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Second-Order Duality for Variational Problems

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Abstract. A Mond-Weir type second-order dual to a variational problem is constructed and the notion of second-order invexity and second order generalized invexity are introduced in variational problems. Under these second-order pseudoinvexity and second-order quasi-invexity assumptions, weak, strong and converse duality results are established. It is pointed out that our duality results can be viewed as dynamic generalizations of corresponding (static) duality results in nonlinear programming.

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1. Introduction

The calculus of variation has been one of the prominent branches of analysis, for more than two centuries. It is a tool of great power that can be used to wide variety of problems, in pure mathematics. It can also be used to express basic principles of mathematical physics in forms of utmost simplicity and elegance. Hanson [6] pointed out that some of the duality results in the mathematical programming have the analogues in calculus of variations. Exploring this relationship between mathematical programming and classical calculus of variation, Mond and Hanson [8] formulated a constrained variational problem as mathematical programming problem in abstract space and using Valentine [10] optimality conditions for the same, presented its Wolfe dual variational problem for validating various duality results under usual convexity. Later Bector, Chandra and Husain [2] studied Mond-Weir type duality for the problem of Mond and Hanson [8] for relaxing its convexity requirements. In [3] Chandra, Craven and Husain studied optimality and duality for a class of nondifferentiable variational problems in which the integrand of the objective functional contains a term of a square root of the quadratic form, while in [5], Husain and Jabeen studied optimality criteria and duality for variational problems in which integrand of objective and constraint functions contains terms of support functions.

Second-order duality in mathematical programming has been extensively studied in recent years. Mangasarian [7] was the first to identify a second-order dual formulation for non-linear programs under the assumptions that are complicated and somewhat difficult to verify. Mond [9] introduced the concept of second-order convex functions (named as bonvex functions by Bector and Chandra [1]) and studied second-order duality for nonlinear programs.

Recently Chen [4] formulated Wolfe type second-order dual problem to the orthodox variational problem and studied usual duality results under invexity assumptions on the functions that occur in the formulation of the problem along with some strange assumptions. Mond [9] has pointed out that the second-order dual for a nonlinear programming gives a tighter bound and has computational advantage over a first order dual. Motivated with this of Mond [9] in this exposition, we construct Mond-Weir type second-order dual to the variational problem and derive usual duality results under second-order pseudo-invexity and second order quasi-invexity assumptions.

The relationship of our results to second-order duality results in nonlinear programming reported in [1] is indicated. In essence it is shown that our duality results can be viewed as dynamic generalizations of corresponding (static) duality theorems of nonlinear programming already in the literature.

2. Definitions and Related Pre-requisites

Let $I = [a, b]$ be a real interval, $f : I \times R^n \times R^n \to R$ and $g : I \times R^n \times R^n \to R^m$ be twice continuously differentiable functions. In order to consider $f(t, x(t), \dot{x}(t))$, where $x : I \to R^n$ is differentiable with derivative $\dot{x}$, denoted by f_x and $f_{\dot{x}}$ the partial derivative of f with respect to x and $\dot{x}$, respectively, that is,

$$f_x = \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \frac{\partial f}{\partial x^2} \\ \vdots \\ \frac{\partial f}{\partial x^n} \end{pmatrix}, \quad f_{\dot{x}} = \begin{pmatrix} \frac{\partial f}{\partial \dot{x}^1} \\ \frac{\partial f}{\partial \dot{x}^2} \\ \vdots \\ \frac{\partial f}{\partial \dot{x}^n} \end{pmatrix}$$

denote by f_{xx} the Hessian matrix of f with respect to x, that is,

$$f_{xx} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \partial x^1} & \frac{\partial^2 f}{\partial x^1 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^1 \partial x^n} \\ \frac{\partial^2 f}{\partial x^2 \partial x^1} & \frac{\partial^2 f}{\partial x^2 \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^2 \partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x^n \partial x^1} & \frac{\partial^2 f}{\partial x^n \partial x^2} & \cdots & \frac{\partial^2 f}{\partial x^n \partial x^n} \end{pmatrix}_{n\times n}$$

It is obvious that f_{xx} is a symmetric $n \times n$ matrix. Denote by g_x the $m \times n$ matrix with respect to x, that is,

$$g_x = \begin{pmatrix} \frac{\partial g_1}{\partial x^1} & \frac{\partial g_1}{\partial x^2} & \cdots & \frac{\partial g_1}{\partial x^n} \\ \frac{\partial g_2}{\partial x^1} & \frac{\partial g_2}{\partial x^2} & \cdots & \frac{\partial g_2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x^1} & \frac{\partial g_m}{\partial x^2} & \cdots & \frac{\partial g_m}{\partial x^n} \end{pmatrix}_{m\times n}$$

Similarly $f_{\dot{x}}, f_{\dot{x}x}, f_{x\dot{x}}$ and $g_{\dot{x}}$ can be defined.

