



K-theory, Chamber Homology and Base Change for $GL(2)$

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Abstract. In this work on $GL(2)$ we have found that it is hard to compute the chamber homology groups from the quotient space $\beta^1 GL(2)/GL(2)$ (Möbius band), so we introduced a new way to compute the chamber homology groups by restricting to the original quotient space (edge) before taking the real line \mathbb{R} . We have not yet given a full description of what happens under base change when we work on the cuspidal representation but, we somehow, gave a way to compute the base change effect of some type of cuspidal representations which are the admissible pairs. The base change of a principal series representation is always a principal series. Similarly, the base change of a twist of Steinberg representation is again a twist of Steinberg. However, an irreducible Galois representation can certainly restrict to a reducible one. Thus it is possible for the base change of a cuspidal to be principal series. In fact, if π is any irreducible admissible representation of $GL(2, F)$ then one can find an extension E/F such that $BC(\pi)$ is either unramified or Steinberg. .

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1. Introduction

Let $G = GL(n, F)$ and let C_r^*G denote the reduced C^* -algebra of G . According to the Baum-Connes correspondence, we have a canonical isomorphism [2]

$$\mu_F : K_j^{top} \beta^1 G \rightarrow K_j C_r^*G,$$

where $\beta^1 G$ denotes the enlarged building of G . In noncommutative geometry, isomorphisms of C^* -algebras are too restrictive to provide a good notion of isomorphisms of noncommutative spaces, and the correct notion is provided by strong Morita equivalence of C^* -algebras. The noncommutative C^* -algebra C_r^*G is strongly Morita equivalent to the commutative C^* -algebra $C_0(Irr^t G)$ where $Irr^t G$ denotes the tempered dual of G [18]. Consequently, we have

$$K_j C_r^*G \cong K^j Irr^t G$$

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and this leads to the following formulation of the Baum-Connes correspondence:

$$K_j^{top} \beta^1 G \cong K^j Irr^t G.$$

This in turn leads to the following diagram

$$\begin{array}{ccc} K_j^{top}(\beta^1 G(E)) & \xrightarrow{\mu_E} & K^j(Irr^t G(E)) \\ \downarrow & & \downarrow K^j(BC_{E/F}) \\ K_j^{top}(\beta^1 G(F)) & \xrightarrow{\mu_F} & K^j(Irr^t G(F)) \end{array}$$

where the left-hand vertical map is the unique map which makes the diagram commutative. This work will be concerned with presenting an explicit construction of the local Langlands correspondence between so-called *cuspidal* representations of $GL_2(F)$ and certain 2-dimensional representations of \mathcal{W}_F , where the residue characteristic $char \neq 2$. We will focus on the classification and the construction of the cuspidal representations for $GL_2(F)$, it was originally treated by [12] and [13].

2. Chamber Homology for $GL(2)$

Consider the pair (L, κ) where L is a Levi subgroup of a parabolic subgroup of G , and κ is an irreducible cuspidal representation of L . Two pairs $(L_1, \kappa_1), (L_2, \kappa_2)$ are called inertially equivalent if there exist $g \in G$ and an unramified character χ of L_2 such that

$$L_2 = L_1^g, \text{ and } \kappa_1^g = \kappa_2 \otimes \chi,$$

where $L_1^g := g^{-1}L_1g$ and $\kappa_1^g(x) = \kappa_1(gxg^{-1})$ for all $x \in L_1^g$. Let $[L, \kappa]_G$ be the inertial equivalence class of the pair (L, κ) and let $\mathfrak{B}(G)$ be the set of all inertial equivalence classes. This set is called the *Bernstein spectrum* of G .

Definition 1. Let $s \in \mathfrak{B}(G)$, an *s-type* is a pair (J, σ) consisting of a compact open subgroup J of G and an irreducible smooth representation σ of J such that for any irreducible smooth representation π of G , the restriction of π to J contains σ if and only if π is an object of $\mathfrak{A}^s(G)$, [8]. It has been proved by [6] and [9] that there exists an *s-type* for each point $s \in \mathfrak{B}(G)$.

Now, let \mathcal{O}_F denote the ring of integers of F , $\varpi \in F$ be a uniformizer, and \mathfrak{p} be the maximal ideal of \mathcal{O}_F . Also, let

$$\Pi = \Pi_n = \begin{pmatrix} 0 & I_{n-1} \\ \varpi_F & 0 \end{pmatrix}, \text{ and } s_i = \begin{pmatrix} I_{i-1} & & & \\ & 0 & 1 & \\ & & 1 & 0 \\ & & & I_{n-i-1} \end{pmatrix},$$

for every $i \in \{1, \dots, n - 1\}$, and $s_0 = \Pi s_1 \Pi^{-1}$ denotes the standard involutions in G . The finite Weyl group is $\mathbb{W}_0 = \langle s_1, s_2, \dots, s_{n-1} \rangle$, and the affine Weyl group defined as follows

$\mathbb{W} = \langle s_0, s_1, \dots, s_{n-1} \rangle$. The extended affine Weyl group is denoted by $\overline{\mathbb{W}} = \mathbb{W} \rtimes \langle \Pi \rangle$. Its clear that $\overline{\mathbb{W}} \cap GL(n, \mathcal{O}_F) = \mathbb{W}_0$. The standard Iwahori subgroup is

$$I = \begin{pmatrix} \mathcal{O}_F^\times & \mathcal{O}_F & \dots & \mathcal{O}_F \\ \mathfrak{p}_F & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O}_F \\ \mathfrak{p}_F & \dots & \mathfrak{p}_F & \mathcal{O}_F^\times \end{pmatrix}.$$

