



## Comparative Analysis of the Modified SOR and BGC Methods Applied to the Poleness Conservative Finite Difference Scheme

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**Abstract.** The poleness conservative finite difference scheme based on the weak solution of Poisson equation in polar coordinates is studied. Due to the singularity at  $r = 0$  in the considered polar domain  $\Omega_{r,\varphi}$ , a special technique of deriving the finite difference scheme in the neighbourhood of the pole point  $r = 0$  is described. The constructed scheme has the order of approximation  $O\left(\frac{(h_r^2 + h_\varphi^2)}{r}\right)$ . In the second part of the paper the structure of the corresponding non-symmetric sparse block matrix is analyzed. A special algorithm based on SOR-method is presented for the numerical solution of the corresponding system of linear algebraic equations. The theoretical result are illustrated by numerical examples for continuous as well as discontinuous source function.

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### 1. Introduction

In this paper we consider the following Dirichlet problem for the Poisson equation in polar coordinates  $(r, \varphi)$ :

$$\begin{cases} Au := -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = F(r, \varphi), & (r, \varphi) \in \Omega_R, \\ u(R, \varphi) = 0, & (r, \varphi) \in \Gamma_\varphi, \\ u(r, 0) = u(r, \beta), & r \in (0, R), \end{cases} \quad (1)$$

where  $\Omega_R := \{(r, \varphi) \in \mathbb{R}^2 : r \in [0, R), \varphi \in [0, \beta)\}$ ,  $\Gamma_\varphi := \{(R, \varphi) : \varphi \in [0, \beta)\}$ , and  $\beta \in (0, 2\pi]$ .

This problem is a mathematical model of various physical and engineering problems arising in steady state flow of an incompressible viscous fluid in a duct of circular cross-section [15, 13], in the determination of a potential in electrostatics [3] and in the elasticity theory [8]. The two circumstances may lead to singularities: the geometrical singularity related to

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the corner  $\beta \in (0, 2\pi)$  of the polar domain and the pole point  $r = 0$ . The first type of singularity means that for some values of the parameter  $\beta \in (0, \pi)$  the second (and higher) derivative of the solution  $u(r, \varphi)$  with respect to  $r \in [0, R]$  is singular at  $r = 0$ . As show examples below the regularity of the weak solution  $u \in H^2(\Omega) \cap \dot{H}^1(\Omega)$  of problem (1) depends on the value of the angle  $\beta \in (0, 2\pi)$ . This behaviour is usual for second order elliptic problems and is related to the  $C^2$ -regularity property of the boundary  $\partial\Omega_{R\beta}$  of the considered domain [1, 11]. The singularity at the re-entrant corner makes the numerical solution of these problems challenging. The second type of singularity is a reason of many difficulties in constructing the standard finite difference (FD) schemes, when the pole  $r = 0$  is treated as a computational boundary. To avoid these difficulties various FD and pseudo-spectral (PS) methods have been suggested in literature [see 4, 14]. These schemes include the necessity of special boundary closures, which leads to undesirable clustering of grid points in PS schemes [see 4, 6]. The treatment of the singularities related to the situations  $r \rightarrow 0$  and  $\sin \varphi \rightarrow 0$  have been given in [9, 10].

This paper is devoted to fill in the lack of result for conservative finite difference scheme for problem (1) and nonsymmetric sparse block matrices related to finite difference equations in polar coordinates. We present a conservative finite difference scheme for this problem and prove its convergence. Our approach is based on the Lax-Wondroff theorem [7], which guarantees convergence of a conservative FD schemes in the class of weak solutions, as the polar mesh is refined. Note that a similar technique was used in [12] for problem (1), where the classical solution of the boundary value problem (1) is considered. Since we are interested in bounded weak solutions  $u \in H^1(\Omega_R)$ , we require that the solution of problem (1) satisfies the boundedness at  $r = 0$  condition

$$\lim_{r \rightarrow 0} r \frac{\partial u}{\partial r} = 0. \quad (2)$$

Hence one needs to approximate not only the elliptic equation (1), but also condition (2). This condition will be used for obtaining the conservative finite difference scheme in the neighbourhood of the pole point  $r = 0$ .

Further, the FD approximations of problem (1)-(2) lead to large sparse system of linear equations, with the special nonsymmetric matrix, due to the periodicity condition  $u(r, 0) = u(r, \beta)$ . These type of linear systems require time-consuming algorithms for their effective numerical solution [2, 5]. We use the special structure of the obtained matrix [5] and construct a fast iteration algorithm, based on SOR-method.

The paper is organized as follows. In section 2 the weak solution of problem (1)-(2) is defined. The piecewise uniform polar mesh and the conservative FD scheme for the problem is constructed in Section 3. In Section 4 the structure of the corresponding sparse matrix and iteration algorithm is discussed. Numerical solution of problem (1)-(2) for different types of source function  $F(r, \varphi)$  and results are presented in Section 5.