Denote by X, the space of piecewise smooth functions $x : I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty + \|D^2 x\|_\infty$, where the differentiation operator D is given by

$$u = D\,x \Leftrightarrow x(t) = \alpha + \int_a^t u(s)\,ds,$$

where α is given boundary value; thus $\dfrac{d}{dt} = D$ except at discontinuities.

We introduce the following definitions which are needed for the duality results to hold.

Definition 2.1 (Second-order Invexity). *If there exists a vector function $\eta = \eta(t, x, \bar{x}) \in R^n$ where $\eta : I \times R^n \times R^n \to R^n$ and with $\eta = 0$ at $t = a$ and $t = b$, such that for the* *functional* $\int_I \phi(t,x,\dot{x})dt$ *where* $\phi : I \times R^n \times R^n \to R$ *satisfies*

$$\int_I \phi(t,x,\dot{x})dt - \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\} dt$$

$$\geq \int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\} dt,$$

then $\int_I \phi(t,x,\dot{x})dt$ *is second-order invex with respect to* η *where* $G = \phi_{xx} - D\phi_{x\dot{x}} + D^2\phi_{\dot{x}\dot{x}}$ *and* $\beta \in C(I,R^n)$*, the space of continuous n-dimensional vector function. The function* β *is analogous to the auxiliary vector* p *in [1].*

Definition 2.2 (Second-order Pseudoinvex)**.** *If the functional* $\int_I \phi(t,x,\dot{x})dt$ *satisfies*

$$\int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\} dt \geq 0$$

$$\Rightarrow \quad \int_I \phi(t,x,\dot{x})dt \geq \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\}dt,$$

then $\int_I \phi(t,x,\dot{x})dt$ *is said to be second-order pseudoinvex with respect to* η*.*

Definition 2.3 (Second-order Quasi-invex)**.** *If the functional* $\int_I \phi(t,x,\dot{x})dt$ *satisfies*

$$\int_I \phi(t,x,\dot{x})dt \leq \int_I \left\{\phi(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T G\beta(t)\right\}dt$$

$$\Rightarrow \quad \int_I \left\{\eta^T \phi_x + (D\eta)^T \phi_{\dot{x}} + \eta^T G\beta(t)\right\}dt \leq 0,$$

then $\int_I \phi(t,x,\dot{x})$ *is said to be second-order quasi-invex with respect to* η*.*

If ϕ does not depend on t, then the above definitions reduce to those given in [1] for static cases.

Consider the following constrained variational problem:

$$\text{(VP)}: \qquad \text{Minimize} \int_I f(t,x,\dot{x})dt$$

Subject to

$$x(a) = 0 = x(b),$$

$$g(t,x,\dot{x}) \leq 0, \quad t \in I,$$

$$h(t,x,\dot{x}) = 0, \quad t \in I,$$

where $f : I \times R^n \times R^n \to R$, $g : I \times R^n \times R^n \to R^m$ and $h : I \times R^n \times R^n \to R^k$ are continuously differentiable.

Proposition 2.1 ([3] (Fritz-John Conditions)). *If (CP) attains a local (or) global minimum at $x = \bar{x} \in X$ then there exist Lagrange multiplier $\tau \in R, z : I \to R^k$ and piecewise smooth $y : I \to R^m$ such that*

$$\tau f_x(t,\bar{x},\dot{\bar{x}}) + y(t)^T g_x(t,\bar{x},\dot{\bar{x}}) + z(t)^T h_x(t,\bar{x},\dot{\bar{x}})$$

$$-D[f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + y(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + z(t)^T h_{\dot{x}}(t,\bar{x},\dot{\bar{x}})] = 0,, \quad t \in I,$$

$$y(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0, \quad t \in I$$

$$(\tau, y(t)) \geq 0, \quad t \in I$$

$$(\tau, y(t), z(t)) \neq 0, \quad t \in I$$

The Fritz John necessary conditions for (CP), become the Karush-Kuhn-Tucker conditions [7] if $\tau = 1$. If $\tau = 1$, the solution $\bar{x}$ is said to be normal.