Let Σ be the apartment attached to the diagonal torus and let Δ be the unique chamber in this apartment which is stabilized by $\langle \Pi \rangle I$. Let J_i be the maximal standard parahoric subgroups of G ,

$$J_i = I \langle s_0, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{n-1} \rangle I$$

where $J_0 = GL(n, \mathcal{O}_F)$. We see that J_i are the stabilizers of the vertices, the stabilizer of the facets of dimension $n - 1$ of Δ are K_0, K_1, \dots, K_{n-1} , where $K_i = I \langle s_i \rangle I$. The enlarged building $\beta^1 G$ is labelled, this means there exists a simplicial map

$$\mathfrak{J} : \beta^1 G \rightarrow \Delta,$$

this map is dimensions preserver. This labelling is unique and it allows us to fix an orientation of the simplices. The chamber homology groups are obtained by totalizing the bicomplex:

$$\begin{array}{ccccccc} 0 \leftarrow \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) \oplus \dots \oplus \mathfrak{R}(J_{n-1}) & \leftarrow \dots & \leftarrow \mathfrak{R}(K_0) \oplus \mathfrak{R}(K_1) \oplus \dots \oplus \mathfrak{R}(K_{n-1}) & \leftarrow \mathfrak{R}(I) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 \leftarrow \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) \oplus \dots \oplus \mathfrak{R}(J_{n-1}) & \leftarrow \dots & \leftarrow \mathfrak{R}(K_0) \oplus \mathfrak{R}(K_1) \oplus \dots \oplus \mathfrak{R}(K_{n-1}) & \leftarrow \mathfrak{R}(I) \end{array}$$

the vertical maps are given by $1 - \sqsupset \Pi$.

We assume that $\mathcal{C} = \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) \oplus \dots \oplus \mathfrak{R}(J_{n-1})$, $\mathcal{C}' = \mathfrak{R}(K_0) \oplus \mathfrak{R}(K_1) \oplus \dots \oplus \mathfrak{R}(K_{n-1})$ and $\mathcal{C}'' = \mathfrak{R}(I)$. By totalizing the above bicomplex, we obtain this chain complex

$$0 \longleftarrow \mathcal{C} \longleftarrow \dots \longleftarrow \mathcal{C}'_{i-1} \oplus \mathcal{C}'_i \longleftarrow \mathcal{C}'_i \oplus \mathcal{C}'_{i+1} \longleftarrow \dots \longleftarrow \mathcal{C}'' \longleftarrow 0.$$

Definition 2 ([3]). *The homology groups of this totalized complex are the chamber homology groups.*

Now, for each point $s \in \mathfrak{B}(G)$ let

$$\mathcal{C}(s) = \mathfrak{R}(J_0(s)) \oplus \mathfrak{R}(J_1(s)) \oplus \dots \oplus \mathfrak{R}(J_{n-1}(s)),$$

$$\mathcal{C}'(s) = \mathfrak{R}(K_0(s)) \oplus \mathfrak{R}(K_1(s)) \oplus \dots \oplus \mathfrak{R}(K_{n-1}(s)),$$

and

$$\mathcal{C}''(s) = \mathfrak{R}(I(s)).$$

We associate a sub-bicomplex

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \mathcal{C}(s) & \longleftarrow & \cdots & \longleftarrow & \mathcal{C}'(s) & \longleftarrow & \mathcal{C}''(s) \\
 & & \downarrow & & & \downarrow & \downarrow & & \downarrow \\
 0 & \longleftarrow & \mathcal{C}(s) & \longleftarrow & \cdots & \longleftarrow & \mathcal{C}'(s) & \longleftarrow & \mathcal{C}''(s)
 \end{array}$$

in which each vertical map is 0. The homology groups of the chain complex

$$0 \longleftarrow \mathcal{C}(s) \longleftarrow \cdots \longleftarrow \mathcal{C}''(s) \longleftarrow 0$$

is denoted by $h_j(s)$, we call this complex the *little complex*. When we totalize the associated bicomplex, we get the chain complex

$$0 \longleftarrow \mathcal{C}(s) \longleftarrow \cdots \longleftarrow \mathcal{C}'_{i-1}(s) \oplus \mathcal{C}'_i(s) \longleftarrow \mathcal{C}'_i(s) \oplus \mathcal{C}'_{i+1}(s) \longleftarrow \cdots \longleftarrow \mathcal{C}''(s) \longleftarrow 0.$$

Theorem 1 ([1]). *The homology groups $H_j(s)$ of this complex are given by*

$$\begin{aligned}
 H_0(s) &= h_0(s), & H_n(s) &= h_{n-1}(s) \\
 H_{i+1}(s) &= h_i(s) \oplus h_{i+1}(s), & 0 \leq i &\leq n-2 \\
 H_{ev}(s) &= h_0(s) \oplus h_1(s) \oplus \cdots \oplus h_{n-1}(s) = H_{odd}(s)
 \end{aligned}$$

The even (resp. odd) chamber homology is precisely the total homology of the little complex.