## 2. The Weak Solution and its Regularity Depending on the Parameter $\beta \in (0, 2\pi)$

Let  $v \in \dot{H}^1(\Omega_{R\beta})$  be an arbitrary function, where

$$\dot{H}^1(\Omega_{R\beta}) := \{v \in H^1(\Omega_{R\beta}) : u(R, \varphi) = 0, \varphi \in (0, \beta); u(r, 0) = u(r, \beta), r \in (0, R)\}$$

and  $H^1(\Omega_{R\beta})$  is the Sobolev space of functions  $v = v(r, \varphi)$  with the norm

$$\|u\|_1 := \left\{ \int \int_{\Omega_{R\beta}} [u^2 + |\nabla u|^2] r dr d\varphi \right\}^{1/2}.$$

Let us multiply the both sides of equation (1) to  $rv(r, \varphi)$ ,  $r \in (0, \beta)$  and integrate on  $\Omega_{R\beta}$ :

$$- \int \int_{\Omega_{R\beta}} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r} \frac{\partial^2 u}{\partial \varphi^2} \right] v dr d\varphi = \int \int_{\Omega_{R\beta}} F(r, \varphi) v r dr d\varphi.$$

Applying here by part integration, using the boundary and periodicity conditions (2) we get

$$\int \int_{\Omega_{R\beta}} \nabla u(r, \varphi) \cdot \nabla v(r, \varphi) r dr d\varphi = \int \int_{\Omega_{R\beta}} F(r, \varphi) v r dr d\varphi, \quad \forall v \in \dot{H}^1(\Omega_{R\beta}). \quad (3)$$

Here  $\nabla u$  is the gradient vector in polar coordinates,

$$\nabla u = \frac{\partial u}{\partial r} \mathbf{e}_1 + \frac{1}{r} \frac{\partial u}{\partial \varphi} \mathbf{e}_2, \quad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \cos \varphi, & \sin \varphi \\ \sin \varphi, & \cos \varphi \end{pmatrix} \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \end{pmatrix},$$

and  $\mathbf{i}_1, \mathbf{i}_2$  are unit coordinate vector in Cartesian coordinates. The solution  $u \in \dot{H}^1(\Omega_{R\beta})$  of the integral identity (3) is defined as a weak solution of the boundary value problem (1)-(2).

The integral identity (3) shows that if the function  $u \in C^2(\Omega_{R\beta}) \cap C^1(\overline{\Omega_{R\beta}})$  is the solution of the boundary value problem (1)-(2), then for all  $v \in \dot{H}^1(\Omega_{R\beta})$  this identity holds. Hence the weak solution of problem (1)-(2) can be defined as function  $u \in \dot{H}^1(\Omega_{R\beta})$  satisfying this integral identity for all  $v \in \dot{H}^1(\Omega_{R\beta})$ . According to the general theory for linear elliptic boundary value problems the regular weak solution of problem (1)-(2) belongs to  $H^2(\Omega_{R\beta}) \cap \dot{H}^1(\Omega_{R\beta})$ , if the boundary  $\partial\Omega_{R\beta}$  is of class  $C^2$  [4, 11]. The following example shows that depending on the values  $\beta \in (\pi, 2\pi)$  of the parameter  $\beta$  the weak solution of the boundary value problem (1)-(2) may not belong to the class  $H^2(\Omega_{R\beta})$ .

**Example 1.** The function

$$u(r, \varphi) = r^{\pi/\beta} \sin(\pi\varphi/\beta), \quad (r, \varphi) \in \Omega_{R\beta} \quad (4)$$

satisfies the Laplace equation in polar coordinates and the Dirichlet condition

$$u(R, \varphi) = R^{\pi/\beta} \sin(\pi\varphi/\beta), \quad \varphi \in (0, \beta).$$

This function belongs to the Sobolev space  $H^2(\Omega_{R\beta}) \cap H^1(\overline{\Omega_{R\beta}})$ , for all  $\beta \in (0, \pi)$ . However for the values  $\beta \in (\pi, 2\pi)$  this solution doesn't belong to  $H^2(\Omega_{R\beta})$ , since  $r^\alpha \notin H^2(\Omega_{R\beta})$  for  $\alpha < 1$ . The reason, as shown in [11], is that for  $\beta \in (\pi, 2\pi)$  the boundary  $\partial\Omega_{R\beta}$  doesn't belong to the class  $C^2$ , as requires the Agmon-Nirenberg regularity theorem [1].

Note that the above loss of regularity of the boundary  $\partial\Omega_{R\beta}$  is a result of the introduced angle  $\alpha = \pi\varphi/\beta$ . When the parameter  $\beta$  changes in  $(0, 2\pi)$  the angle  $\alpha$  always remains in  $(0, \pi)$ . This also implies that for the function  $u(r, \varphi)$ , given by (4), the periodicity condition

$$\frac{\partial u(r, 0)}{\partial \varphi} = \frac{\partial u(r, \beta)}{\partial \varphi}, \quad r \in (0, R) \tag{5}$$

doesn't hold. The next example shows that by introducing the parameter  $\alpha = n\pi\varphi/\beta$ ,  $n = 2$ , the fulfilment of this condition can be achieved.