3. Second-Order Duality

Consider the following variational problem (CP) by ignoring the equality constraint of (VP):

$$\text{(CP)}: \quad \text{Minimize} \int_I f(t,x,\dot{x})dt$$

Subject to

$$x(a) = 0 = x(b), \tag{3.1}$$

$$g(t, x, \dot{x}) \leq 0, \qquad t \in I, \tag{3.2}$$

Chen [4] presented the following Wolfe type second-order dual problem for (CP) analogous to that for nonlinear programming by Mangasarian [7] and established various duality results under somewhat strange invexity-like conditions.

$$\text{Maximize} : \int_a^b \Big\{ f(t, u(t), \dot{u}(t)) + \alpha(t)^T g(t, u(t), \dot{u}(t))$$

$$\frac{1}{2}\beta(t)^T [f_{uu}(t, u(t), \dot{u}(t)) + (g_u(t, u(t), \dot{u}(t))^T \alpha(t))_u$$

$$-2D(f_{u\dot{u}}(t, u(t), \dot{u}(t)) + (g_u(t, u(t), \dot{u}(t))^T \alpha(t))_{\dot{u}})$$

$$+D^2(f_{\dot{u}\dot{u}}(t, u(t), \dot{u}(t)) + (g_{\dot{u}}(t, u(t), \dot{u}(t))^T \alpha(t))_{\dot{u}})]\beta(t)\Big\} dt$$

Subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b)$$

$$f_u(t, u(t), \dot{u}(t)) + g_u(t, u(t), \dot{u}(t))^T \alpha(t)$$

$$-D[f_{\dot{u}}(t, u(t), \dot{u}(t)) + g_{\dot{u}}(t, u(t), \dot{u}(t))^T \alpha(t)]$$

$$+[f_{uu}(t, u(t), \dot{u}(t)) + (g_u(t, u(t), \dot{u}(t))^T \alpha(t))_u$$

$$-2D(f_{u\dot{u}}(t, u(t), \dot{u}(t)) + (g_u(t, u(t), \dot{u}(t))^T \alpha(t))_{\dot{u}})$$

$$+D^2(f_{\dot{u}\dot{u}}(t, u(t), \dot{u}(t)) + (g_{\dot{u}}(t, u(t), \dot{u}(t))^T \alpha(t))_{\dot{u}})]\beta(t) = 0,$$

$$t \in I,$$

$$\alpha(t) \in R_+^m, \quad \beta(t) \in R^n, \quad t \in I$$

where R_+^m designates the non-negative orthant of the Euclidean space R^n.
Let

$$\begin{aligned} H \;=\; & f_{uu}(t,u(t),\dot{u}(t)) + (g_u(t,u(t),\dot{u}(t))^T\alpha(t))_u \\ & -2D(f_{u\dot{u}}(t,u(t),\dot{u}(t)) + (g_u(t,u(t),\dot{u}(t))^T\alpha(t))_{\dot{u}}) \\ & +D^2(f_{\dot{u}\dot{u}}(t,u(t),\dot{u}(t)) + (g_{\dot{u}}(t,u(t),\dot{u}(t))^T\alpha(t))_{\dot{u}}). \end{aligned}$$

Then the above dual problem can be expressed in a much simpler form which is given below.

$$\text{(VD)} \qquad \text{Maximize} : \int_a^b \{f(t,u(t),\dot{u}(t)) + \alpha(t)^T g(t,u(t),\dot{u}(t)) - \frac{1}{2}\beta(t)^T H(t,u(t),\dot{u}(t),\alpha(t),\beta(t))\}dt$$

Subject to

$$u(a) = 0 = u(b), \quad \dot{u}(a) = 0 = \dot{u}(b)$$

$$\begin{aligned} & f_u(t,u(t),\dot{u}(t)) + g_u(t,u(t),\dot{u}(t))^T\alpha(t) \\ & \quad -D[f_{\dot{u}}(t,u(t),\dot{u}(t)) + g_{\dot{u}}(t,u(t),\dot{u}(t))^T\alpha(t)] \\ & \quad +H(t,u(t),\dot{u}(t))\alpha(t)\beta(t) = 0, \quad t \in I \end{aligned}$$

$$\alpha(t) \in R_+^m, \quad \beta(t) \in R^n, \quad t \in I$$

It is remarked here that if f and g are independent of t, then (VD) becomes second-order dual problem studied by Mangasarian [7].

