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Vector Space–groupoids

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Abstract. We define the notion of vector space-groupoid. The main purpose of this paper is to give the basic properties of vector space-groupoids.

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1. Introduction

In the category theoretical approach, a groupoid is a small category in which every morphism is an isomorphism [6].

The concept of groupoid was first introduced by H. Brandt [1] and it is developed by P. J. Higgins in [6]. The topological and differentiable versions of the groupoids were defined by C. Ehresmann [5]. The notion of group-groupoid was defined by R. Brown and Spencer in the paper [4].

In this paper, the group-groupoid is extended to notion of vector space-groupoid. Another algebraic concept considered in this paper is the vector groupoid. This new mathematical structure was defined by V. Popuţa and Gh. Ivan [13, 14].

The groupoids, group- groupoids and their generalizations (topological groupoids, Lie groupoids etc.) are mathematical structures that have proved to be useful in many areas of science (see for instance [2, 7, 9–12, 15].

The paper is organized as follows. In Section 2 we present some concepts and main results related to groupoids and group-groupoids [4]. In Section 3 we introduce the concept of vector space-groupoid. This is viewed as a groupoid object in the category of vector spaces. The useful properties of vector space-groupoids are established. Finally, we prove that each vector space-groupoid is a vector groupoid.

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2. Preliminaries about Group-groupoids

We begin with the presentation of some necessary backgrounds on groupoids (for further details see e.g. [8, 9]).

Definition 1 ([9]). *A groupoid G over G_0 is a pair (G, G_0) of sets endowed with two surjective maps $\alpha, \beta : G \to G_0$ (***source*** and* **target***), a partially binary operation (***multiplication***)* $m : G_{(2)} := \{(x, y) \in G \times G | \beta(x) = \alpha(y)\} \to G$, $(x, y) \to m(x, y) := x \cdot y$, *($G_{(2)}$ is the* **set of composable pairs***), an injective map $\varepsilon : G_0 \to G$ (***inclusion map***) and a map $i : G \to G$, $x \to i(x) := x^{-1}$ (***inversion***).*

These maps must verify the following conditions:

(G1) (**associativity**)*: if $(x, y) \in G_{(2)}$ and $(y, z) \in G_{(2)}$, then so $(x \cdot y, z) \in G_{(2)}$ and $(x, y \cdot z) \in G_{(2)}$, and the relation, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ is satisfied;*

(G2) (**units**)*: $\alpha \circ \varepsilon = \beta \circ \varepsilon = Id_{G_0}$ and $\varepsilon(\alpha(x)) \cdot x = x = x \cdot \varepsilon(\beta(x))$, $(\forall) x \in G$;*

(G3) (**inverses**)*: for each $x \in G$ we have $\alpha(x^{-1}) = \beta(x)$, $\beta(x^{-1}) = \alpha(x)$, $x^{-1} \cdot x = \varepsilon(\beta(x))$ and $x \cdot x^{-1} = \varepsilon(\alpha(x))$.*

We sometimes use the notation xy instead of the product $x \cdot y$. Whenever we write a product in a given groupoid, we are assuming that it is defined.

The element $\varepsilon(\alpha(x))$ (resp., $\varepsilon(\beta(x))$) is called the *left unit* (resp., *right unit*) of x; $\varepsilon(G_0)$ is called the *unit set*; x^{-1} is called the *inverse* of x.

For a groupoid we use the notation $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ or (G, G_0) or G. The functions $\alpha, \beta, m, \varepsilon, i$ are called *structure functions*. For each $u \in G_0$, the set $\alpha^{-1}(u)$ (resp., $\beta^{-1}(u)$) is called $\alpha-$*fibre* (resp., $\beta-$*fibre*) of G at $u \in G_0$. For any $u \in G_0$, the set $G(u) := \alpha^{-1}(u) \cap \beta^{-1}(u)$ is a group under the restriction of the multiplication, called the *isotropy group at u* of the groupoid (G, G_0). The map $(\alpha, \beta) : G \to G_0 \times G_0$ defined by $(\alpha, \beta)(x) := (\alpha(x), \beta(x))$, $(\forall) x \in G$ is called the *anchor map* of G. A groupoid is *transitive*, if its anchor map is surjective.

In particular, if $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ is a groupoid such that $G_0 \subseteq G$ and $\varepsilon : G_0 \to G$ is the inclusion map, then $(G, \alpha, \beta, m, i, G_0)$ is a Brandt groupoid, called G_0-*groupoid*.

Some elementary properties of groupoids are contained in the following proposition.

Theorem 1. *[8] In a groupoid (G, G_0) the following assertions hold:*

(i) $\alpha(xy) = \alpha(x)$ and $\beta(xy) = \beta(y)$ for any $(x, y) \in G_{(2)}$;

(ii) $\alpha \circ i = \beta$, $\beta \circ i = \alpha$ and $i \circ i = Id_G$;

(iii) $i \circ \varepsilon = \varepsilon$ and $\varepsilon(u) \cdot \varepsilon(u) = \varepsilon(u)$ for each $u \in G_0$;

(iv) $i(x \cdot y) = i(y) \cdot i(x)$, for all $(x, y) \in G_{(2)}$;

(v) $\varphi : G(\alpha(x)) \to G(\beta(x))$, $\varphi(z) := x^{-1} z x$ is an isomorphism of groups.