**Example 2.** Consider now the function

$$u(r, \varphi) = r^{2\pi/\beta} \text{Sin}(2\pi\varphi/\beta), \quad (r, \varphi) \in \Omega_{R\beta}, \tag{6}$$

which evidently belongs to  $H^2(\Omega_{R\beta})$ ,  $\forall \beta \in (0, 2\pi)$ . This function satisfies the Laplace equation and the boundary conditions (1). Observe that the angle  $\alpha = 2\pi\varphi/\beta$  always remains in  $(0, 2\pi)$ , when the parameter  $\beta$  changes in  $(0, 2\pi)$ . This moment removes the lack of smoothness of the boundary and as a result the solution  $u(r, \varphi) \in H^2(\Omega_{R\beta}) \cap H^1(\overline{\Omega_{R\beta}})$ ,  $\forall \beta \in (0, \pi)$ , given by (6), also satisfies the periodicity condition (5).

These two solutions show the main distinguished features of the boundary value problem (1)-(2), and they will be used for testing of the presented finite difference scheme.

### 3. The Conservative FD Scheme on a Piecewise Uniform Polar Mesh

We assume here  $\beta = 2\pi$  and introduce the following uniform meshes with respect to variables  $r$  and  $\varphi$

$$\begin{aligned} \overline{w}_r &:= \{r_n = (n - 0.5)h_r : n = 1, 2, \dots, N + 1, h_r = (2R)/(2N + 1)\}, \\ \overline{w}_\varphi &:= \{\varphi_m = (m - 1)h_\varphi : m = 1, 2, \dots, M + 1, h_\varphi = 2\pi/M\}, \end{aligned}$$

with mesh steps  $h_r, h_\varphi > 0$ . Then we obtain the piecewise uniform polar mesh  $\overline{w}_{r\varphi} := w_r \times w_\varphi$  (Fig. 1):

$$\overline{w}_{r\varphi} := \{(r_n, \varphi_m) \in \overline{\Omega_{R\varphi}} : r_n \in \overline{w}_r, \varphi_m \in \overline{w}_\varphi\}, \quad \dim \overline{w}_{r\varphi} = (N + 1) \times (M + 1),$$

where  $\overline{w}_{r\varphi} := w_{r\varphi} \cup c\gamma_\varphi \cup \gamma_0$ , and  $w_{r\varphi} := \{(r_n, \varphi_m) \in \Omega_{R\varphi} : n = \overline{2, N}, m = \overline{2, M}\}$ . The boundary mesh points are defined as follows

$$\gamma_\varphi := \{(R, \varphi_m) \in \Gamma_\varphi : m = \overline{1, M}\}, \quad \gamma_0 := \{(r_n, 0) \in \Gamma_0 : n = \overline{1, N}\}.$$

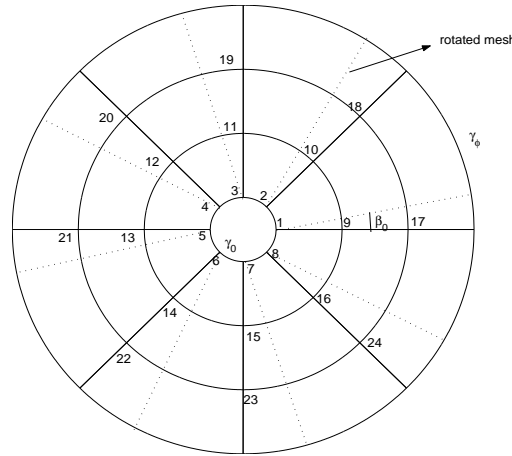


Figure 1: Geometry of the poleness polar mesh and its rotated form

Due to the periodicity condition  $u(r, 0) = u(r, 2\pi)$  we will not include the values  $u(r, 0)$ ,  $r \in [0, R]$  to the list of unknowns in the discrete problem. The introduced mesh  $\overline{w}_{r\varphi}$  doesn't include the pole point  $r = 0$ , that is in the presented discrete model the domain  $\Omega_{R\varphi}$  is approximated by the circular disc  $\overset{\circ}{\Omega}_{R\varphi}$ . Thus our discrete model does not deal with the singularity at  $r = 0$ , that is usual for the differential problem. Instead we will derive an approximation of the boundedness condition (2) at the central circle with radius  $r = r_1$ ,  $r_1 = 0.5h_r$ .

Denote by  $e_{nm} = \{(r, \varphi) \in \Omega_{R\beta} : r_n \leq r \leq r_{n+1}, \varphi_m \leq \varphi \leq \varphi_{m+1}\}$  the polar finite element with four nodes. We derive an error approximation for each element. Introducing the half-nodes  $r_n^\pm = r_n \pm h_r/2$ ,  $\varphi_m^\pm = \varphi_m \pm h_\varphi/2$  and integrating equation (1) on the finite element  $\tilde{e}_{nm} := [r_n^-, r_n^+] \times [\varphi_m^-, \varphi_m^+]$  we obtain the following balance equation:

$$\int_{\varphi_m^-}^{\varphi_m^+} \int_{r_n^-}^{r_n^+} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) dr d\varphi + \int_{r_n^-}^{r_n^+} \int_{\varphi_m^-}^{\varphi_m^+} \frac{1}{r} \frac{\partial^2 u}{\partial \varphi^2} d\varphi dr = - \int_{r_n^-}^{r_n^+} \int_{\varphi_m^-}^{\varphi_m^+} r F(r, \varphi) d\varphi dr. \quad (7)$$