Now if we back to our case, let F be non-archimedean p-adic local field, $G = GL(2, F)$ and $\beta^1GL(2)$ be the enlarged building of G . The enlarged building of G can be defined as

$$\beta^1G = \beta SL(2) \times \mathbb{R}$$

with an action

$$\begin{aligned}
 GL(2) \times \beta^1GL(2) &\longrightarrow \beta^1GL(2) \\
 GL(2) \times \beta SL(2) \times \mathbb{R} &\longrightarrow \beta^1GL(2) \\
 (x, y, t) &\longmapsto (xy, t + val_F(det x)).
 \end{aligned}$$

The enlarged building $\beta^1GL(2)$ has the structure of polysimplicial complex, but we have $\beta^1G = \beta SL(2) \times \mathbb{R}$. The action of $SL(2)$ on its tree could be extended to an action of $GL(2)$. We will investigate the chamber homology $H_j(\beta^1GL(2))$ of G acting on its enlarged building properly. The quotient $\beta^1GL(2)/GL(2)$ is a Mobius band (an identification space of a chamber) which is a compact space.

Let

$$\Pi = \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } s_0 = \begin{pmatrix} 0 & \varpi_F^{-1} \\ \varpi_F & 0 \end{pmatrix}$$

be the standard involutions in $GL(2)$. Restricted to the affine line \mathbb{R} in the enlarged building $\beta^1GL(2) = \beta SL(2) \times \mathbb{R}$, Π sends t to $t + 1$. It is a bit hard to calculate the chamber homology group for $GL(2)$ from a Mobius band but its not that difficult to construct a complex to compute the chamber homology of $GL(2)$ if we restrict to the original quotient space before taking the real line copy which is an edge of the tree of $SL(2)$. Let I, J_0 and J_1 are the stabilizer groups of the edge and the two vertices in the above chamber.

$$J_0 \circ \xrightarrow{I} \bullet J_1$$

2.1. The trivial type $(I, \mathbb{1}_I)$

Let

$$T = \begin{pmatrix} F^\times & 0 \\ 0 & F^\times \end{pmatrix}$$

be the diagonal subgroup of $G = GL(2, F)$ and let $\mathbb{1}$ be the trivial representation of T . Then the pair $(T, \mathbb{1})$ is a cuspidal pair. Let us discuss the special case when $s = [T, \mathbb{1}]_G$, the s -type in this case will be the trivial type $(I, \mathbb{1})$. We will construct the little complex created by $(I, \mathbb{1}_I)$.

Theorem 2. *Let I be the Iwahori subgroup of $GL(2)$, and let St_2 be the Steinberg representation of $GL(2, F)$, and let χ_1, χ_2, χ be unramified unitary characters. Then the unramified unitary representation of $GL(2)$ can be written as follows:*

(i) $Ind_B^G(\chi_1 \times \chi_2) \simeq Ind_B^G(\chi_2 \times \chi_1)$.

(ii) $\chi \otimes St_2$.

Proof: see [18]

We have $Ind_I^{J_0} \mathbb{1}_I = \mathbb{1}_{J_0} \oplus St_2^{J_0}$ and $Ind_I^{J_1} \mathbb{1}_I = \mathbb{1}_{J_1} \oplus St_2^{J_1}$. Then the little complex determined by this type is

$$\begin{array}{ccccc} 0 & \longleftarrow & \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) & \longleftarrow & \mathfrak{R}(I) \\ & & \downarrow & & \downarrow \\ 0 & \longleftarrow & \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) & \longleftarrow & \mathfrak{R}(I) \end{array}$$

where $\mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)$ is the free abelian group on the two elements

$$(\mathbb{1}_{J_0}, \mathbb{1}_{J_1}), (St_2^{J_0}, St_2^{J_1}) \in \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)$$

and $\mathfrak{R}(I)$ is the free abelian group on the single generator

$$\mathbb{1}_I \in \mathfrak{R}(I).$$

The above bicomplex chain implies that we need to consider only invariant elements. Therefore, we need to restrict to the invariant elements so we have the following:

$$\mathbb{1}_{J_0} \oplus St_2^{J_0} \sim 0 \text{ i.e. } \mathbb{1}_{J_0} \sim -St_2^{J_0}$$

$$\mathbb{1}_{J_1} \oplus St_2^{J_1} \sim 0 \text{ i.e. } \mathbb{1}_{J_1} \sim -St_2^{J_1}$$

This means we have one element

$$(\mathbb{1}_{J_0}, \mathbb{1}_{J_1}) \sim -(St_2^{J_0}, St_2^{J_1}).$$

By totalizing the above little complex we get

$$0 \longleftarrow \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1) \longleftarrow \mathfrak{R}(I) \longleftarrow 0 ,$$

hence by Theorem 1 we have

$$H_0 = h_0, \quad H_1 = h_0 + h_1, \quad H_2 = h_1.$$

Now, $h_0 = \mathbb{Z}, h_1 = \mathbb{Z}$, therefore $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^2$ and $H_2 = \mathbb{Z}$. i.e.

$$H_{even} = \mathbb{Z}^2 = H_{odd}.$$

Let λ be a unitary character of $GL(1, F) \simeq F^\times$, and let $\tau = \lambda \circ \det : I \rightarrow \mathcal{U}(\mathbb{1})$. Theorem 9 in [1] shows that the totalised little complex created by the (I, τ) is isomorphic to the totalized little complex created by the trivial type $(I, \mathbb{1}_I)$. Therefore, the homology groups $H_{even} = \mathbb{Z}^2 = H_{odd}$

2.2. The Irreducible Components of Reducible Principal Series

Let $s = [T, \sigma]_G$. Consider the s -type (J, τ) where J is compact open subgroup of G and τ is an irreducible smooth representation of J .