(vi) if (G, G_0) is transitive, then all isotropy groups are isomorphic.

Example 1.

(i) *A nonempty set G_0 may be considered to be a groupoid over G_0, called the* **null groupoid** *associated to G_0. For this, we take $\alpha = \beta = \varepsilon = i = Id_{G_0}$ and $u \cdot u = u$ for all $u \in G_0$.*

(ii) *A group G having e as unity has a structure of $\{e\}-$groupoid with respect to maps: $\alpha(x) = \beta(x) := e$, $G_{(2)} = G \times G$, $m(x, y) := xy$, $\varepsilon(e) := e$ and $i(x) := x^{-1}$. Conversely, a groupoid with one unit (i.e., $G_0 = \{e\}$) is a group.*

(iii) *For the groupoids $(G_j, \alpha_j, \beta_j, m_j, \varepsilon_j, i_j, G_{j,0})$, $j = 1, 2$, one may construct the groupoid $G_1 \times G_2$ whose its structure functions are given by: $\alpha := \alpha_1 \times \alpha_2$, $\beta := \beta_1 \times \beta_2$, $\varepsilon := \varepsilon_1 \times \varepsilon_2$, $i := i_1 \times i_2$ and $m((g_1, g_2), (g_1', g_2')) = (m_1(g_1, g_1'), m_2(g_2, g_2'))$ for all $(g, g_1') \in G_{(2)}$, $(g_2, g_2') \in G_{(2)}$. Then $(G_1 \times G_2, \alpha, m, \varepsilon, i, G_{1,0} \times G_{2,0})$ is a groupoid, called the* **direct product** *of $(G_1, G_{1,0})$ and $(G_2, G_{2,0})$.*

Definition 2 ([8]). *Let $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ be a groupoid.*

(i) *A pair (H, H_0) of nonempty subsets where $H \subseteq G$ and $H_0 \subseteq G_0$, is called* ***subgroupoid*** *of G, if:*

- *(1)* *$\alpha(H) = H_0$ and $\beta(H) = H_0$;*
- *(2)* *H is closed under partially multiplication and inversion, that is:*
  - *(a)* *$(\forall)$ $x, y \in H$ such that $(x, y) \in G_{(2)}$ we have $x \cdot y \in H$;*
  - *(b)* *$x^{-1} \in H$, for all $x \in H$.*

(ii) *A subgroupoid (H, H_0) of (G, G_0) is said to be* **wide**, *if $H_0 = G_0$.*

(iii) *A wide subgroupoid (N, N_0) of (G, G_0) is called* **normal**, *if for all $x \in G$ and $a \in N$ we have $x \cdot a \cdot x^{-1} \in N$.*

Definition 3 ([9]). *Let (G, α, β, G_0) and $(G', \alpha', \beta', G_0')$ be two groupoids.*

(i) *A* **morphism of groupoids** *or* **groupoid morphism** *from G into G' is a pair (f, f_0) of maps $f : G \to G'$ and $f_0 : G_0 \to G_0'$ such that the following conditions hold:*

- *(i1)* *$\alpha' \circ f = f_0 \circ \alpha$, $\beta' \circ f = f_0 \circ \beta$;*
- *(i2)* *$f(m(x, y)) = m'(f(x), f(y))$ for all $(x, y) \in G_{(2)}$.*

(ii) *If $G_0 = G_0'$ and $f_0 = Id_{G_0}$, we say that f is a G_0-***morphism of groupoids**.

(iii) *A groupoid morphism $(f, f_0) : (G, G_0) \to (G', G_0')$ such that f and f_0 are bijective maps, is called* **isomorphism of groupoids**.

If $(f, f_0) : (G, G_0) \to (G', G_0')$ is a groupoid morphism, then [8]:

$$f \circ \varepsilon = \varepsilon' \circ f_0 \text{ and } f \circ i = i' \circ f. \tag{1}$$

In the sequel we describe the notion of group-groupoid as algebraic structure (for definition see [4]).

A group structure on a nonempty set is regarded as an universal algebra determined by a binary operation, an nullary operation and an unary operation.

Let $(G,\alpha,\beta,m,\varepsilon,i,G_0)$ be a groupoid. We suppose that on G is defined a group structure $\omega : G\times G\to G$, $(x,y)\mapsto \omega(x,y):=x\oplus y$. The unit element of the group G is denoted by e, that is $\nu:\{\lambda\}\to G$, $\lambda\mapsto \nu(\lambda):=e$ (here $\{\lambda\}$ is a singleton). The inverse of $x\in G$ is denoted by $\bar{x}$, that is $\sigma : G\to G$, $x\mapsto\sigma(x):=\bar{x}$. Also, we suppose that on G_0 is defined a group structure $\omega_0 : G_0\times G_0\to G_0$, $(u,v)\mapsto\omega_0(u,v):=u\oplus v$. The neutral element of the group G_0 is denoted by e_0, that is $\nu_0:\{\lambda\}\to G_0$, $\lambda\mapsto\nu_0(\lambda):=e_0$. The inverse of $u\in G_0$ is denoted by $\bar{u}$, that is $\sigma_0 : G_0\to G_0$, $u\mapsto\sigma_0(u):=\bar{u}$.