Now we present the following Mond-Weir type second-order dual (CD) in the spirit of [1] to relax second-order invexity requirements and establish various duality results between the problems (CP) and (CD) under generalized second-order invexity hypothesis.

(CD): Maximize $\int_I \{f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F\beta(t)\}dt$

Subject to

$$u(a) = 0 = u(b) \tag{3.3}$$

$$f_u + y(t)^T g_u - D(f_{\dot{u}} + y(t)^T g_{\dot{u}}) + (F+H)\beta(t) = 0,\ t \in I \tag{3.4}$$

$$\int_I \left\{ y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t) \right\} dt \geq 0, \tag{3.5}$$

$$y(t) \geq 0, \quad t \in I \tag{3.6}$$

where $F = f_{uu} - Df_{u\dot{u}} + D^2 f_{\dot{u}\dot{u}}$ and $H = (y(t)^T g_u)_u - D(y(t)^T g_u)_{\dot{u}} + D^2(y(t)^T g_{\dot{u}})_{\dot{u}}$ and define $D = \dfrac{d}{dt}$ as defined earlier.

If f and g are independent of t then $F = f_{uu}$ and $H = (y^T g_u)_u$ and consequently (CD) will reduce to the second-order dual problem introduced in [1].

Theorem 3.1 (Weak duality). *Let $x(t) \in X$ be a feasible solution of (CP) and $(u(t), y(t), \beta(t))$ be feasible solution of (CD).If $\int_I f(t,.,.)dt$ be second-order pseudoinvex and $\int_I y(t)^T g(t,.,.)dt$ be second-order quasi-invex with respect to the same η : $I \times R^n \times R^n \to R^n$ satisfying $\eta = 0$ at $t = a$ and $t = b$, then*

$$\int_I f(t,x,\dot{x})dt \geq \int_I \left\{ f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F\beta(t) \right\} dt$$

Proof. The relations, $g(t,x,\dot{x}) \leq 0$, $y(t) \geq 0$, $t \in I$ and (3.5) imply

$$\int_I y(t)^T g(t,x,\dot{x})dt \leq \int_I \left\{ y(t)^T g(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T H\beta(t) \right\} dt,$$

This, because of second-order quasi-invexity of $\int_I y(t)^T g(t,.,.)dt$, implies that

$$\int_I \{\eta^T (y(t)^T g_u) + (D\eta)^T (y(t)^T g_{\dot{u}}) + \eta^T H\beta(t)\} dt \leq 0$$

i.e.,

$$\int_I \eta^T(y(t)^T g_u)dt + \int_I (D\eta)^T(y(t)^T g_{\dot{u}})dt + \int_I \eta^T H\beta(t)dt \leq 0$$

This, by integration by parts, this inequality yields,

$$\int_I \eta^T(y(t)^T g_u)dt + \eta y(t)^T g_{\dot{u}}|_a^b - \int_I \eta^T D(y(t)^T g_{\dot{u}})dt + \int_I \eta^T H\beta(t)dt \leq 0$$

Using $\eta = 0$ at $t = a$ and $t = b$ in the above inequality, we obtain,

$$\int_I \eta[(y(t)^T g_u) - D(y(t)^T g_{\dot{u}}) + \eta^T H\beta(t)]dt \leq 0,$$

Using (3.4), this gives

$$\int_I [\eta^T(f_u - Df_{\dot{u}}) + \eta^T F\beta(t)]dt \geq 0.$$

Integrating by parts, gives

$$\int_I (\eta^T f_u + (D\eta)^T f_{\dot{u}} + \eta^T F\beta(t)dt) \geq 0.$$

This, in view of second-order pseudoinvexity of $\int_I f(t,.,.)dt$ implies

$$\int_I f(t,x,\dot{x})dt \geq \int_I \left\{ f(t,u,\dot{u}) - \frac{1}{2}\beta(t)^T F\beta(t) \right\} dt.$$

This implies,

infimum(CP) $\geq$ supremum(CD).