Lemma 1. *Let $\chi = Ind_J^I \tau$. Then χ is irreducible.*

Proof. See [1]

Lemma 2. *We have*

$$Ind_I^{J_0} \chi = \alpha_0 \oplus \gamma_0$$

$$Ind_I^{J_1} \chi = \alpha_1 \oplus \gamma_1.$$

Let $\mathfrak{R}(J_0(\tau)) \oplus \mathfrak{R}(J_1(\tau))$ be the free abelian group on the element $(\alpha_0, \alpha_1) \sim (\gamma_0, \gamma_1) \in \mathfrak{R}(J_0) \oplus \mathfrak{R}(J_1)$ and let $\mathfrak{R}(I(\tau))$ be the free abelian group on the element χ . The little complex is then

$$0 \longleftarrow \mathfrak{R}(J_0(\tau)) \oplus \mathfrak{R}(J_1(\tau)) \longleftarrow \mathfrak{R}(I(\tau)) \longleftarrow 0 .$$

Hence

$$H_0(\tau) = h_0(\tau) = \mathbb{Z}, \quad H_1(\tau) = h_0(\tau) + h_1(\tau) = \mathbb{Z}^2, \quad H_2(\tau) = h_1(\tau) = \mathbb{Z},$$

i.e. $H_{even} = \mathbb{Z}^2 = H_{odd}$.

2.3. Cuspidal Representations

Let $s = [G, \pi]$, where $G = GL(2, F)$ and π is an irreducible cuspidal representation of G . Let (J, σ) be the maximal simple type contained in π [6]. It follows that $J(s) = GL(2, \mathcal{O}_F) = J_0$ [1]. By Lemma 1, it follows that $\lambda = Ind_J^{J_0} \sigma$ is irreducible. Now, the pair (J_0, λ) is an s -type. The restriction of a smooth irreducible representation ρ of G to J_0 contains λ if and only if

$$\rho \cong \pi \otimes \chi \circ \det$$

where χ is an unramified character of F^\times , i.e. π contains λ with multiplicity 1. Therefore, the representation λ is the unique smooth irreducible representation τ of J_0 such that (J_0, τ) is an s -type [17]. The little complex determined by λ is

$$0 \longleftarrow \mathcal{C}(s) \longleftarrow 0$$

where $\mathcal{C}(s)$ is the free abelian group on the invariant 0-cycle τ . The total homology of the little complex is given by $h_0(s) = \mathbb{Z}$. Therefore,

$$H_{even} = \mathbb{Z} = H_{odd}.$$

2.4. The Principal Series

Let (J, τ) be s -type, $J_0 = GL(2, \mathcal{O}_F)$. If $J \subset J_0$ then the only double J -coset representative which G -intertwines τ is 1_G . Therefore,

$$Ind_J^I \tau \text{ is irreducible}$$

$$Ind_J^{J_i} \tau \text{ is irreducible.}$$

Now, let $\gamma = Ind_J^I \tau$ and $\sigma = Ind_J^{J_i} \tau$. Let $\mathcal{C}(\tau)$ be the free abelian group on the generator σ , and let $\mathcal{C}'(\tau)$ be the free abelian group on the generator γ . The totalized little complex is

$$0 \longleftarrow \mathcal{C}(\tau) \longleftarrow \mathcal{C}'(\tau) \longleftarrow 0 .$$

Then

$$H_0 = \mathbb{Z}, \quad H_1 = \mathbb{Z}^2, \quad H_2 = \mathbb{Z} \text{ and so } H_{even} = \mathbb{Z}^2 = H_{odd}.$$

Theorem 3. (i) *The base change of a twist of Steinberg representation is again a twist of Steinberg.*

(ii) *The base change of a principal series representation is always a principal series.*

(iii) *The base change of a cuspidal representation will never be a twist of Steinberg representation. It is possible for the base change of a cuspidal representation to be principal series representation.*

Proof. Let $\mathcal{L}_F = \mathcal{W}_F \times SL(2, \mathbb{C})$ and $\mathcal{L}_E = \mathcal{W}_E \times SL(2, \mathbb{C})$ be the local Langlands groups and let the two L-parameters corresponding to these groups respectively be

$$\phi : \mathcal{W}_F \times SL(2, \mathbb{C}) \longrightarrow G^\vee = GL_2(\mathbb{C}) ,$$

$$\phi|_{\mathcal{W}_E} : \mathcal{W}_E \times SL(2, \mathbb{C}) \longrightarrow G^\vee = GL_2(\mathbb{C}) .$$

- (i) Let $\phi_F = \psi \otimes St_2^F$, where $\psi \in \Psi^t(\mathcal{W}_F)$. Since the base change works by restricting the L-parameter to \mathcal{W}_E and the restriction is apply only on the Weil group part, therefore the base change of St_2^F is $BC(St_2^F) = St_2^E$ and hence the the base change works on the unitary twist of Steinberg as follows:

$$BC(\phi_F) = \phi_E, \quad BC(\psi \otimes St_2^F) = BC(\psi) \otimes St_2^E = \psi \circ N_{E/F} \otimes St_2^E.$$

- (ii) Let $\phi_F = (\psi_1 \otimes \mathbb{1}) \oplus (\psi_2 \otimes \mathbb{1})$ be the L-parameter, where $\psi_1, \psi_2 \in \Psi^t(\mathcal{W}_F)$ then the base change map works on the reducible principal series as follows:

$$BC(\phi_F) = \phi_E$$

$$BC(\psi_1 \otimes \mathbb{1} \oplus \psi_2 \otimes \mathbb{1}) = BC(\psi_1 \otimes \mathbb{1}) \oplus BC(\psi_2 \otimes \mathbb{1}) = (\psi_1 \circ N_{E/F} \otimes \mathbb{1}) \oplus (\psi_2 \circ N_{E/F} \otimes \mathbb{1}).$$

- (iii) Let $\phi_F = \psi \sigma \otimes \mathbb{1}$, where σ is irreducible representation of \mathcal{W}_F and $\psi \in \Psi^t(\mathcal{W}_F)$.