Definition 4 ([4]). *A* **group-groupoid** *or* $\mathcal{G}-$**groupoid***, is a groupoid (G,G_0) such that the following conditions hold:*

(i) (G,ω,ν,σ) *and* $(G_0,\omega_0,\nu_0,\sigma_0)$ *are groups.*

(ii) The maps $(\omega,\omega_0):(G\times G,G_0\times G_0)\to(G,G_0)$, $\nu:\{\lambda\}\to G$ *and* $(\sigma,\sigma_0):(G,G_0)\to(G,G_0)$ *are groupoid morphisms.*

We shall denote a group-groupoid by $(G,\alpha,\beta,m,i,\varepsilon,\oplus,G_0)$ or $(G,\alpha,\beta,m,\oplus,G_0)$.

Theorem 2. *If* $G,\alpha,\beta,m,\varepsilon,i,\oplus,G_0)$ *is a group-groupoid, then:*

(i) the multiplication m and the binary operation ω are compatible, that is:

$$(x\cdot y)\oplus(z\cdot t)=(x\oplus z)\cdot(y\oplus t),\quad (\forall)(x,y),(z,t)\in G_{(2)};\tag{2}$$

(ii) the structure functions $\alpha,\beta:(G,\oplus)\to(G_0,\oplus)$, $\varepsilon:(G_0,\oplus)\to(G,\oplus)$ *and* $i:(G,\oplus)\to(G,\oplus)$ *are morphisms of groups;*

(iii) the multiplication m and the unary operation σ are compatible, that is:

$$\sigma(x\cdot y)=\sigma(x)\cdot\sigma(y),\quad (\forall)(x,y)\in G_{(2)}.\tag{3}$$

Proof. By Definition 3, since (ω,ω_0) is a groupoid morphism it follows that:

(a) $\alpha\circ\omega=\omega_0\circ(\alpha\times\alpha)$ and $\beta\circ\omega=\omega_0\circ(\beta\times\beta)$;

(b) $\omega(m_{G\times G}((x,y),(z,t)))=m_G(\omega(x,z),\omega(y,t))$, $(\forall)(x,y)$, $(z,t)\in G_{(2)}$.

(i) We have

$$\omega(m_{G\times G}((x,y),(z,t)))=\omega(m_G(x,y),m_G(z,t))=\omega(x\cdot y,z\cdot t)=(x\cdot y)\oplus(z\cdot t)$$

and

$$m_G(\omega(x,z),\omega(y,t))=m_G(x\oplus z,y\oplus t)=(x\oplus z)\cdot(y\oplus t).$$

Using (b) one obtains $(x\cdot y)\oplus(z\cdot t)=(x\oplus z)\cdot(y\oplus t)$, and (2) holds.

(ii) For each $(x, y) \in G \times G$, we have

$$\alpha(\omega(x, y)) = \alpha(x \oplus y)$$

and

$$\omega_0((\alpha \times \alpha)(x, y)) = \omega_0(\alpha(x), \alpha(y)) = \alpha(x) \oplus \alpha(y).$$

According to the first equality (a), it follows $\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y)$, and α is a group morphism. Similarly, we prove that β is a group morphism.

Since (ω, ω_0) is a groupoid morphism, from (1) it follows.

(c) $\omega \circ (\varepsilon \times \varepsilon) = \varepsilon \circ \omega_0$ and $i \circ \omega = \omega \circ (i \times i)$.

For all $u, v \in G_0$, we have $\omega((\varepsilon \times \varepsilon)(u, v)) = \omega(\varepsilon(u), \varepsilon(v)) = \varepsilon(u) \oplus \varepsilon(v)$ and $\varepsilon(\omega_0(u, v)) = \varepsilon(u \oplus v)$. From the first equality (c), it follows $\varepsilon(u \oplus v) = \varepsilon(u) \oplus \varepsilon(v)$. Hence, ε is a group morphism.

For all $x, y \in G$, we have $i(\omega(x, y)) = i(x \oplus y)$ and $\omega(i(x), i(y)) = i(x) \oplus i(y)$. Using the second equality (c), it follows $i(x \oplus y) = i(x) \oplus i(y)$, and i is a group morphism.

(iii) Since (σ, σ_0) is a groupoid morphism, for all $(x, y) \in G_{(2)}$ we have

$$\sigma(m(x, y)) = m(\sigma(x), \sigma(y));$$

i.e., $\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y)$. Hence (3) holds.

□

The relation (2) (resp., (3)) is called the *interchange law* between groupoid multiplication m and group operation ω (resp., σ).

We say that the group-groupoid $(G, \alpha, \beta, m, i, \varepsilon, \oplus, G_0)$ is a *commutative group-groupoid*, if the groups G and G_0 are commutative.

Remark 1.

