EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 6, No. 3, 2013, 340-351
ISSN 1307-5543 – www.ejpam.com

A New Generalization of the Operator-Valued Poisson Kernel

Sharifa Al-Sharif[1], Fatima Salem[1], Basem Frasin[2]

[1] *Department of Mathematics, Yarmouk University, Irbed, Jordan*
[2] *Department of Mathematics, Al al-Bayt University, Almafrag, Jordan*

Abstract. The purpose of this paper is to give a new generalization of the operator-valued Poisson kernel and discuss integral formulas for them.

2010 Mathematics Subject Classifications: 45P05, 47A60; 46E40, 47B38

Key Words and Phrases: Poisson kernel, Operator-valued Poisson kernel

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space and $\mathfrak{L}(\mathscr{H})$ denote the algebra of all bounded linear operators from $\mathscr{H}$ into $\mathscr{H}$. For $T \in \mathfrak{L}(\mathscr{H})$, its spectrum $\sigma(T)$ is the non-empty compact subset of the complex plane $\mathbb{C}$ consisting of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is non-invertible in $\mathfrak{L}(\mathscr{H})$, where I is the identity operator on $\mathscr{H}$. We write $\mathbb{D}$ for the open unit disk in $\mathbb{C}$, $\mathbb{D} = \{z : |z| < 1\}$.

Let $A \in \mathfrak{L}(\mathscr{H})$. For a complex valued function f analytic on a domain $\mathbb{E}$ of the complex plane containing the spectrum $\sigma(A)$ of A, we recall Riesz-Dunford integral $f(A)$ which is given by

$$f(A) = \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1} dz, \tag{1}$$

where C is a positively oriented simple closed rectifiable contour containing $\sigma(A)$.

By differentiating the integral in equation (1) with respect to A we get

$$f'(A) = \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-2} dz. \tag{2}$$

Email addresses: `sharifa@yu.edu.jo` (S. Sharif) `fatimabayui@yahoo.com` (F. Salem), `bafrasin@yahoo.com` (B. Frasin)

If we differentiate the integral in equation (2) with respect to A, $(n-1)$ times, we get

$$f^{(n)}(A) = \frac{n!}{2\pi i}\int_C f(z)(zI - A)^{-n-1}\,dz, \quad (n = 0, 1, 2, \ldots). \tag{3}$$

Note that, expression (3) is an extension of the Riesz-Dunford integral in equation (1).

For $re^{it} \in \mathbb{D}$, the (scalar) Poisson kernel $P_{r,t}$ is defined by

$$\begin{aligned} P_{r,t}\left(e^{i\theta}\right) &= \frac{1 - r^2}{\left(1 - re^{it}e^{-i\theta}\right)\left(1 - re^{-it}e^{i\theta}\right)} \\ &= \frac{1}{1 - re^{it}e^{-i\theta}} + \frac{1}{1 - re^{-it}e^{i\theta}} - 1 \\ &= \sum_{n\geq 0} r^n e^{int} e^{-in\theta} + \sum_{n\geq 0} r^n e^{-int} e^{in\theta} - 1. \end{aligned} \tag{4}$$

The integral formula of the (scalar) Poisson kernel

$$\frac{1}{2\pi}\int_0^{2\pi} P_{r,t}\left(e^{i\theta}\right) d\theta = 1,$$

holds, where r is a real parameter satisfying $|r| < 1$, see[3].

For $T \in \mathfrak{L}(\mathscr{H})$, $\sigma(T) \subset \overline{\mathbb{D}}$ and $re^{it} \in \mathbb{D}$, the author in [2], define the operator-valued Poisson kernel $K_{r,t}(T)$ as follows

$$K_{r,t}(T) = \left(I - re^{it}T^*\right)^{-1} + \left(I - re^{-it}T\right)^{-1} - I, \tag{5}$$

and prove the following theorem.