- (a) If the L-parameter ϕ_E remains irreducible after restriction, then this determines a cuspidal representation of $GL(2, E)$. Base change in this case will send one cuspidal representation of $GL(2, F)$ to a cuspidal representation of $GL(2, E)$. Therefore, the map BC works as follows:

$$BC(\phi_F) = \phi_E$$

$$BC(\psi \sigma \otimes \mathbb{1}) = BC(\psi) \sigma^* \otimes \mathbb{1} = \psi \circ N_{E/F} \sigma^* \otimes \mathbb{1}.$$

- (b) If the L-parameter ϕ_E is reducible after restriction, then this representation split into two one-dimensional representations say σ_1 and σ_2 . This means that the restriction of the cuspidal representation is a principal series. Therefore, the map BC works as follows:

$$BC(\phi_F) = \phi_E$$

$$BC(\psi \sigma \otimes \mathbb{1}) = \psi \circ N_{E/F} (\sigma_1 \oplus \sigma_2) \otimes \mathbb{1}.$$

□

In fact, if π is any irreducible admissible representation of $GL(2, F)$ then one can find an extension E/F such that $BC(\pi)$ is either unramified or Steinberg.

3. K-theory for $GL(2)$

Let F be a non-archimedean local field with characteristic 0 and $p \neq 2$. Such a field has a norm, denoted by mod_F [19]. The representations in $GL(2, F)$ can be view as a one of the following:

- (i) The irreducible admissible representations of G fall into three classes: principal series, twists of Steinberg and cuspidal.
- (ii) The unramified representations of G are exactly the principal series representations coming from unramified characters. These are parameterized by (unordered) pairs of complex numbers.

Let E/F be a finite Galois extension, and let the corresponding Weil groups be denoted $\mathcal{W}_E, \mathcal{W}_F$. Let \mathbb{T} denote the circle group

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

and let $\Psi^t(\mathcal{W}_F)$ denote the group of unramified unitary characters of \mathcal{W}_F . Then we have

$$\Psi^t(\mathcal{W}_F) \cong \mathbb{T}, \quad \psi \mapsto \psi(\varpi)$$

where ϖ_F is a uniformizer in F .

Now, let \mathcal{L}_F denote the local Langlands group: $\mathcal{L}_F := \mathcal{W}_F \times SL(2, \mathbb{C})$. A Langlands parameter (or L -parameter) is a continuous homomorphism

$$\phi : \mathcal{L}_F \rightarrow GL(2, \mathbb{C}),$$

($GL(2, \mathbb{C})$ is given the discrete topology) such that $\phi(\Phi_F)$ is semisimple, where Φ_F is a geometric Frobenius in \mathcal{W}_F . Two Langlands parameters are equivalent if they are conjugate under $GL(2, \mathbb{C})$. The set of equivalence classes of Langlands parameters is denoted by $\Phi(GL(2))$. Now the base change is defined by the restriction of L -parameter from \mathcal{L}_F to \mathcal{L}_E . Consider first the single L -parameter $\phi = \rho \otimes \tau(j_1) \oplus \rho \otimes \tau(j_2)$. In this formula, ρ is an irreducible representation of \mathcal{W}_F , $\tau(j)$ is the j -dimensional complex representation of $SL(2, \mathbb{C})$. We define the compact orbit of ϕ as follows:

$$\mathcal{O}^t(\phi) = \left\{ \bigoplus_{r=1}^2 \psi_r \otimes \rho \otimes \tau(j_r) : \psi_r \in \Psi^t(\mathcal{W}_F), \quad 1 \leq r \leq 2 \right\} / \sim,$$

where as before, \sim denotes the equivalence relation of conjugacy in $GL(2, \mathbb{C})$. Each partition $j_1 + j_2 = 2$ determines an orbit. The disjoint union of these orbits, one of each partition of 2, creates a complex affine algebraic variety with finitely many irreducible components. This variety is smooth by [4].