(i) Let (G, G_0) be a $\mathcal{G}-$groupoid. For all $x, y \in G$, we have

$$\sigma(x \oplus y) = \sigma(y) \oplus \sigma(x) \text{ and } \sigma(\sigma(x)) = x;$$

(ii) If (G, G_0) is a commutative group-groupoid, then

$$\overline{x \oplus y} = \bar{x} \oplus \bar{y}, \quad (\forall)\ x, y \in G.$$

Theorem 3. *If $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a $\mathcal{G}-$groupoid, then:*

$$e \cdot y = y, \quad (\forall) y \in \alpha^{-1}(e_0) \text{ and } x \cdot e = x, \quad (\forall) x \in \beta^{-1}(e_0); \tag{4}$$

$$x \cdot (y \oplus t) = x \cdot y \oplus t, \quad (\forall)(x, y) \in G_{(2)} \text{ and } t \in \alpha^{-1}(e_0); \tag{5}$$

$$(x \oplus z) \cdot y = x \cdot y \oplus z, \quad (\forall)(x, y) \in G_{(2)} \text{ and } z \in \beta^{-1}(e_0). \tag{6}$$

Proof. If $y \in \alpha^{-1}(e_0)$, then $\alpha(y) = e_0$. We have $\beta(\varepsilon(e_0)) = e_0$, since $\beta \circ \varepsilon = Id_{G_0}$. So $(\varepsilon(e_0), y) \in G_{(2)}$. Using the condition $(G2)$ from Definition 1, one obtains
$e \cdot y = \varepsilon(e_0) \cdot y = \varepsilon(\alpha(y)) \cdot y = y$. Hence the first relation of (4) holds. Similarly, we prove that the second relation of (4) hold.

For to prove the relation (5) we apply the interchange law (2) and (4). Indeed, if in (2) we replace z with e, one obtains

$$(x \cdot y) \oplus (e \cdot t) = (x \oplus e) \cdot (y \oplus t), \quad (\forall)(x, y), (e, t) \in G_{(2)}.$$

It follows $(x \cdot y) \oplus t = x \cdot (y \oplus t)$, since $x \oplus e = x, \beta(e) = e_0$ and $t \in \alpha^{-1}(e_0)$. Hence, the relation (5) holds. Similarly, if in (2) we replace t with e, one obtains

$$(x \cdot y) \oplus (z \cdot e) = (x \oplus y) \cdot (y \oplus e), \quad (\forall)(x, y), (y, e) \in G_{(2)}.$$

It follows $(x \cdot y) \oplus z = (x \oplus z) \cdot y$, since $y \oplus e = y$, $\alpha(e) = e_0$ and $z \in \beta^{-1}(e_0)$. Hence, the relation (6) holds. □

Theorem 4. *[3] If $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a $\mathcal{G}-$groupoid, then:*

$$x \cdot y = x \oplus \overline{\varepsilon(\beta(x))} \oplus y, \quad (\forall)(x, y) \in G_{(2)}; \tag{7}$$

$$x^{-1} = \varepsilon(\alpha(x)) \oplus \bar{x} \oplus \varepsilon(\beta(x)), \quad (\forall)x \in G. \tag{8}$$

Proof. Let $(x, y) \in G_{(2)}$. Then $\beta(x) = \alpha(y)$. We have

$$x \cdot y = (x \oplus (\overline{\varepsilon(\beta(x))} \oplus \varepsilon(\beta(x)))) \cdot (e \oplus y),$$

since $\overline{\varepsilon(\beta(x))} \oplus \varepsilon(\beta(x)) = e$, $x \oplus e = x$ and $e \oplus y = y$.

From the associativity of the law $\oplus$ and $\beta(x) = \alpha(y)$, one obtains

$$x \cdot y = ((x \oplus \overline{\varepsilon(\beta(x))}) \oplus \varepsilon(\alpha(y))) \cdot (e \oplus y).$$

Applying the interchange law (2), the relations (4) and $(G.2)$, we have

$$x \cdot y = ((x \oplus (\overline{\varepsilon(\beta(x))} \cdot e) \oplus (\varepsilon(\alpha(y)) \cdot y) \quad \Rightarrow \quad x \cdot y = x \oplus \overline{\varepsilon(\beta(x))} \oplus y.$$

Hence, the relation (7) holds. Applying the fact that α is a group morphism and the relation $\alpha \circ \varepsilon = Id_{G_0}$, one obtains

$$\alpha(a) = \alpha(\varepsilon(\alpha(x))) \oplus \alpha(\bar{x}) \oplus \alpha(\varepsilon(\beta(x))) = \alpha(x) \oplus \alpha(\bar{x}) \oplus \beta(x) = \alpha(x \oplus \bar{x}) \oplus \beta(x) = \beta(x).$$

From $\alpha(a) = \beta(x)$ it follows that the product $x \cdot a$ is defined.