Let us transform the first left integral  $I_{nm}^r$ .

$$I_{nm}^r = \int_{\varphi_m^-}^{\varphi_m^+} \left( r \frac{\partial u}{\partial r} \right)_{r=r_n^-}^{r=r_n^+} d\varphi \cong h_\varphi \left[ \left( r \frac{\partial u}{\partial r} \right) (r_n^+, \varphi_m) - \left( r \frac{\partial u}{\partial r} \right) (r_n^-, \varphi_m) \right]$$

We use here the central finite difference formula for approximation of derivatives on the right hand side, by using the mesh points  $r_n, r_{n+1/2}, r_{n+1}$ , with mesh step  $h_{r/2} = h_r/2$ :

$$\begin{aligned} \left( r \frac{\partial u}{\partial r} \right) (r_n^+, \varphi_m) &\cong r_n^+ \frac{u(r_{n+1}, \varphi_m) - u(r_n, \varphi_m)}{h_r}, \\ \left( r \frac{\partial u}{\partial r} \right) (r_n^-, \varphi_m) &\cong r_n^- \frac{u(r_n, \varphi_m) - u(r_{n-1}, \varphi_m)}{h_r}. \end{aligned}$$

Then we have the following variational finite difference approximation of the integral operator  $I_{nm}^r$ :

$$I_{nm}^r \cong h_\varphi \left[ r_n^+ u_{r,nm} - r_n^- u_{\bar{r},nm} \right].$$

By the same way we can derive an approximation of the second integral operator  $I_{nm}^\varphi$  on the left hand side of (7):

$$\begin{aligned} I_{nm}^\varphi u &\cong \frac{h_r}{r_n} \left[ \frac{\partial u}{\partial \varphi}(r_n, \varphi_m^+) - \frac{\partial u}{\partial \varphi}(r_n, \varphi_m^-) \right] \\ &\cong \frac{h_r}{r_n} \left[ u_\varphi(r_n, \varphi_m) - u_{\bar{\varphi}}(r_n, \varphi_m) \right]. \end{aligned}$$

Applying to the right hand side of (7) the numerical integration (rectangle) formula, finally we have

$$-h_\varphi [r_n^+ y_{r,nm} - r_n^- y_{\bar{r},nm}] - \frac{h_r}{r_n} [y_{\varphi,nm} - y_{\bar{\varphi},nm}] = h_r h_\varphi r_n F(r_n, \varphi_m)$$

Dividing by  $h_r h_\varphi r_n > 0$  we obtain the following finite difference equation

$$\mathcal{A}_{nm}^r u := - \left( \frac{1}{r} (r y_{\bar{r}}) \right)_{r,nm} - \left( \frac{1}{r^2} y_{\bar{\varphi}} \right)_{nm} = F(r_n, \varphi_m), \quad (r_n, \varphi_m) \in \omega_{r,\varphi}, \quad n \neq 1. \quad (8)$$

The finite dimensional operators

$$\mathcal{A}_{nm}^r y := \left\{ \frac{1}{r} (r y_{\bar{r}}) \right\}_{r,nm}, \quad \mathcal{A}_{nm}^\varphi y \left( \frac{1}{r^2} y_{\bar{\varphi}} \right)_{nm}$$

are the finite difference approximations of the differential operators

$$A^r u := \frac{1}{r} \frac{\partial}{\partial r} \left( k(r) \frac{\partial u}{\partial r} \right), \quad A^\varphi u := \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2}, \quad (r, \varphi) \in \Omega_{R\beta},$$

correspondingly. Note that the same approximations can also be obtained from the direct finite difference approximation of the Poisson equation (1).

The finite difference equation corresponding to the layer  $r = r_1 = h_r/2$  can be derived by using the same balance equation (7), substituting  $r_n^- = \varepsilon$ ,  $r_n^+ = h_r$ :

$$\int_{\varphi_m^-}^{\varphi_m^+} \int_\varepsilon^{h_r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) dr d\varphi + \int_{\varphi_m^-}^{\varphi_m^+} \int_\varepsilon^{h_r} \frac{1}{r} \frac{\partial^2 u}{\partial \varphi^2} dr d\varphi = - \int_{\varphi_m^-}^{\varphi_m^+} \int_\varepsilon^{h_r} r F(r, \varphi) dr d\varphi.$$

Going to the limit  $\varepsilon \rightarrow 0$  here and using condition (2) we obtain

$$\int_{\varphi_m^-}^{\varphi_m^+} \frac{\partial u(h_r, \varphi)}{\partial r} d\varphi + \int_0^{h_r} \frac{1}{r} \left[ \frac{\partial u(r, \varphi_m^+)}{\partial \varphi} - \frac{\partial u(r, \varphi_m^-)}{\partial \varphi} \right] dr + \int_0^{h_r} r \int_{\varphi_m^-}^{\varphi_m^+} F(r, \varphi) d\varphi dr = 0.$$