Also, let $\mathcal{G}_2^0(F)$ be the set of equivalence classes of irreducible 2-dimensional smooth (complex) representations of \mathcal{W}_F . Let $\mathcal{A}_2^0(F)$ be the subset of $\mathcal{A}_2^t(F)$ consisting of equivalence

classes of irreducible cuspidal representations of $GL(2, F)$. The local Langlands correspondence gives a bijection,

$$\tau : \mathcal{G}_2^0(F) \longrightarrow \mathcal{A}_2^0(F).$$

We will use the local Langlands correspondence for $GL(2)$ [7, 10, 11, 14]:

$$\pi_F : \Phi(GL(2)) \rightarrow Irr(GL(2)).$$

Lemmas 1.1 and 1.2 in [15] explain the formula of the base change. Now let $z_j = \psi_j(\varpi_F)$, we have the map:

$$\psi_1 \otimes \tau(j_1) \oplus \psi_2 \otimes \tau(j_2) \longmapsto (z_1, z_2).$$

This map gives a bijection

$$\mathfrak{D}^t(\phi) \longrightarrow sym^2(\mathbb{T}).$$

So we will write the L-parameter

$$\phi = \psi_1 \otimes \tau(j_1) \oplus \psi_2 \otimes \tau(j_2)$$

as

$$z_1 \cdot \tau(j_1) \oplus z_2 \cdot \tau(j_2).$$

After base change has been applied, this L-parameter becomes

$$z_1^f \cdot \tau(j_1) \oplus z_2^f \cdot \tau(j_2).$$

The Steinberg representation St_2 has L-parameter $1 \otimes \tau(2)$.

Theorem 4. *Let $\phi = 1 \otimes \tau(2)$ and let $\mathfrak{D}^t(\phi)$ be the compact orbit of ϕ . Then we have*

$$BC : \mathbb{T} \rightarrow \mathbb{T}, \quad z \mapsto z^f.$$

(i) *This map has degree f , and so at the level of the K-theory group K^1 , BC induces the map*

$$\mathbb{Z} \rightarrow \mathbb{Z}, \quad \alpha_1 \rightarrow f \cdot \alpha_1$$

of multiplication by the residue degree f , where α_1 denotes a generator of the group $K^1(\mathbb{T}) \cong \mathbb{Z}$.

(ii) *At the level of the K-theory group K^0 , BC induces the identity map*

$$\mathbb{Z} \rightarrow \mathbb{Z}, \quad \alpha_0 \mapsto \alpha_0,$$

where α_0 denotes a generator of $K^0(\mathbb{T}) \cong \mathbb{Z}$.

Proof. Since this map has degree f then 1 has been proved. Since α_0 is the trivial bundle of rank 1 over \mathbb{T} then 2 has been showed.

$$\begin{array}{ccc} \mathbb{1} \otimes \tau(2) & \longmapsto & St_2^F \\ \downarrow & & \downarrow \\ \mathbb{1} \otimes \tau(2) & \longmapsto & St_2^E \end{array}$$

□

We can generalize this in the following form:

$$\begin{array}{ccc} \sigma \otimes \tau(2) & \longmapsto & St_2^F \\ \downarrow & & \downarrow \\ \sigma \otimes \tau(2) & \longmapsto & St_2^E \end{array}$$

Next we define the L -parameter ϕ to be: $\phi = \rho \otimes 1 \oplus \rho \otimes 1$ where ρ is a unitary character of \mathcal{W}_F . The unitary characters of \mathcal{W}_F factor through F^\times and we have $F^\times \cong \langle \varpi_F \rangle \times \mathcal{U}_F$. We will take ρ to be trivial on $\langle \varpi_F \rangle$, and then regard ρ as a unitary character of \mathcal{U}_F . The group \mathcal{U}_F admits countably many such characters ρ . In this case the compact orbit is symmetric square of the circle \mathbb{T} :

$$\mathcal{O}^t(\phi) \cong \mathcal{O}^t(BC(\phi)) \cong Sym^2(\mathbb{T}) := \mathbb{T}^2 / \mathbb{Z}/2\mathbb{Z} = \mathbb{T}^2 / \mathbb{W}.$$

Lemma 3. *The symmetric square $\mathbb{T}^2 / \mathbb{W}$ has the homotopy type of a circle*

$$\mathbb{T}^2 / \mathbb{W} \sim \mathbb{T}$$

$$(z_1, z_2) \longmapsto z_1 z_2.$$

Proof. By sending the pair $z = (z_1, z_2)$ to a unique monic polynomial

$$(z_1, z_2) \longmapsto z^2 + a_1 z + a_0, \quad a_0 \neq 0$$

with roots z_1, z_2 . It follows that

$$Sym^2(\mathbb{T}) \cong \{z^2 + a_1 z + a_0 : a_0 \neq 0\} \sim_h \mathbb{T},$$

since the space of coefficients a_1, a_0 is contractible. Therefore,

$$Sym^2(\mathbb{T}) \sim_h \mathbb{T}$$

using the map $(z_1, z_2) \mapsto z_1 \cdot z_2$.

□

Let π_F be the local Langlands correspondence

$$\pi_F : \Phi(GL(2)) \rightarrow IrrGL(2)$$

and let $x = \text{diag}(t_1, t_2)$ be a diagonal element in the standard maximal torus T of $GL(2)$. Then

$$\chi : x \mapsto \pi_F(\rho)(t_1, t_2)$$

is a unitary character of T . Let σ be an unramified unitary character of T , and form the induced representation $Ind_{TU}^G(\sigma \otimes \chi)$ which is an irreducible unitary representation of G . Let σ vary over all unramified unitary characters of T , then we obtain a subset of the unitary dual of G . This subset has the structure of a symmetric square of \mathbb{T} . The consequence for \mathcal{U}_F admits countably many unitary characters is the unitary dual of G contains countably many subspaces (in the Fell topology) each with the structure $Sym^2(\mathbb{T})$. We are concerned with the effect of base change E/F on each of these compact spaces.