Applying the interchange law (2) and (4), we have

$$\begin{aligned} x \cdot a =& (e \oplus x) \cdot ((\varepsilon(\alpha(x)) \oplus \bar{x}) \oplus \varepsilon(\beta(x))) = (e \cdot (\varepsilon(\alpha(x)) \oplus \bar{x})) \oplus (x \cdot \varepsilon(\beta(x))) \\ =& \varepsilon(\alpha(x)) \oplus \bar{x} \oplus x = \varepsilon(\alpha(x)). \end{aligned}$$

Hence, $x \cdot a = \varepsilon(\alpha(x))$. Similarly, we verify that $a \cdot x = \varepsilon(\beta(x))$. Then $a = x^{-1}$ and the relation (8) holds. □

Corollary 1. *If $(G,\alpha,\beta,m,\varepsilon,i,\oplus,G_0)$ is a $\mathcal{G}-$groupoid, then:*

$$x\cdot y = x\oplus y \text{ and } x^{-1}=\bar{x}, \quad (\forall)x,y\in G(e_0). \tag{9}$$

Proof. Let $x,y\in G(e_0)$. Then $\alpha(x)=\alpha(y)=\beta(x)=\beta(y)=e_0$ and $(x,y)\in G_{(2)}$. Applying (7), we have $x\cdot y=x\oplus y$, since $\overline{\varepsilon(\beta(x))}=\overline{\varepsilon(e_0)}=e$. Hence, the first equality from (9) holds. Also, we have $\varepsilon(\alpha(x))=\varepsilon(\beta(x))=\varepsilon(e_0)=e$. Applying now (8), we have $x^{-1}=\bar{x}$. Hence, the second equality from (9) holds. □

3. Category of Vector Space-groupoids

Let $(V,\alpha,\beta,m,\varepsilon,i,V_0)$ be a groupoid. We suppose that V (resp., V_0) is a vector space over a field K. For the binary operation and unary operation in the group V (resp., V_0) we will use the notations $\omega:=+$ (resp., $\omega_0:=+$) and $\sigma(x):=-x$, $x\in V$ (resp., $\sigma_0(u):=-u$, $u\in V_0$). The null vector of V (resp., V_0) is e (resp., e_0). The scalar multiplication $\varphi:K\times V\to V$ (resp., $\varphi_0:K\times V_0\to V_0$) is given by

$$(k,x)\mapsto\varphi(k,x):=kx \text{ (resp., } (k,u)\mapsto\varphi_0(k,u):=ku).$$

Consider the direct product $(K\times V, Id\times\alpha, Id\times\beta, Id\times m, Id\times\varepsilon, Id\times i, K\times V_0)$ of the null groupoid associated to K and groupoid (V,V_0). Its set of composable elements is $(K\times V)_{(2)}=\{((k_1,x),(k_2,y))\in(K\times V)^2 \mid k_1=k_2,\ \beta(x)=\alpha(y)\}$.

The multiplication in $K\times V$ is given by

$$(k,x)\cdot(k,y):=(k,x\cdot y), \quad (\forall)(x,y)\in V_{(2)}, k\in K.$$

Definition 5. *A* **vector space-groupoid** *or* $\mathcal{VS}-$**groupoid**, *is a groupoid (V,V_0) such that the following conditions hold:*

(5.1) $(V,+,\varphi)$ *and* $(V_0,+,\varphi_0)$ *are vector spaces;*

(5.2) $(V,\alpha,\beta,m,\varepsilon,i,+,V_0)$ *is a commutative group-groupoid;*

(5.3) The pair $(\varphi,\varphi_0):(K\times V,K\times V_0)\to(V,V_0)$ *is a groupoid morphism.*

We shall denote a vector space-groupoid by $(V,\alpha,\beta,m,i,\varepsilon,+,\varphi,V_0)$ or (V,V_0).

Theorem 5. *If $(V,\alpha,\beta,m,i,\varepsilon,+,\varphi,V_0)$ is a vector space-groupoid, then:*

(i) the multiplication m and the additive operation ω are compatible, that is:

$$(x\cdot y)+(z\cdot t)=(x+z)\cdot(y+t), \quad (\forall)(x,y),(z,t)\in V_{(2)}; \tag{10}$$

(ii) the structure functions $\alpha,\beta:(V,+)\to(V_0,+)$, $\varepsilon:(V_0,+)\to(V,+)$ and $i:(V,+)\to(V,+)$ are linear maps;

(iii) the multiplication m and the scalar multiplication φ are compatible, that is:

$$k(x\cdot y)=(kx)\cdot(ky),\quad (\forall)(x,y)\in V_{(2)} \text{ and } k\in K; \tag{11}$$

(iv) the multiplication m and the unary operation σ are compatible, that is:

$$-(x\cdot y)=(-x)\cdot(-y),\quad (\forall)(x,y)\in V_{(2)}. \tag{12}$$

Proof. (i) and (iv). Since $(V,\alpha,\beta,m,\varepsilon,i,+,V_0)$ is a group-groupoid, it follows that the relation (10) holds and the structure functions $\alpha,\beta,\varepsilon,i$ are morphisms from the corresponding additive groups. Also, Theorem 2(iii) it implies that the equality (12) is verified.

From the fact that (φ,φ_0) is a groupoid morphism, we have:

(a) $\alpha\circ\varphi=\varphi_0\circ(Id\times\alpha)$ and $\beta\circ\varphi=\varphi_0\circ(Id\times\beta)$;

(b) $\varphi((Id\times m)((k,x),(k,y)))=m(\varphi(k,x),\varphi(k,y))$, $(\forall)(x,y)\in V_{(2)},k\in K$.

(ii) and (iii). For each $(k,x)\in K\times V$, we have $\alpha(\varphi(k,x))=\alpha(kx)$ and $\varphi_0((Id\times\alpha)(k,x))=\varphi_0(k,\alpha(x))=k\alpha(x)$.