Again using the numerical integration formula in the first integral we get

$$h_r h_\varphi \frac{\partial u(h_r, \varphi_m)}{\partial r} + 2 \left[ \frac{\partial u}{\partial \varphi} \left( \frac{h_r}{2}, \varphi_m + \frac{h_\varphi}{2} \right) - \frac{\partial u}{\partial \varphi} \left( \frac{h_r}{2}, \varphi_m - \frac{h_\varphi}{2} \right) \right] + \frac{h_r^2 h_\varphi}{2} F \left( \frac{h_r}{2}, \varphi_m \right) \approx 0$$

To derive this finite difference equation in canonical form we use the standard definitions

$$\begin{aligned} y_{r,1m} &:= \frac{1}{h_r} [y(r_1 + h_r, \varphi_m) - y(r_1, \varphi_m)], \\ y_{\bar{\varphi},1m} &:= \frac{1}{h_\varphi} [y(r_1, \varphi_m) - y(r_1, \varphi_{m-1})], \\ y_{\varphi,1m} &:= \frac{1}{h_\varphi} [y(r_1, \varphi_{m+1}) - y(r_1, \varphi_m)]. \end{aligned}$$

Then we have

$$h_r h_\varphi y_r(r_1, \varphi_m) + 2 h_\varphi y_{\bar{\varphi},\varphi}(r_1, \varphi_m) + \frac{h_r^2 h_\varphi}{2} F(r_1, \varphi_m) = 0, \quad m = 1, 2, \dots, M$$

Dividing the both sides to  $h_r^2 h_\varphi / 2$  finally we obtain the finite difference equation corresponding to the layer  $r_1 = h_r / 2$ :

$$-\frac{2}{h_r} y_r(r_1, \varphi_m) - \frac{4}{h_r^2} y_{\bar{\varphi},\varphi}(r_1, \varphi_m) = F(r_1, \varphi_m) = 0, \quad m = 1, 2, \dots, M. \tag{9}$$

Equations (8)-(9) represent the finite difference analogue of the Poisson equation (1) in the constructed polar mesh  $\omega_{r\varphi}$ .

**Lemma 1.** *If  $u \in C^4(\Omega_{R\beta})$  then the order of the approximation error of the finite difference schemes (8)-(9) in C-norm is  $\psi_{r\varphi} : \psi_{r\varphi} = O\left(\frac{h_r^2 + h_\varphi^2}{r_n}\right)$ , where*

$$\psi_{r\varphi} = \frac{h_r^2}{6r_m} \left[ \frac{\partial^3 u}{\partial r^3} + \frac{r_n + h_r/2}{4} \frac{\partial^4 u(\tilde{r}_n, \varphi_m)}{\partial r^4} + \frac{r_n - h_r/2}{4} \frac{\partial^4 u(\tilde{r}_n, \varphi)}{\partial r^4} \right] + \frac{h_\varphi^2}{12r_m} \frac{\partial^4 u(r_n, \tilde{\varphi}_m)}{\partial \varphi^4}. \tag{10}$$

The explicit form (10) of  $\psi(h_r, \varphi_m)$  and the proof of this result is given in [11].

The lemma shows that as  $r \rightarrow r_1$  the function  $\psi_{r\varphi}$  increases, and at the first layer  $r = r_1$  ( $r_1 = h_r / 2$ ) becomes  $\psi_{r\varphi} = O(h_r + h_\varphi^2 / h_r)$ .

Note that the boundedness condition (2) is necessary for the above approximation, and hence for the convergence of the finite difference schemes (8)-(9), although this condition is not used explicitly on deriving the approximation error. To show this, consider the following:

**Example 3.** *The function*

$$u(r, \varphi) = \ln \frac{1}{r} \text{Sin} \varphi, \quad (r, \varphi) \in \Omega_{R,\varphi},$$

satisfies the Poisson equation (1) in  $\Omega_{R\beta}$ , with for  $R = 1$ ,  $\beta = 2\pi$ , and the right hand side  $F(r, \varphi) = \frac{1}{r^2} \sin \varphi$ . Evidently this solution also satisfies the boundary and periodicity conditions (1), but does not satisfy the boundedness condition (2), since  $r \partial u / \partial r = \sin \varphi \not\rightarrow 0$ , as  $r \rightarrow 0$ . Calculating the right hand side of (10) for  $n = 1$  we obviously observe that for  $r = r_1$

$$\psi(r_1, \varphi_m) = \frac{h_\varphi^2}{12r_1^3} \sin \varphi_m.$$

This shows that  $\psi(r_1, \varphi_m) \not\rightarrow 0$ , as  $r_1 \rightarrow 0$ , and there is no approximation in the neighbourhood of the pole point  $r = 0$ .

To formulate the discrete problem we need to add to equations (8)-(9), the equations, obtained from the periodicity condition (1).

