

By means of (46), we have thus proved the assertion of Theorem 5 that

$$f * \mathfrak{q} \in \mathscr{S}_k^c(p, \lambda; h). \tag{47}$$

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