According to the first equality (a), it follows $\alpha(kx)=k\alpha(x)$, and α is a linear map. Similarly, we prove that β is a linear map.

We have $\varphi((Id\times m)((k,x),(k,y)))=\varphi(k,m(x,y))=km(x,y)=k(x\cdot y)$ and $m(\varphi(k,x),\varphi(k,y))=m(kx,ky)=(kx)\cdot(ky)$. Using (b) one obtains $k(x\cdot y)=(kx)\cdot(ky)$, and (11) holds.

Since (φ,φ_0) is a groupoid morphism, from (1) it follows

(c) $\varphi\circ(Id\times\varepsilon)=\varepsilon\circ\varphi_0$ and $i\circ\varphi=\varphi\circ(Id\times i)$.

For all $u\in V_0$ and $k\in K$, we have $\varphi((Id\times\varepsilon)(k,u))=\varphi(k,\varepsilon(u))=k\varepsilon(u)$ and $\varepsilon(\varphi_0(k,u))=\varepsilon(ku)$. From the first equality (c), it follows $\varepsilon(ku)=k\varepsilon(u)$. Hence, ε is a linear map.

For all $x\in V$ and $k\in K$, we have $i(\varphi(k,x))=i(kx)$ and $\varphi(k,i(x))=ki(x)$. Using the second equality (c), it follows $i(kx)=ki(x)$, and i is a linear map. □

The relation (10) (resp., (11)) is called the *interchange law* between groupoid multiplication m and scalar multiplication ω (resp., φ). The relation (12) is called the *interchange law* between groupoid multiplication m and group operation σ.

From the Theorems $5,1,3,4$ follows the following corollary.

Corollary 2. *Let $(V,\alpha,\beta,m,i,\varepsilon,+,\varphi,V_0)$ be a $\mathcal{VS}-$groupoid. Then:*

(i) The source and target $\alpha,\beta:V\to V_0$ are surjective linear maps, and

$$\alpha(e)=\beta(e)=e_0,\quad \alpha(-x)=-\alpha(x) \text{ and } \beta(-x)=-\beta(x),\quad (\forall)\ x\in V;$$

(ii) The inclusion map $\varepsilon:V_0\to V$ is an injective linear map, and

$$\varepsilon(e_0)=e,\quad \varepsilon(-u)=-\varepsilon(u),\quad (\forall)\ u\in V_0;$$

(iii) The inversion $i : V \to V$ is a linear automorphism, and

$$i(e) = e, \quad i(-x) = -i(x), \quad (\forall)\ x \in V;$$

(iv) The following assertions hold:

$$e \cdot y = y, \quad (\forall) y \in \alpha^{-1}(e_0) \text{ and } x \cdot e = x, \quad (\forall) x \in \beta^{-1}(e_0); \tag{13}$$

$$x \cdot (y + t) = x \cdot y + t, \quad (\forall)(x, y) \in V_{(2)} \text{ and } t \in \alpha^{-1}(e_0); \tag{14}$$

$$(x + z) \cdot y = x \cdot y + z, \quad (\forall)(x, y) \in V_{(2)} \text{ and } z \in \beta^{-1}(e_0); \tag{15}$$

$$x \cdot y = x + y - \varepsilon(\beta(x)), \quad (\forall)(x, y) \in V_{(2)}; \tag{16}$$

$$x^{-1} = \varepsilon(\alpha(x)) + \varepsilon(\beta(x)) - x, \quad (\forall) x \in V. \tag{17}$$

Corollary 3. *If $(V, \alpha, \beta, m, i, \varepsilon, +, \varphi, V_0)$ is a $\mathcal{VS}-$groupoid, then:*

$$x \cdot y = x + y \text{ and } x^{-1} = -x, \quad (\forall) x, y \in V(e_0). \tag{18}$$

Proof. It follows immediately from (16) and (17). □

Theorem 6. *Let $(V, \alpha, \beta, m, \varepsilon, i, V_0)$ be a groupoid. If the following conditions are satisfied:*

(i) $(V, +, \varphi)$ and $(V_0, +, \varphi_0)$ are vector spaces;

(ii) $\alpha, \beta : V \to V_0$, $\varepsilon : V_0 \to V$ and $i : V \to V$ are linear maps;

(iii) the interchange law (10) between the operations m and ω holds,

then $(V, \alpha, \beta, m, \varepsilon, i, +, \varphi, V_0)$ is a vector space-groupoid.

Proof. By hypothesis, the condition (5.1) from Definition 5 is verified. We prove now the condition (5.2) from Definition 5 is satisfied. The condition (i) from Definition 4 holds, since (V, ω, ν, σ) and $(V_0, \omega_0, \nu_0, \sigma_0)$ are commutative groups.