#### 4. The Algebraic Problem with Nonsymmetric Sparse Matrix: Iteration Algorithm

The finite difference equations (8)-(9) compose  $K := NM$  number of algebraic equations with  $K$  unknowns

$$y = (y_{11}, y_{12}, \dots, y_{1M}, y_{21}, \dots, y_{NM})^T, \quad \dim y = K.$$

We can rewrite these equations in the form of the system of linear algebraic equations  $\mathcal{A}y = \mathcal{F}$  with the following positive band matrix  $\mathcal{A}$ ,  $\dim \mathcal{A} = K \times K$ :

$$\mathcal{A} := \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & \dots & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & \dots & 0 & 0 & 0 \\ 0 & A_{32} & A_{33} & A_{34} & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & A_{N-1,N-2} & A_{N-1,N-1} & A_{N-1,N} \\ 0 & 0 & 0 & 0 & \dots & 0 & A_{N,N-1} & A_{N,N} \end{bmatrix}.$$

Here  $M \times M$ -dimensional block matrices  $A_{ij}$  are of the following structure:

$$A_{ii} = \begin{bmatrix} \bullet & \bullet & 0 & 0 & \dots & 0 & 0 & \bullet \\ \bullet & \bullet & \bullet & 0 & \dots & 0 & 0 & 0 \\ 0 & \bullet & \bullet & \bullet & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \bullet & \bullet & \bullet \\ \bullet & 0 & 0 & 0 & \dots & 0 & \bullet & \bullet \end{bmatrix}$$

and

$$A_{ii+1} = \alpha_i I, \quad A_{i-1i} = \beta_i I.$$



The parameters  $\alpha_i$  and  $\beta_i$  can be defined from the finite difference schemes (8)-(9) as follows:

$$\alpha_1 = \frac{2}{h_r^2}; \quad \alpha_i = \frac{k_i}{r_i h_r^2}, \quad i = \overline{2, N}; \quad \beta_i = \frac{k_{i+1}}{r_i h_r^2}, \quad i = \overline{1, N}.$$

Since  $\alpha_i \neq \beta_i$ , the matrix  $\mathcal{A}$  is not symmetric one. Moreover,  $\mathcal{A}$  is a sparse matrix of special structure, corresponding to the polar mesh and periodicity condition. Evidently such a system of linear algebraic equations with non-symmetric sparse matrix needs to be solved by iteration methods [12, 13, 14]. However, as the computational experiments below show, use of compact storage for the band matrix  $\mathcal{A}$  and then an application of any effective iteration method requires large enough time for the solution of the linear system of algebraic equations  $\mathcal{A}y = \mathcal{F}$ . The reason is that the bandwidth of the non-symmetric matrix  $\mathcal{A}$  is

$$bw = 3M$$

and there are many null terms in the band. Specifically, the above block matrices  $A_{ii}, A_{i+1,i}, A_{i-1,i}$  contain  $M^2 - 3M, M^2 - M, M^2 - M$  zero elements, correspondingly. Hence for  $M = N$  the number of zero and non-zero elements of the band are  $3N^3 - 4N^2 - 4N$  and  $5N^2 + N$ , respectively. For the mesh with  $N = 30$ , this means that the number of non-zero elements of the band matrix  $\mathcal{A}$  is about 4.3% of all band elements. Therefore one needs to construct a special algorithm which can store and operate with only nonzero elements  $\alpha_i$  and  $\beta_i$ , realising the multiplications  $Uy$  and  $Ly$ , to minimize the time required for the solution of the considered problem by any iteration method. Here the  $U$  and  $L$  are the upper and lower triangular matrices:  $\mathcal{A} = D + U + L$ . Note that for some class of systems, arising from the finite-element discretization, similar algorithm was constructed in [15].

Table 1 illustrates the comparative analysis of the standard SOR method

$$y^{k+1} = (D + \omega L)^{-1}[(1 - \omega)D - \omega U]y^k + \omega(D + \omega L)^{-1}\mathcal{F}, \tag{11}$$

by using MATLAB code, and SOR method with the constructed here special algorithm. As a test example the analytical solution given in Example 2, with  $\beta = 2\pi$ , is used. The iteration parameter  $\omega \in (0, 2)$  in (11) was defined by the formula

$$\omega = \frac{2}{1 + \sqrt{\lambda_{min}(2 - \lambda_{min})}},$$

where  $\lambda_{min}$  is the minimal eigenvalue of the Laplace operator, and  $\lambda_{min} = 2\sin^2(\pi/2N)$ , for the square mesh  $M = N$ .

Table 1 shows that direct application of the SOR method is expensive in the sense of the required CPU time. This time increases as the number  $K = NM$  of mesh points increases. Computational results show that this increase has the character  $2^n$ , i.e.  $n$ -times increase of the number of mesh points leads  $2^n$ -times increase of the CPU time. This is due to the moderately ill-conditionedness, according to [12], of the matrix  $\mathcal{A}$ , as the fourth column of the table shows. At the same time, as the second column of Table 1 shows, the SOR method with the special algorithm requires less than 1 second for all considered meshes.