Theorem 3.2 (Strong Duality). *If $\bar{x}(t) \in X$ is an optimal solution of (CP) and meets the normality conditions, then there exists a piece wise smooth $\bar{y} : R \to R^m$ such that $(\bar{x}(t), \bar{y}(t), \beta(t) = 0)$ is a feasible for (CD) and the two objective values are equal. Furthermore, if the hypothesis of Theorem 1 holds, then $(\bar{x}(t), \bar{y}(t), \beta(t))$ is an optimal solution for (CD).*

Proof. From Proposition 1, there exists a piece wise smooth function $\bar{y} : R \to R^m$ satisfying the following conditions:

$$(f_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}})) - D(f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})) = 0, \quad t \in I$$

i.e,

$$(f_x(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_x(t,\bar{x},\dot{\bar{x}}))$$
$$- D(f_{\dot{x}}(t,\bar{x},\dot{\bar{x}}) + \bar{y}(t)^T g_{\dot{x}}(t,\bar{x},\dot{\bar{x}})) + (F+H)\beta(t) = 0, \tag{3.7}$$

where

$$\beta(t) = 0, \quad t \in I$$

$$\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) = 0$$

i.e.,

$$\int_I \{\bar{y}(t)^T g(t,\bar{x},\dot{\bar{x}}) - \frac{1}{2}\beta(t)^T H\beta(t)\}dt = 0, \quad \text{where } \beta(t) = 0,\ t \in I \tag{3.8}$$

$$\bar{y}(t) \geq 0, \quad t \in I \tag{3.9}$$

From (3.7), (3.8) and (3.9), it implies that $(\bar{x}(t), \bar{y}(t), \beta(t) = 0)$ is feasible for (CD) and the objective value of (CP) and (CD) are equal. The optimality of $(\bar{x}(t), \bar{y}(t), \beta(t))$ follows by an application of Theorem 1.

Theorem 3.3 (Converse duality)**.** *Suppose that f and g are thrice continuously differentiable. Let $(\bar{x}(t), \bar{y}(t), \beta(t))$ be an optimal solution of (CD) at which*

(A_1): *the Hessian matrices F and H are not the multiple of each other.*

(A_2): *$y(t)^T g_x - Dy(t)^T g_{\dot{x}} \neq 0$,*

(A_3): *(i) $\int_I \beta(t)^T (y(t)^T g_x - Dy(t)^T g_{\dot{x}})dt \geq 0$ and $\int_I \beta(t)^T H\beta(t)dt > 0$*

or

(ii) $\int_I \beta(t)^T(y(t)^T g_x - Dy(t)^T g_{\dot{x}})dt \leq 0$ *and* $\int_I \beta(t)^T H\beta(t)dt < 0$

If, for all feasible $(x(t), y(t), \beta(t))$, $\int_I f(t,.,.)dt$ *be second order pseudoinvex and* $\int_I y(t)^T g(t,.,.)dt$ *be second-order quasi-invex with respect to the same* η*, then* $\bar{x}(t)$ *is an optimal solution of* (P)*.*

Proof. Since $(\bar{x}(t), \bar{y}(t), \beta(t))$ is an optimal solution for (CD), by proposition 1, there exist Lagrange multiplier $\alpha \in R$, and piece wise smooth $\lambda : I \to R^n$, $\gamma \in R$ and $\mu : I \to R^m$ such that Fritz-John conditions hold at $(\bar{x}(t), \bar{y}(t), \beta(t))$:

$$
\begin{aligned}
&-\alpha\left[\left(f_x - \frac{1}{2}(\beta(t)^T F\beta(t))_x\right) - D\left(f_{\dot{x}} - \frac{1}{2}(\beta(t)^T F\beta(t))_{\dot{x}}\right)\right] \\
&+\lambda(t)^T\Big\{f_{xx} + (y(t)^T g_x)_x - D(f_{\dot{x}x} + (y(t)^T g_{\dot{x}})_x) + ((F+H)\beta(t))_x \\
&-D(f_{x\dot{x}} + (y(t)^T g_x)_{\dot{x}}) - (f_{\dot{x}\dot{x}} + (y(t)^T g_{\dot{x}})_{\dot{x}}) + ((F+H)\beta(t))_{\dot{x}}\Big\} \\
&\gamma\Big\{y(t)^T g_x - \frac{1}{2}(\beta(t)^T F\beta(t))_x \\
&\qquad - D(y(t)^T g_{\dot{x}} - \frac{1}{2}(\beta(t)^T F\beta(t))_{\dot{x}})\Big\} = 0, \quad t \in I,
\end{aligned} \tag{3.10}
$$