(a) We prove that $(\omega, \omega_0) : (V \times V, V_0 \times V_0) \to (V, V_0)$ is a morphism of groupoids. Since α is a morphism of groups, it follows $\alpha(x + y) = \alpha(x) + \alpha(y)$, for all $x, y \in V$. Then $\alpha(\omega(x, y)) = \omega_0(\alpha(x), \alpha(y))$, and it follows

$$\alpha(\omega(x, y)) = \omega_0((\alpha \times \alpha)(x, y));$$

i.e., $\alpha \circ \omega = \omega_0 \circ (\alpha \times \alpha)$. Similarly, we prove that $\beta \circ \omega = \omega_0 \circ (\beta \times \beta)$. Hence the condition $(i1)$ from Definition 3(i) is satisfied.

We suppose that the interchange law (10) holds. Then, for all (x, y) and (z, t) in $G_{(2)}$ we have $(x \cdot y) + (z \cdot t) = (x + z) \cdot (y + t)$. From the last equality it follows

$$m(x, y) \oplus m(z, t) = \omega(x, z) \cdot \omega(y, t) \quad \Rightarrow \quad \omega(m(x, y), m(z, t)) = m(\omega(x, z), (\omega(y, t)).$$

Then $\omega(m_{G \times G}((x, y), (z, t))) = m(\omega(x, z), (\omega(y, t))$, and the condition $(i2)$ from Definition 3(i) holds. Hence, (ω, ω_0) is a groupoid morphism.

(b) We prove that (ν,ν_0) is a morphism of groupoids (here $\{\lambda\}$ is regarded as null groupoid with the structure functions α'_0, β'_0, ε'_0, i'_0 and multiplication m'_0). Since α and ε are group morphisms, we have $\alpha(e)=e_0$ and $\varepsilon(e_0)=e$. From $\alpha(\nu(\lambda))=\alpha(e)=e_0$ and $\nu_0(\lambda)=e_0$, it follows $\alpha\circ\nu=\nu_0\circ Id$. Similarly, we have $\beta\circ\nu=\nu_0\circ Id$. Also, we have $\nu(m'_0(\lambda,\lambda))=\nu(\lambda)=e$ and $m(\nu(\lambda),\nu(\lambda))=e\cdot e=\varepsilon(\alpha(e))\cdot e=e$. Then, $\nu(m'_0(\lambda,\lambda))=m(\nu(\lambda),\nu(\lambda))$. Hence, (ν,ν_0) is a groupoid morphism.

(c) We prove that (σ,σ_0) is a groupoid morphism. Applying the fact that α is group morphism, we have $\alpha(\sigma(x))=\alpha(-x)=-\alpha(x)$ and $\sigma_0(\alpha(x))=-\alpha(x)$. Then $\alpha\circ\sigma=\sigma_0\circ\alpha$. Similarly, we have $\beta\circ\sigma=\sigma_0\circ\beta$. We shall prove that:

(c1) $-x\cdot y=(-x)\cdot(-y)$, $(\forall)\ (x,y)\in V_{(2)}$.

From $(x,y)\in V_{(2)}$ we have $\beta(x)=\alpha(y)$. Then $\beta(-x)=\alpha(-y)$. Therefore $(-x,-y)\in V_{(2)}$. Using now (10) one obtains

(c2) $(x\cdot y)+((-x)\cdot(-y))=(x+(-x))\cdot(y+(-y))$ and

(c3) $((-x)\cdot(-y))+(x\cdot y)=((-x)+x)\cdot((-y)+y)$.

Since $a+(-a)=(-a)+a=e$, and $e\cdot e=e$, from $(c2)$ and $(c3)$, we have

(c4) $(x\cdot y)+((-x)\cdot(-y))=e$ and $((-x)\cdot(-y))+(x\cdot y)=e$.

From $(c4)$ one obtains that the equality $(c1)$ holds. The relation $(c1)$ is equivalently with $\sigma(x\cdot y)=sigma(x)\cdot\sigma(y)$. Then $\sigma(m(x,y))=m(\sigma(x),\sigma(y))$. Hence, (σ,σ_0) is a groupoid morphism. Therefore, $(V,\alpha,\beta,m,\varepsilon,i,+,V_0)$ is a commutative group-groupoid and the condition (5.2) from Definition 5 holds.

We shall prove that $(\varphi,\varphi_0):(K\times V,K\times V_0)\to(V,V_0)$ is a groupoid morphism.

Applying the fact that α is a linear map, for all $x\in V$ and $k\in K$ we have $\alpha(\varphi(k,x))=\alpha(kx)=k\alpha(x)$ and $\varphi_0((Id\times\alpha)(k,x))=\varphi_0(k,\alpha(x))=k\alpha(x)$. Then $\alpha\circ\varphi=\varphi_0\circ(Id\times\alpha)$. Similarly, we have $\beta\circ\varphi=\varphi_0\circ(Id\times\beta)$.

We consider $x,y\in V$ such that $(x,y)\in V_{(2)}$. We have also $(kx,ky)\in V_2$. Indeed, using the linearity of α and β, from $\beta(x)=\alpha(y)$ follows $\beta(kx)=\alpha(ky)$.