Table 1: Comparison of the standard and modified SOR algorithms

$N \times M$	SOR with special algorithm CPU time(sec.)	Standard SOR CPU time(sec.)	Condition number of the matrix $\mathcal{A}$
$20 \times 20$	0	9	$1.5 \times 10^4$
$30 \times 30$	0	48	$8.0 \times 10^4$
$40 \times 40$	0	126	$2.5 \times 10^5$

### 5. Numerical Solution of the Elliptic Problem (1)-(2) by the Poleness Conservative Schemes (8)-(9)

In this section we discuss results of computational experiments related to numerical solution of the Dirichlet problem (1)-(2) by the poleness conservative scheme (8)-(9). In the first series of the computational experiments, the convergence and accuracy of the numerical solution of the Dirichlet problem (1)-(2), with the analytical solution

$$u(r, \varphi) = (r - 1)\text{Sin}\varphi, \quad (r, \varphi) \in \Omega_{R\beta}, \quad R = 1,$$

is studied. Note that the corresponding source function

$$F(r, \varphi) = \frac{1}{r^2} \text{Sin}\varphi, \quad (r, \varphi) \in \Omega_{R\beta}, \tag{12}$$

has singularity at  $r = 0$ . The two appropriate iterative methods - SOR method and Bi-Congugate Gradient (BCG) method - are applied for the iterative solution of the linear system of algebraic equations, corresponding to the finite difference schemes (8)-(9). In all cases the MATLAB codes of this methods with the above mentioned special algorithm is used. Results are presented in the Table 2. For the comparison, the numerical results obtained by the Gauss-Seidel method, are also presented. Here and below the value of the stopping parameter  $\delta > 0$  in

$$\|u^{n_i-1} - u^{n_i}\|_0 \leq \delta,$$

where  $\|\cdot\|_0$  is the  $L_2$ -norm, is taken  $\delta = 10^{-5}$ .

Table 2 shows numbers of iterations corresponding to all three iteration methods, with  $H^1$ -relative and  $L_\infty$ -absolute errors. Results given in the table show that for relatively coarse meshes BGC method is more effective than the SOR method. But for the meshes  $45 \times 45$  and higher, SOR method is more effective in the sense of iterations. Moreover, the number of iterations  $n_i$  in SOR method increases slowly, as increases the number of mesh points. Thus  $n_i = 234$  and  $n_i = 277$ , for the meshes  $40 \times 40$  and  $50 \times 50$ , respectively. As show the last three columns of the table, the  $H^1$ -relative and  $L_\infty$ -absolute errors are small enough, although the source function (12) has singularity at the pole point  $r = 0$ .

The next series of computational experiments is realized for the smooth continuous source function

$$\tilde{F}(x, y) = F(r, \varphi) = \begin{cases} 50\text{exp}\left(-\frac{\varepsilon^2}{\varepsilon^2-r^2}\right), & 0 < r < \varepsilon \\ 0, & \varepsilon < r < R, \quad R = 1, \end{cases} \tag{13}$$

Table 2: Comparative analysis of the modified SOR and BGC methods applied to the poleness conservative finite difference scheme (8)-(9)

Methods	$N \times M$	Time (sec.)	Number of iterations $n_i$	Rel. error $H^1$ -norm	Abs. error $L_\infty$ -norm $r = r_1$	Abs. error $L_\infty$ -norm $r = R/2$
<b>SOR</b>	$30 \times 30$	0	185	$8.0 \times 10^{-6}$	$3.8 \times 10^{-3}$	$1.3 \times 10^{-3}$
	$40 \times 40$	1	234	$2.7 \times 10^{-6}$	$2.5 \times 10^{-3}$	$8.1 \times 10^{-4}$
	$50 \times 50$	2	277	$1.4 \times 10^{-6}$	$1.9 \times 10^{-3}$	$5.9 \times 10^{-4}$
<b>BCG</b>	$30 \times 30$	0	110	$8.0 \times 10^{-6}$	$3.4 \times 10^{-3}$	$1.2 \times 10^{-3}$
	$40 \times 40$	2	224	$2.5 \times 10^{-6}$	$2.0 \times 10^{-3}$	$6.8 \times 10^{-4}$
	$50 \times 50$	4	385	$1.0 \times 10^{-6}$	$1.3 \times 10^{-3}$	$4.3 \times 10^{-4}$
<b>Gauss-Seidel</b>	$30 \times 30$	33	473	$8.5 \times 10^{-6}$	$4.9 \times 10^{-3}$	$1.2 \times 10^{-3}$
	$40 \times 40$	95	762	$5.0 \times 10^{-6}$	$3.0 \times 10^{-3}$	$7.8 \times 10^{-4}$
	$50 \times 50$	219	1105	$1.1 \times 10^{-5}$	$2.0 \times 10^{-3}$	$1.6 \times 10^{-3}$

with  $\varepsilon = 1/2$ , approximating in weak sense the Dirac  $\delta$ -function (Figure 2a). This function is taken as a given data for the Dirichlet problem (1)-(2). The right pane, Figure 2b, illustrates the numerical solution  $\tilde{u}_h(x, y) = u_h(r, \varphi)$  of problem (1)-(2) by the poleness conservative schemes (8)-(9), for the mesh size  $30 \times 30$ .