$$(\lambda(t) + \alpha\beta(t))F + (\lambda(t) + \gamma\beta(t))H = 0, \quad t \in I \tag{3.11}$$

$$
\begin{aligned}
&\lambda(t)^T\left[g_{jx} - Dg_{j\dot{x}} + (g_{jxx} - Dg_{jx\dot{x}} + D^2 g_{\dot{x}\dot{x}})\beta(t)\right] \\
&\quad + \gamma\left[g_j + \frac{1}{2}\beta(t)^T(g_{jxx} - Dg_{jx\dot{x}} + D^2 g_{\dot{x}\dot{x}})\beta(t)\right] + \mu_j(t) = 0, \quad t \in I
\end{aligned} \tag{3.12}
$$

$$(f_x + y(t)^T g_{\dot{x}}) - D(f_{\dot{x}} + y(t)^T g_{\dot{x}}) + (F+H)\beta(t) = 0, \quad t \in I \tag{3.13}$$

$$\gamma\int_I\left\{y(t)^T g - \frac{1}{2}\beta(t)H\beta(t)\right\}dt = 0, \quad t \in I \tag{3.14}$$

$$\mu^T(t)\bar{y}(t) = 0, \quad t \in I \tag{3.15}$$

$$(\alpha, \gamma, \mu(t)) \geq 0, \quad t \in I \tag{3.16}$$

$$(\alpha, \gamma, \lambda(t), \mu(t)) \neq 0, \quad t \in I \tag{3.17}$$

In view of hypothesis (A_1), the equation (3.11) yields,

$$\left.\begin{aligned} &\lambda(t) + \alpha\beta(t) = 0, \quad t \in I \\ &\lambda(t) + \gamma\beta(t) = 0, \quad t \in I \end{aligned}\right\} \tag{3.18}$$

Multiplying (3.12) by $y_j(t)$ and summing over j, we have

$$\begin{aligned} &\lambda(t)^T \left[y(t)^T g_x - D(y(t)^T g_{\dot{x}}) + ((y(t)^T g_x)_x - D(y(t)^T g_x)_{\dot{x}} + D^2(y(t)^T g_{\dot{x}})_{\dot{x}})\beta(t) \right] \\ &\quad -\gamma \left[y(t)^T g_x - \frac{1}{2}\beta(t)^T((y(t)^T g_x)_x - D(y(t)^T g_x)_{\dot{x}} + D^2(y(t)^T g_{\dot{x}})_{\dot{x}})\beta(t) \right] \\ &\quad +\mu^T(t)y(t) = 0, \end{aligned}$$

Using (3.15) and then integrating, we have

$$\begin{aligned} &\int_I \lambda(t)^T \{y(t)^T g_x - D(y(t)^T g_{\dot{x}}) + H\beta(t)\} dt \\ &\qquad -\gamma \int_I \{y(t)^T g_x - \frac{1}{2}\beta(t)^T H\beta(t)\} dt = 0 \end{aligned}$$

This, because of (3.14), yields,

$$\int_I \lambda(t)^T \{y(t)^T g_x - D(y(t)^T g_{\dot{x}}) + H\beta(t)\} dt = 0 \tag{3.19}$$

If $(\alpha, \gamma) = 0$ i.e. $\alpha = 0 = \gamma$, then (3.18) implies $\lambda(t) = 0$, $t \in I$ and $\mu(t) = 0$ from (3.12).

Thus, we have

$$(\alpha, \gamma, \lambda(t), \mu(t)) = 0.$$

This contradicts (3.17). Hence

$$(\alpha, \gamma) \neq 0 \quad \text{i.e.} \quad \alpha > 0 \ \text{ or } \ \gamma > 0.$$

We claim $\beta(t) = 0$, $t \in I$. Suppose that $\beta(t) \neq 0$, $t \in I$.