Theorem 1. *For $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, we have*

$$\begin{aligned} K_{r,t}(T) &= \left(I - re^{it}T^*\right)^{-1}\left(I - r^2T^*T\right)\left(I - re^{-it}T\right)^{-1} \\ &= \sum_{n\geq 0} r^n e^{int} T^{*n} + \sum_{n\geq 0} r^n e^{-int} T^n - I. \end{aligned}$$

Afterwards, in [1] Bulut proved the following theorem.

Theorem 2. *For $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, we have*

$$\frac{1}{2\pi}\int_0^{2\pi} K_{r,t}(T)\,dt = I, \tag{6}$$

where r is a real parameter satisfying $|r| < 1$.

A generalization of the (scalar) Poisson kernel, (4) in [3] is given by

$$Q_{a,b,t}\left(e^{i\theta}\right) = \frac{1-ab}{\left(1-ae^{it}e^{-i\theta}\right)\left(1-be^{-it}e^{i\theta}\right)}, \tag{7}$$

where a and b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

In [1], Bulut introduced a generalization of the operator-valued Poisson kernel $K_{r,t}(T)$ for $T \in \mathfrak{L}(\mathscr{H})$, $\sigma(T) \subset \overline{\mathbb{D}}$ and $re^{it} \in \mathbb{D}$ in the following way

$$Q_{a,b,t}(T) = \left(I - ae^{it}T^*\right)^{-1} + \left(I - be^{-it}T\right)^{-1} - I, \tag{8}$$

where a and b are complex parameters satisfying $|a| < 1$ and $|b| < 1$ and prove the following theorem.

Theorem 3 ([1]). *Let $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} Q_{a,b,t}(T)\,dt = I, \tag{9}$$

where a and b are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Remark 1. *We note that* (8) *and* (9) *are generalizations of* (5) *and* (6)*, respectively, by taking $a = b = r$.*

2. A New Generalization of the Operator-Valued Poisson Kernel

In this section, we set the following definition and open problem.

Definition 1. *Let $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. For $n = 0, 1, 2, \ldots$, let*

$$I_n = def\,\frac{1}{2\pi}\int_0^{2\pi} Q^{n+1}_{a,b,t}(T)\,dt,$$

where a, b, are complex parameters satisfying $|a| < 1$ and $|b| < 1$.

Open Problem: Compute I_n, $n = 0, 1, 2, \ldots$.

In the following theorem we give a partial answer to the open problem to certain class of operators in $\mathfrak{L}(\mathscr{H})$.

Theorem 4. *Let $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$ and $(I - ae^{it}T^*)$ is self adjoint. Then*

$$\int_0^{2\pi} Q_{a,\overline{a},t}(T)^{n+1}\,dt = \sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}(-I)^l, \tag{10}$$

for $n = 0, 1, 2, \ldots$, and a complex parameter a satisfying $|a| < 1$.

Proof. Let

$$
\begin{aligned}
I_n = \int_0^{2\pi} Q_{a,\overline{a},t}(T)^{n+1}\,dt &= \frac{1}{2\pi}\int_0^{2\pi}\left(\left(I-ae^{it}T^*\right)^{-1}+\left(I-\overline{a}e^{-it}T\right)^{-1}-I\right)^{n+1}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(I-ae^{it}T^*\right)^{-n-1+k}\left(\left(I-\overline{a}e^{-it}T\right)^{-1}+(-I)\right)^{k}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}\left(I-ae^{it}T^*\right)^{-n-1+k}\left(I-\overline{a}e^{-it}T\right)^{-k+l}(-I)^l dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}\left(I-ae^{it}T^*\right)^{-n-1+k}\left(\left(I-ae^{it}T^*\right)^*\right)^{-k+l}(-I)^l dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}\left(I-ae^{it}T^*\right)^{-n-1+l}(-I)^l dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}e^{-(n+1-l)it}\left(e^{-it}I-aT^*\right)^{-n-1+l}(-I)^l dt. \qquad (11)
\end{aligned}
$$