Finally we consider the weak solution of the Dirichlet problem (1)-(2), when the source  $F(r, \varphi)$  is a discontinuous at  $r = 1/2$  function

$$\tilde{F}(x, y) = F(r, \varphi) = \begin{cases} 50\varepsilon \frac{\exp(-\varepsilon^2/4r)}{\sqrt{4\pi r^3}} & , \quad 0 < r < \varepsilon \\ 50\varepsilon \frac{\exp(-\varepsilon^2/4)}{\sqrt{4\pi}} & , \quad \varepsilon < r < R, \quad R = 1, \end{cases}$$

given in the left pane of Figure 3. This function with  $\varepsilon = 1/2$  is taken as a given data for the Dirichlet problem (1)-(2). The numerical solution  $\tilde{u}_h(x, y) = u_h(r, \varphi)$  of problem (1)-(2) obtained for the mesh  $30 \times 30$  is plotted in the right pane, Figure 3b. To estimate an accuracy of the numerical solution, in particular at the discontinuity point  $r = 1/2$ , the numerical solutions  $u_h^{(1)}(r, \varphi) = u_h^{(2)}(r, \varphi)$  corresponding to two different meshes  $w_{r\varphi}^{(1)} = w_{r\varphi}^{(2)}$  is compared. The relative error

$$\varepsilon_h = \left\| \frac{u_h^{(1)}(r, \varphi) - u_h^{(2)}(r, \varphi)}{0.5 \left( u_h^{(1)}(r, \varphi) + u_h^{(2)}(r, \varphi) \right)} \right\|_\infty ,$$

is about  $\varepsilon_h = 10^{-3} \div 10^{-3}$ , including the discontinuity point. This shows high accuracy of the numerical method in the case of discontinuous source function, also.

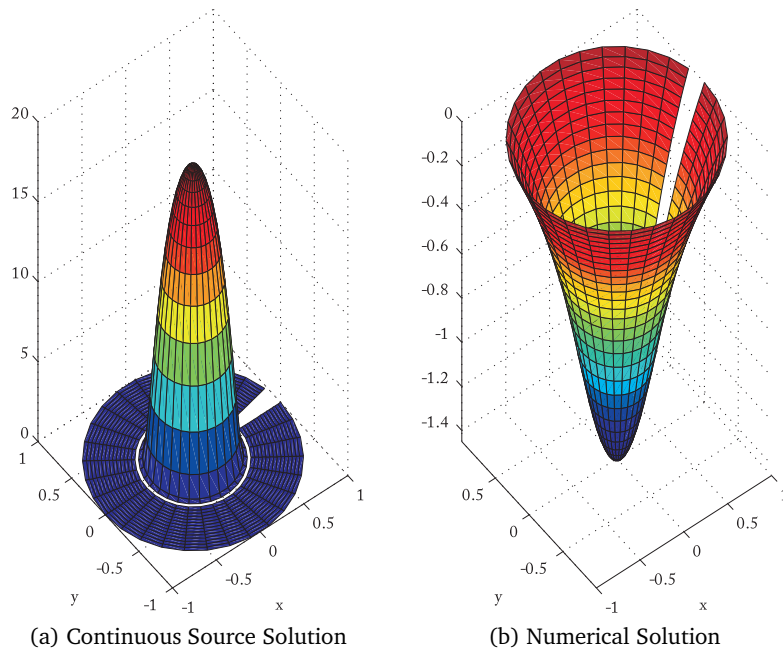


Figure 2: Dirichlet problem in polar coordinates

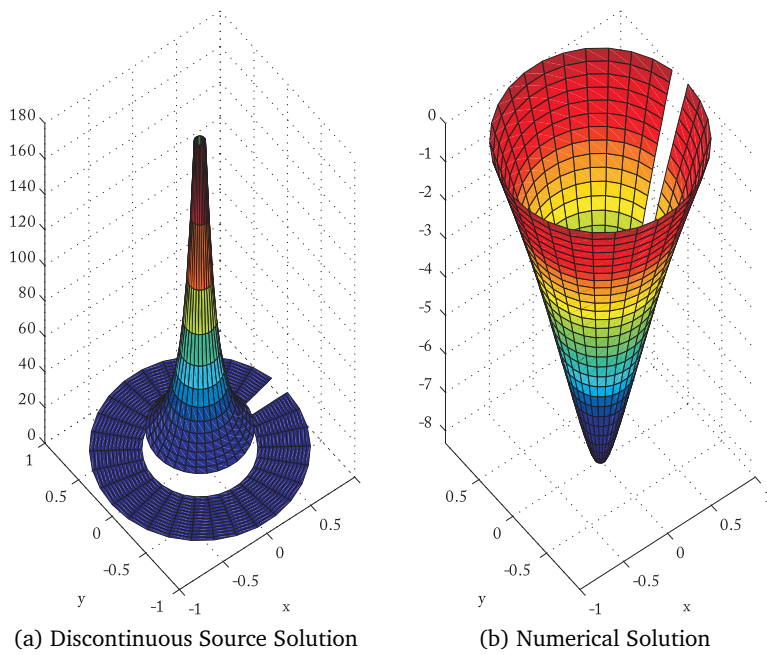


Figure 3: Dirichlet problem in polar coordinates

## 6. Conclusion

We studied poleness conservative finite difference scheme for Laplace operator in polar coordinates. The scheme with the modification of the SOR method allows to construct an effective numerical method for solving the Dirichlet problem in the polar coordinates, based on the weak solution approach. Numerical results presented for discontinuous source function shows high accuracy of the method on acceptable meshes.

Extension of results given here can be made for positive elliptic operators with discontinuous coefficients, and for nonlinear monotone operators of Plateau type, as well. This require some additional techniques that will be done in next studies.

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