From (3.18) we have

$$(\alpha - \gamma)\beta(t) = 0$$

implying $\alpha = \gamma > 0$. Using (3.18) in (3.19), we have

$$\int_I \alpha\beta(t)^T\{y(t)^T g_x - D(y(t)^T g_{\dot{x}}) + H\beta(t)\}dt = 0$$

implies

$$\int_I \beta(t)^T\{y(t)^T g_x - D(y(t)^T g_{\dot{x}})\}dt + \int_I \beta(t)^T H\beta(t)dt = 0 \tag{3.20}$$

In view of the hypothesis (A_3) i.e.,

$$\int_I \{\beta(t)^T y(t)^T g_x - D(y(t)^T g_{\dot{x}})\}dt \geq 0$$

and

$$\int_I \beta(t)^T H\beta(t)dt > 0.$$

We have

$$\int_I \beta(t)^T\{y(t)^T g_x - D(y(t)^T g_{\dot{x}}) + H\beta(t)\}dt > 0$$

This contradicts (3.20). Hence $\beta(t) = 0$, $t \in I$. Consequently (3.18) implies $\lambda(t) = 0$, $t \in I$.

From (3.10), we have

$$-\alpha(f_x - Df_{\dot{x}}) + \gamma(y(t)^T g_x - Dy(t)^T g_{\dot{x}}) = 0 \tag{3.21}$$

Also from (3.4), we have

$$(f_x - Df_{\dot{x}}) = -(y(t)^T g_x - D(y(t)^T g_{\dot{x}}))$$

Using this in (3.21), we have

$$(\alpha - \gamma)(y(t)^T g_x - Dy(t)^T g_{\dot{x}}) = 0$$

In view of the hypothesis (A_2), this gives

$$\alpha = \gamma > 0.$$

From (3.12) we have

$$\gamma g_j + \mu_j(t) = 0$$

Because $\gamma > 0$, this gives

$$g_j = -\frac{\mu_j(t)}{\gamma} \leq 0$$

$$g(t, \bar{x}, \dot{\bar{x}}) \leq 0$$

$\bar{x}$ is feasible to (CP). In view of $\beta(t) = 0, t \in I$ gives the equality of two objective values follows. The optimality of $\bar{x}$ for (CP) follows from Theorem 1.

4. Natural Boundary Values

In this section, we formulate dual variational problem with natural boundary values rather than fixed end points.

(CP_0): Minimize $\int_I f(t, x, \dot{x})dt$

Subject to

$$g(t, x, \dot{x}) \leq 0, \qquad t \in I$$

(CD_0): Maximize $\int_I \{F(t, x, \dot{x}) - \frac{1}{2}\beta(t)^T F \beta(t)\}dt$

Subject to

$$f_x + y(t)^T g_x - D(f_{\dot{x}} + y(t)^T g_{\dot{x}}) + (F + H)\beta(t) = 0 \quad t \in I$$

$$y(t) \geq 0, \quad t \in I$$

$$y(t)^T g_{\dot{x}}|_{t=a} = 0,$$

$$y(t)^T g_{\dot{x}}|_{t=b} = 0,$$

We shall not repeat the proofs of Theorem 3.1-3.3, as these follow on the lines of the analysis given in [1].

5. Nonlinear Programming

If all functions in (CP_0) and (CD_0) are independent of t, then these problems will reduce to following pair of dual problems, treated by Bector and Chandra [1].

(P_1) : Minimize $f(x)$

Subject to

$$g(x) \leq 0,$$

(D_1) : Maximize $f(x) - \frac{1}{2} p^T \nabla^2 f(x) p$

Subject to

$$\nabla(f + y^T g) + \nabla^2(f + y^T g)p = 0$$

$$y^T g(x) - \frac{1}{2} p^T \nabla^2 (y^T g(x)) p \geq 0$$

$$y \geq 0$$

where

$$f_x(x) = \nabla f(x),\ y^T g(x) = \nabla(y^T g),\ f_{xx}(x) = \nabla^2 f(x)$$

$$\nabla^2(y^T g(x)) = (y^T g_x)_x \quad \text{and} \quad \beta = p$$

6. Conclusion

We have considered a pair of Mond-Weir type second order dual that relaxes the invexity requirement in Chen [4] to validate duality theorems. The approach chosen here is to render the problem analogous to the second-order dual problems introduced in [1] as a mathematical programming problem in infinite dimensional space. Our dual model presents simpler dual objective function and also allows the weaking of the invexity assumption of [4]. There is a rich scope to study this problem in multi-objective setting. One can also formulate a fractional analogue of our model to study duality results.

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