By the change of variables, with $z=e^{-it}$, (11) becomes

$$
\begin{aligned}
I_n &= \frac{-1}{2\pi i}\oint_{|z|=1}\sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}\left(zI-aT^*\right)^{-n-1+l}(-I)^l z^{n-l}dz \\
&= \frac{-1}{2\pi i}\sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}\oint_{|z|=1}\left(zI-aT^*\right)^{-n-1+l}(-I)^l z^{n-l}dz,
\end{aligned}
$$

where the integral along $|z|=1$ is taken in the negative direction. Hence, by the Riesz-Dunford integral (3), we have

$$
I_n=\sum_{k=0}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}(-I)^l, \quad (n=0,1,2,\ldots).
$$

□

Corollary 1. *For $T\in\mathfrak{L}(\mathscr{H})$ such that $\sigma(T)\subset\overline{\mathbb{D}}$ and $\left(I-ae^{it}T^*\right)$ is self adjoint, we have*

$$
\frac{1}{2\pi}\int_0^{2\pi} Q_{a,\overline{a},t}(T)^2\,dt = I, \qquad (12)
$$

where a is complex parameter satisfying $|a| < 1$.

Remark 2. *By taking $n = 0$ in (10), we obtain (9).*

Definition 2. *For $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, we set a generalization of the operator-valued Poisson kernel, $Q_{a,b,t}(T)$ in the following way:*

$$R_{a,b,c,d,t}(T) = \left(I - ae^{it}T^*\right)^{-1} + \left(I - be^{-it}T\right)^{-1} + \left(I - ce^{it}T^*\right)^{-1} - \left(I - de^{-it}T\right)^{-1} - I, \quad (13)$$

where a, b, c, and d are complex parameters satisfying $|a| < 1$, $|b| < 1$, $|c| < 1$, and $|d| < 1$.

Remark 3. *Note that $R_{a,b,c,d,t}(T) \in \mathfrak{L}(\mathscr{H})$.*

Lemma 1. *For $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, we have*

$$R_{a,b,c,d,t}(T) = \sum_{n\geq 0} a^n e^{int} T^{*n} + \sum_{n\geq 0} b^n e^{-int} T^n + \sum_{n\geq 0} c^n e^{int} T^{*n} - \sum_{n\geq 0} d^n e^{-int} T^n - I. \quad (14)$$

Proof. Since $\left\|ae^{it}T^*\right\| < 1$, $\left\|be^{-it}T\right\| < 1$, $\left\|ce^{it}T^*\right\| < 1$, and $\left\|de^{-it}T\right\| < 1$, we have

$$\sum_{n\geq 0} a^n e^{int} T^{*n} = \left(I - ae^{it}T^*\right)^{-1},$$

$$\sum_{n\geq 0} b^n e^{-int} T^n = \left(I - be^{-it}T\right)^{-1},$$

$$\sum_{n\geq 0} c^n e^{int} T^{*n} = \left(I - ce^{it}T^*\right)^{-1},$$

and

$$\sum_{n\geq 0} d^n e^{-int} T^n = \left(I - de^{-it}T\right)^{-1}.$$

By the above four equalities and (13), we get (14). □.

For an operator $T \in \mathfrak{L}(\mathscr{H})$ and a polynomial $r(z) = \sum_{k=0}^{s} c_k z^k \in \mathbb{C}[z]_{|\overline{\mathbb{D}}}$, $r(T) \in \mathfrak{L}(\mathscr{H})$ is defined by

$$r(T) = \sum_{k=0}^{s} c_k T^k.$$

Lemma 2. *Let $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. For $r(z) \in \mathbb{C}[z]_{|\overline{\mathbb{D}}}$, we have*

$$r(bT) - r(dT) + c_0 I = \frac{1}{2\pi}\int_0^{2\pi} r\left(e^{it}\right) R_{a,b,c,d,t}(T)\, dt,$$

where a, b, c, and d are complex parameters satisfying $|a| < 1$, $|b| < 1$, $|c| < 1$, and $|d| < 1$.