Theorem 5. *Let \mathbb{T}^2/\mathbb{W} denote one of the compact subspaces of the unitary principal series of $GL(2)$. Then we have*

$$BC : \mathbb{T}^2/\mathbb{W} \rightarrow \mathbb{T}^2/\mathbb{W}, \quad (z_1, z_2) \mapsto (z_1^f, z_2^f)$$

(i) *At the level of the K-theory group K^1 , BC induces the map*

$$\mathbb{Z} \rightarrow \mathbb{Z}, \quad \alpha_1 \mapsto f \cdot \alpha_1$$

of multiplication by f , where f is the residue degree and α_1 denotes a generator of $K^1(\mathbb{T}) = \mathbb{Z}$.

(ii) *At the level of the K-theory group K^0 , BC induces the identity map*

$$\mathbb{Z} \rightarrow \mathbb{Z}, \quad \alpha_0 \mapsto \alpha_0,$$

where α_0 denotes a generator of $K^0(\mathbb{T}) = \mathbb{Z}$.

Proof. From Lemma 3 we have this commutative diagram:

$$\begin{array}{ccc} Sym^2(\mathbb{T}) & \xrightarrow{h} & \mathbb{T} \\ BC \downarrow & & \downarrow BC^* \\ Sym^2(\mathbb{T}) & \xrightarrow{h} & \mathbb{T} \end{array}$$

where $BC(z_1, z_2) = (z_1^f, z_2^f)$, $BC^*(z) = z^f$ and $h(z_1, z_2) = z_1 \cdot z_2$. Since

$$(z_1 \cdot z_2)^f = z_1^f \cdot z_2^f$$

we have $K^j(BC) = K^j(BC^*)$, but BC^* is a map of degree f . Therefore,

$$K^1(BC)(\alpha_1) = f \cdot \alpha_1 \text{ and } K^0(BC)(\alpha_0) = \alpha_0$$

where α_1 is a generator of $K^1(\mathbb{T}) = \mathbb{Z}$ and α_0 is a generator of $K^0(\mathbb{T}) = \mathbb{Z}$.

Therefore, the K-theory for the trivial type $(I, \mathbb{1}_I)$ would be as follows:

Theorem 6.

$$K_j C_r^*(s) = K^j(\text{Sym}^2(\mathbb{T}) \bigsqcup \mathbb{T}) \cong \mathbb{Z}^2$$

Proof. Proof immediately follows from Theorems 4 and 5 □.

Definition 3. Let E/F be a quadratic extension and let χ be a character of E^\times . The pair $\vartheta = (E/F, \chi)$ is called admissible if

- (i) χ does not factor through the norm map $N_{E/F} : E^\times \rightarrow F^\times$ and,
- (ii) if $\chi \mid \mathcal{W}_E^1$ does factor through $N_{E/F}$, then E/F is unramified.

Let $\mathcal{P}_2(F)$ be the set of isomorphism classes of admissible pairs ϑ . The map

$$\mathcal{P}_2(F) \rightarrow \mathcal{G}_2^0(F), \quad \vartheta \mapsto \text{Ind}_{E/F} \chi$$

is bijection according to [5, p. 215], where χ is a character of \mathcal{W}_E via the class field theory isomorphism $\mathcal{W}_E^{ab} \cong E^\times$ and $\text{Ind}_{E/F}$ is the functor of induction from representations of \mathcal{W}_E to representations of \mathcal{W}_F .

The tempered dual of $GL(2)$ consists of the cuspidal representations with unitary central character, the unitary twists of the Steinberg representation, and the unitary principal series. It is clear that in the admissible pairs we can describe what is happening so we further restrict ourselves to admissible pairs ϑ for which E/F is totally ramified and χ is a unitary character. This ensures that $\pi := \text{Ind}_{E/F} \chi$ is unitary. Therefore $\det(\pi)$ is unitary and $\tau(\pi)$ has unitary central character. The cuspidal representations of $GL(2)$ with unitary central character arrange themselves in the tempered dual as a countable union of circles. For each circle \mathbb{T} , we select an admissible pair ϑ for which $\tau(\pi) \in \mathbb{T}$ and label this circle as \mathbb{T}_ϑ .

Theorem 7. Let E'/F be an unramified extension of odd degree. Then we have:

- (i) Base change is a proper map.
- (ii) When we restrict base change to one circle we get the following:

$$BC : \mathbb{T}_\vartheta \rightarrow \mathbb{T}_{(E'/E, \chi_{E'})}, \quad z \mapsto z^{f(E'/F)}$$

with

$$\chi_{E'} = \chi \circ N_{E'/E}.$$

Proof. Since we are considering circles indexed by characters of $\widehat{\mathcal{U}_F}$, then the base change maps each circle into one precise circle.

Let \mathbb{D} be a compact subset of $\mathbb{T}_{\chi_{E'}}$ which is a closed arc in $\mathbb{T}_{\chi_{E'}}$. Then we may write

$$\mathbb{D} = \{e^{i\theta} \in \mathbb{T}_{\chi_{E'}} : \theta_0 \leq \theta \leq \theta_1, \quad \theta \in [0, 2\pi]\},$$

and we have the pre-image of this arc

$$BC^{-1}(\mathbb{D}) = \{e^{i\theta} \in \mathbb{T}_{\chi_F} : \theta_0/f \leq \theta \leq \theta_1/f, \theta \in [0, 2\pi]\}$$

which is closed arc in \mathbb{T}_{χ_F} . It follows that $BC^{-1}(\mathbb{D})$ is compact. Therefore, the base change map BC is a proper map and then (1) has been proved. Now, let $\rho \in \mathcal{G}_2^0(F)$, then the order of the cyclic group of all unramified characters χ such that $\chi\rho \simeq \rho$ is called a torsion number of ρ and denotes by $\nu(\rho)$. Put

$$\sigma = \text{Ind}_{E/F} \chi, \quad \pi = \tau(\sigma) \text{ and } \sigma_{E'} = \text{Ind}_{EE'/E'} \chi_{E'} = \sigma|_{\mathcal{W}_{E'}}.$$