Proof. Let $r(z)=\sum_{k=0}^{s}c_k z^k$. By (14), and since $\int_0^{2\pi} e^{ilt}dt=0$ for $l\in\mathbb{Z}/\{0\}$, we get

$$\begin{aligned}
&\int_0^{2\pi} r\left(e^{it}\right)R_{a,b,c,d,t}(T)\,dt\\
&=\int_0^{2\pi}\sum_{k=0}^{s}c_k e^{ikt}\left(\begin{array}{c}\sum_{n\geq 0}a^n e^{int}T^{*n}+\sum_{n\geq 0}b^n e^{-int}T^n\\ +\sum_{n\geq 0}c^n e^{int}T^{*n}-\sum_{n\geq 0}d^n e^{-int}T^n-I\end{array}\right)dt\\
&=2\pi c_0 I+\sum_{k=0}^{s}\int_0^{2\pi}c_k b^k T^k dt+2\pi c_0 I-\sum_{k=0}^{s}\int_0^{2\pi}c_k d^k T^k dt-2\pi c_0 I\\
&=2\pi\sum_{k=0}^{s}c_k b^k T^k-2\pi\sum_{k=0}^{s}c_k d^k T^k+2\pi c_0 I\\
&=2\pi r(bT)-2\pi r(dT)+2\pi c_0 I.
\end{aligned}$$

□

Corollary 2. *Note that, if r identically equal to 1, then*

$$\frac{1}{2\pi}\int_0^{2\pi}R_{a,b,c,d,t}(T)\,dt=I, \tag{15}$$

for $|a|<1$, $|b|<1$, $|c|<1$, $|d|<1$ and $T\in\mathfrak{L}(\mathscr{H})$ such that $\sigma(T)\subset\overline{\mathbb{D}}$.

In the next theorem, we give a different proof of equation (15) independent of a polynomial. For this purpose we will use the Riesz-Dunford integral formula.

Theorem 5. *Let $T\in\mathfrak{L}(\mathscr{H})$ such that $\sigma(T)\subset\overline{\mathbb{D}}$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi}R_{a,b,c,d,t}(T)\,dt=I,$$

where $a,b,c,$ and d are complex parameters satisfying $|a|<1$, $|b|<1$, $|c|<1$, and $|d|<1$.

Proof. From (13), we have

$$\frac{1}{2\pi}\int_0^{2\pi}R_{a,b,c,d,t}(T)\,dt=\frac{1}{2\pi}\int_0^{2\pi}\left(\begin{array}{c}\left(I-ae^{it}T^*\right)^{-1}+\left(I-be^{-it}T\right)^{-1}+\\ \left(I-ce^{it}T^*\right)^{-1}-\left(I-de^{-it}T\right)^{-1}-I\end{array}\right)dt. \tag{16}$$

We set

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - ae^{it}T^*\right)^{-1} dt, \tag{17}$$

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - be^{-it}T\right)^{-1} dt, \tag{18}$$

$$I_3 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - ce^{it}T^*\right)^{-1} dt, \tag{19}$$

$$I_4 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - de^{-it}T\right)^{-1} dt, \tag{20}$$

and

$$I_5 = \frac{1}{2\pi}\int_0^{2\pi} I dt. \tag{21}$$

Therefore, it follows from (16)- (21) that

$$\frac{1}{2\pi}\int_0^{2\pi} R_{a,b,c,d,t}(T)\, dt = I_1 + I_2 + I_3 - I_4 - I_5. \tag{22}$$

It is clear that

$$I_5 = I. \tag{23}$$

Next, we shall calculate I_1, I_2, I_3 and I_4. Firstly, we have

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - ae^{it}T^*\right)^{-1} dt = \frac{1}{2\pi}\int_0^{2\pi} e^{-it}\left(e^{-it}I - aT^*\right)^{-1} dt.$$

Making substitution $z = e^{-it}$ in the last integral, we get

$$I_1 = \frac{-1}{2\pi i}\int_{|z|=1} \left(zI - aT^*\right)^{-1} dz,$$

where the integral along $|z| = 1$ is taken in the negative direction. Hence, by the Riesz-Dunford integral in the equation (1), we have

$$I_1 = I. \tag{24}$$

Similarly, we get

$$I_3 = I. \tag{25}$$

Secondly, we have

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - be^{-it}T\right)^{-1} dt = \frac{1}{2\pi}\int_0^{2\pi} e^{it}\left(e^{it}I - bT\right)^{-1} dt.$$

If we set $z = e^{it}$, then the last integral is of the form

$$I_2 = \frac{1}{2\pi i}\int_{|z|=1} (zI - bT)^{-1}\, dz,$$

where the integral along $|z| = 1$ is taken in the positive direction. Hence, by the Riesz-Dunford integral (1), we have

$$I_2 = I. \tag{26}$$

Similarly, we get

$$I_4 = I. \tag{27}$$

Therefore, from (22)-(27), we get (15). □

Remark 4. *By taking* $c = 0$ *and* $d = 0$ *in* (13) *and* (15) *we find that* (13) *and* (15) *are generalizations of* (8) *and* (9)*, respectively.*

3. The Finite Sum of the Operator- Valued Poisson Kernel

In this section we definite a new generalization of the operator-valued Poisson kernel $M_{(a_k,b_k)_{k=0}^n,t}(T)$ in $2(n+1)$ complex parameters. Let us begin by the following definition.

Definition 3. *For* $T \in \mathfrak{L}(\mathscr{H})$ *such that* $\sigma(T) \subset \overline{\mathbb{D}}$*, define the finite sum of the operator-valued Poisson kernel in the following way.*

$$\begin{aligned} M_{(a_k,b_k)_{k=0}^n,t}(T) &= \left(I - a_0 e^{it}T^*\right)^{-1} + \left(I - b_o e^{-it}T\right)^{-1} \\ &\quad + \sum_{k=1}^{n}\left(I - a_k e^{it}T^*\right)^{-1} - \sum_{k=1}^{n}\left(I - b_k e^{-it}T\right)^{-1} - I, \end{aligned} \tag{28}$$

where a_k *and* b_k *are complex parameters satisfying* $\left|a_k\right| < 1$ *and* $\left|b_k\right| < 1$*,* $0 \le k \le n$*, and for* $n = 0, 1, 2, \ldots$*.*

Remark 5. *By taking* $n = 0$ *and* $n = 1$ *in* (28)*, we obtain* (8) *and* (13)*, respectively.*

Remark 6. *Note that* $M_{(a_k,b_k)_{k=0}^n,t}(T) \in \mathfrak{L}(\mathscr{H})$*.*

Lemma 3. *For $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$, we have*

$$M_{\left(a_k,b_k\right)_{k=0}^n,t}(T) = \sum_{m\geq 0} a_0^m e^{imt} T^{*m} + \sum_{m\geq 0} b_o^m e^{-imt} T^m$$
$$+ \sum_{m\geq 0}\sum_{k=1}^{n} a_k^m e^{imt} T^{*m} - \sum_{m\geq 0}\sum_{k=1}^{n} b_k^m e^{-imt} T^m - I. \quad (29)$$