The proof of Theorem 3.3 in [7] shows that the representation σ is totally ramified, in the sense that $\nu(\sigma) = 1$. Theorem 4.6 in the same reference shows that the pair $(EE'/E', \chi_{E'})$ is admissible. Also, we have the map

$$\tau(\sigma_{E'}) = BC_{E'/F} \pi.$$

By Proposition 7.2 in [16], EE'/E' is unramified, whenever the extension E'/F is unramified and

$$e_{EE'/F} = e_{EE'/E'} \times e_{E'/F} = e_{EE'/E} \times e_{E/F}.$$

and it follows that

$$e_{EE'/E'} = e_{E/F} = 2.$$

Since EE'/E' is quadratic extension, EE'/E' is totally ramified. Therefore $\sigma_{E'}$ is totally ramified, in another words $\nu(\sigma_{E'}) = 1$. Therefore, the base change maps each circle to another circle and its given by

$$z \longmapsto z^{f(E'/F)}.$$

□

If the extension E'/F is a finite unramified Galois extension, then the cuspidal part of the tempered dual of $GL(2)$ is a countable disjoint union of circles and has the structure of a locally compact Hausdorff space. The base change map

$$BC : \bigsqcup \mathbb{T}_{\vartheta} \rightarrow \bigsqcup \mathbb{T}_{\zeta}$$

is a proper map, where ϑ an admissible pair, E/F totally ramified, χ unitary and $\zeta = (EE'/E', \eta)$. Therefore, there is a functorial map at the level of K -theory groups

$$K^j(BC) : \bigoplus \mathbb{Z}_{\zeta} \rightarrow \bigoplus \mathbb{Z}_{\vartheta}.$$

Each K -group is a countably generated free abelian group:

$$K^j(\bigsqcup \mathbb{T}_{\vartheta}) \cong \bigoplus \mathbb{Z}_{\vartheta}, \quad K^j(\bigsqcup \mathbb{T}_{\zeta}) \cong \bigoplus \mathbb{Z}_{\zeta},$$

where \mathbb{Z}_{ϑ} and \mathbb{Z}_{ζ} denote a copy of \mathbb{Z} , $j = 0, 1$. The base change map selects among the admissible pairs ζ those of the form $(EE'/E', \chi_{E'})$, where

$$\chi_{E'} = \chi \circ N_{EE'/E}.$$

Theorem 8. When we restrict $K^1(BC)$ to the direct summand $\mathbb{Z}_{(EE'/E', \chi_{E'})}$ we get the following map:

$$\begin{aligned} \mathbb{Z}_{(EE'/E', \chi_{E'})} &\longrightarrow \mathbb{Z}_{\partial}, \\ x &\longmapsto f(E'/F) \cdot x. \end{aligned}$$

On the remaining direct summands, $K^1(BC) = 0$. When we restrict $K^0(BC)$ to the direct summand $\mathbb{Z}_{(EE'/E', \chi_{E'})}$ we get the following map:

$$\begin{aligned} \mathbb{Z}_{(EE'/E', \chi_{E'})} &\longrightarrow \mathbb{Z}_{\partial}, \\ x &\longmapsto x. \end{aligned}$$

On the remaining direct summands, $K^0(BC) = 0$.

Here's a summary of the cases in this work:

(i) On the admissible side, if we have the following:

- (a) The twist of Steinberg : $\{\psi \otimes St(2) : \psi \in \Psi^t(\mathcal{W}_F)\} \cong \mathbb{T}$.
- (b) The cuspidal : $\{\psi \otimes \pi : \pi \in \mathcal{A}^0 GL(2)\} \cong \mathbb{T}$.
- (c) U.PS : $\{Ind_B^G \begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto \psi_1 \cdot \psi_2 : \psi_j \in \Psi^t(\mathcal{W}_F)\} \cong \mathbb{T}^2$, when $\psi_1 \neq \psi_2$.
- (d) U.PS : $\{Ind_B^G \begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto \psi_1 \cdot \psi_2 : \psi_j \in \Psi^t(\mathcal{W}_F)\} \cong \mathbb{T}^2 / (\mathbb{Z}/2\mathbb{Z})$, when $\psi_1 = \psi_2$

(ii) Then, the K -theory groups of each of these cases are as follows:

- (a) $K_j C_0(\mathbb{T}) = \mathbb{Z}$.
- (b) $K_j C_0(\mathbb{T}) = \mathbb{Z}$.
- (c) $K_j C_0(\mathbb{T}^2) = K^j(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$.
- (d) $K_j C_0(\mathbb{T}^2 / (\mathbb{Z}/2\mathbb{Z})) = K_j C_0(\mathbb{T}) = K^j(\mathbb{T}) \cong \mathbb{Z}$.

(iii) The homology groups of these cases are as follows:

- (a) $H_{even} = \mathbb{Z}^2 = H_{odd}$.
- (b) $H_{even} = \mathbb{Z} = H_{odd}$.
- (c) $H_{even} = \mathbb{Z}^2 = H_{odd}$.

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