Proof. Since $\left|\left|a_k e^{it} T^*\right|\right| < 1$, and $\left|\left|b_k e^{-it} T\right|\right| < 1$, $0 \leq k \leq n$, we have

$$\sum_{k=0}^{n} \left(I - a_k e^{it} T^*\right)^{-1} = \sum_{m\geq 0}\sum_{k=0}^{n} a_k^m e^{imt} T^{*m},$$

and

$$\sum_{k=0}^{n} \left(I - b_k e^{-it} T\right)^{-1} = \sum_{m\geq 0}\sum_{k=0}^{n} b_k^m e^{-imt} T^m$$

respectively. By the two equalities above and (28), we get (29). □

For an operator $T \in \mathfrak{L}(\mathscr{H})$ and a polynomial $r(z) = \sum_{j=0}^{s} c_j z^j \in \mathbb{C}[z]_{|\overline{\mathbb{D}}}$, $r(T) \in \mathfrak{L}(\mathscr{H})$ is defined by

$$r(T) = \sum_{j=0}^{s} c_j T^j.$$

Lemma 4. *Let $T \in \mathfrak{L}(\mathscr{H})$ such that $\sigma(T) \subset \overline{\mathbb{D}}$. For $r(z) \in \mathbb{C}[z]_{|\overline{\mathbb{D}}}$. Then*

$$r\left(b_0 T\right) - \sum_{k=1}^{n} r\left(b_k T\right) + n c_0 I = \frac{1}{2\pi}\int_0^{2\pi} r\left(e^{it}\right) M_{\left(a_k,b_k\right)_{k=0}^n,t}(T)\, dt,$$

where a_k and b_k are complex parameters satisfying $\left|a_k\right| < 1$ and $\left|b_k\right| < 1$, $0 \leq k \leq n$, and for $n = 0, 1, 2, \ldots$.

Proof. From (29) and since $\int_0^{2\pi} e^{ilt} dt = 0$ for $l \in \mathbb{Z}/\{0\}$, we get

$$\int_0^{2\pi} r(e^{it}) M_{\left(a_k,b_k\right)_{k=0}^n,t}(T)\, dt = \sum_{j=0}^{s}\sum_{m\geq 0} c_j a_0^m T^{*m} \int_0^{2\pi} e^{i(m+j)t} dt$$
$$+ \sum_{j=0}^{s}\sum_{m\geq 0} c_j b_0^m T^m \int_0^{2\pi} e^{i(j-m)t} dt$$

$$
\begin{aligned}
&+\sum_{j=0}^{s}\sum_{m\geq 0}\sum_{k=1}^{n} c_j a_k^m T^{*m}\int_0^{2\pi} e^{i(j+m)t}dt \\
&-\sum_{j=0}^{s}\sum_{m\geq 0}\sum_{k=1}^{n} c_j b_k^m T^{m}\int_0^{2\pi} e^{i(j-m)t}dt - \sum_{j=0}^{s} c_j\int_0^{2\pi} e^{ijt}dt \\
&= 2\pi c_0 I + 2\pi\sum_{j=0}^{s} c_j b_0^j T^j + 2\pi\sum_{k=1}^{n} c_0 I \\
&-2\pi\sum_{j=0}^{s}\sum_{k=1}^{n} c_j b_k^j T^j - 2\pi c_0 I \\
&= 2n\pi c_0 I + 2\pi r\left(b_0 T\right) - 2\pi\sum_{k=1}^{n} r\left(b_k T\right).
\end{aligned}
$$

□

Corollary 3. *Note that if* r *identically equal to* 1*, we have*

$$
\frac{1}{2\pi}\int_0^{2\pi} M_{\left(a_k,b_k\right)_{k=0}^n,t}(T)\,dt = I, \tag{30}
$$

for complex parameters a_k *and* b_k *satisfying* $\left|a_k\right| < 1$ *and* $\left|b_k\right| < 1$, $0 \leq k \leq n$, $n = 0,1,2,\ldots$ *and* $T \in \mathfrak{L}(\mathscr{H})$ *such that* $\sigma(T) \subset \overline{\mathbb{D}}$.

Now, we give a different proof of equation (30) independent of a polynomial.

Theorem 6. *Let* $T \in \mathfrak{L}(\mathscr{H})$ *such that* $\sigma(T) \subset \overline{\mathbb{D}}$*. Then*

$$
\frac{1}{2\pi}\int_0^{2\pi} M_{\left(a_k,b_k\right)_{k=0}^n,t}(T)\,dt = I,
$$

where a_k *and* b_k *are complex parameters satisfying* $\left|a_k\right| < 1$ *and* $\left|b_k\right| < 1$, $0 \leq k \leq n$, $n = 0,1,2,\ldots$.

Proof. From (28), we have

$$
\frac{1}{2\pi}\int_0^{2\pi} M_{\left(a_k,b_k\right)_{k=0}^n,t}(T)\,dt = \frac{1}{2\pi}\int_0^{2\pi}\left(\begin{array}{c}\left(I-a_0 e^{it}T^*\right)^{-1}+\left(I-b_o e^{-it}T\right)^{-1}+\\ \sum\limits_{k=1}^{n}\left(I-a_k e^{it}T^*\right)^{-1}-\sum\limits_{k=1}^{n}\left(I-b_k e^{-it}T\right)^{-1}-I\end{array}\right)dt. \tag{31}
$$

We set

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - a_0 e^{it} T^*\right)^{-1} dt, \tag{32}$$

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi} \left(I - b_0 e^{-it} T\right)^{-1} dt, \tag{33}$$

$$I_3 = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{n} \left(I - a_k e^{it} T^*\right)^{-1} dt, \tag{34}$$

$$I_4 = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{n} \left(I - b_k e^{-it} T\right)^{-1} dt, \tag{35}$$

and

$$I_5 = \frac{1}{2\pi}\int_0^{2\pi} I dt. \tag{36}$$

Therefore, it follows from (32)- (36) that

$$\frac{1}{2\pi}\int_0^{2\pi} M_{\left(a_k, b_k\right)_{k=0}^{n}, t} (T) dt = I_1 + I_2 + I_3 - I_4 - I_5. \tag{37}$$

It is clear that

$$I_5 = I. \tag{38}$$

Following similarly the proof of Theorem 5, we get

$$I_1 = I. \tag{39}$$

$$I_2 = I. \tag{40}$$

Next, we shall calculate I_3 and I_4. First, we have

$$I_3 = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{n} \left(I - a_k e^{it} T^*\right)^{-1} dt = \sum_{k=1}^{n} (\frac{1}{2\pi}\int_0^{2\pi} e^{-it} \left(e^{-it} I - a_k T^*\right)^{-1} dt).$$

Making substitution $z = e^{-it}$ in the last integral, we get

$$I_3 = \sum_{k=1}^{n} (\frac{-1}{2\pi i}\int_{|z|=1} \left(zI - a_k T^*\right)^{-1} dz),$$

where the integral along $|z| = 1$ is taken in the negative direction. Hence, by the Riesz-Dunford integral in the equation (1), we have

$$I_3 = \sum_{k=1}^{n} I = nI. \tag{41}$$

Similarly, we get

$$I_4 = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^{n} \left(I - b_k e^{-it} T\right)^{-1} dt = \sum_{k=1}^{n} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{it} \left(e^{it} I - b_k T\right)^{-1} dt\right).$$

If we set $z = e^{it}$, then the last integral is of the form

$$I_4 = \sum_{k=1}^{n} \left(\frac{1}{2\pi i} \int_{|z|=1} \left(zI - b_k T\right)^{-1} dz\right),$$

where the integral along $|z| = 1$ is taken in the positive direction. Hence, by the Riesz-Dunford integral (1), we have

$$I_4 = \sum_{k=1}^{n} I = nI. \tag{42}$$

Therefore, from (37)-(42) we get (30). □

References

[1] S Bulut. A note on the operator-valued poisson kernel. *European Journal of Pure and Applied Mathematics*, 2(2):296–301, 2009.

[2] I Chalendar. The operator-valued poisson kernel and its application. *Irish Mathematical Society Bulletin*, 51:21–44, 2003.

[3] H Haruki. A new generalization of the poisson kernel. *Journal of Applied Analysis and Stochastic Analysis*, 10(2):191–196, 1997.