Applying now the relation (16), linearity of ε and β and the fact that V is a vector space, we have $k(x\cdot y)=k(x+y-\varepsilon(\beta(x)))=kx+ky-k\varepsilon(\beta(x))$ and

$$(kx)\cdot(ky)=kx+ky-\varepsilon(\beta(kx)))=kx+ky-k\varepsilon(\beta(x)).$$

Then $k(x\cdot y)=(kx)\cdot(ky)$, for all $(x,y)\in V_{(2)}$ and $k\in K$; i.e. the interchange law (11) holds. From (11) it follows

$$\varphi(k,m(x,y))=\varphi(k,x)\cdot\varphi(k,y)\ \Rightarrow\ \varphi((Id\times m)((k,x),(k,y)))=m(\varphi(k,x),\varphi(k,y)).$$

Therefore, (φ,φ_0) is a groupoid morphism. Hence, $(V,\alpha,\beta,m,\varepsilon,i,+,\varphi,V_0)$ is vector space-groupoid.

□

According to Theorems 5 and 6, we can give another definition for the notion of vector space-groupoid (this is equivalent with Definition 5).

Definition 6. *A* **vector space-groupoid** *is a groupoid* $(V, \alpha, \beta, m, \varepsilon, i, V_0)$ *such that the following conditions are satisfied:*

(i) $(V, +, \varphi)$ *and* $(V_0, +, \varphi_0)$ *are vector spaces;*

(ii) $\alpha, \beta : V \to V_0$, $\varepsilon : V_0 \to V$ *and* $i : V \to V$ *are linear maps;*

(iii) the interchange law (10) between the operations m *and* ω *holds.*

If in Definition 6, we consider $V_0 \subseteq V$ and $\varepsilon : V_0 \to V$ is the inclusion map, then $(V, \alpha, \beta, m, i, +, \varphi, V_0)$ is a vector space-groupoid. In this case, we will say that (V, V_0) is a *vector space*$-V_0-$*groupoid*.

Example 2.

(i) Let $(V, +, \varphi)$ *be a vector space. Then* V *has a structure of null groupoid over* V *(see Example 1*(i)*). We have that* $V_0 = V$ *and* $\alpha, \beta, \varepsilon, i$ *are linear maps. It is easy to verify that the interchange law (10) holds. Then* V *is a vector space-groupoid, called the* **null vector space-groupoid** *associated to* V.

(ii) Let $(V, +, \varphi)$ *be a vector space having* $\{e\}$ *as null vector. The set* V *is a* $\{e\}-$*groupoid (see Example 1*(ii)*). In this case,* $m = +$. *We have that* $V_0 := \{e\}$ *is a vector subspace in* V *and* α, β, ε *and* i *are linear maps. The relation (10) holds. Indeed, for* $x, y, z, t \in V$ *we have* $(x + y) + (z + t) = (x + z) + (y + t)$, *since the addition operation is associative and commutative. Hence* $(V, \alpha, \beta, m, \varepsilon, i, +, \varphi, \{e\})$ *is a vector space-groupoid called the* **vector space-groupoid with a single unit** *associated to* V. *Therefore, each vector space* V *has a structure of vector space*$-\{e\}-$*groupoid.*

Definition 7. *Let* $(V, \alpha, \beta, m, \varepsilon, i, +, \varphi, \{e\})$ *be a vector space-groupoid.*

(i) By a **vector space-subgroupoid** *(resp.,* **vector space-wide subgroupoid** *or* **vector space** $-V_0-$**subgroupoid***) of* (V, V_0), *we mean a subgroupoid (resp., wide subgroupoid)* (W, W_0) *of the groupoid* (V, V_0) *with the property that* W *and* W_0 *are vector subspaces in* V *and* V_0, *respectively.*

(ii) A vector space-subgroupoid (N, V_0) *of* (V, V_0) *is called* **vector space-normal subgroupoid***, if* (N, V_0) *is a normal subgroupoid of the groupoid* (V, V_0).

According to the Definition 6, if (W, W_0) is a vector space-subgroupoid of (V, V_0), then the pair (W, W_0) endowed with the restrictions of the functions α, β, i and $+$ to W, the restriction of ε to W_0 and the restriction of m to $W_{(2)}$, is a vector space-groupoid, denoted by (W, W_0).

Theorem 7. *Let* $(V, \alpha, \beta, m, \varepsilon, i, +, \varphi, V_0)$ *be a vector space-groupoid. Then:*

(i) The fibres $\alpha^{-1}(e_0)$ and $\beta^{-1}(e_0)$ are vector subspaces in V.

(ii) The isotropy group $V(e_0)$ is a vector space$-\{e_0\}-$subgroupoid of V.

(iii) $\varepsilon(V_0)$ is a vector space $-$normal subgroupoid of V.

(iv) $Is(V) := \{x \in V \mid \alpha(x) = \beta(x)\}$ is a vector space $-$normal subgroupoid of V.

Proof.

(i) For all $x, y \in \alpha^{-1}(e_0)$ and $k \in K$, we have $\alpha(x-y) = \alpha(x) - \alpha(y) = e_0$ and $\alpha(kx) = k\alpha(x) = ke_0 = e_0$. Then $x - y, kx \in \alpha^{-1}(e_0)$. Hence $\alpha^{-1}(e_0)$ is a vector subspace. Similarly, we prove that $\beta^{-1}(e_0)$ is a vector subspace.

(ii) $V(e_0)$ is a vector subspace, since $V(e_0) = \alpha^{-1}(e_0) \cap \beta^{-1}(e_0)$. Also, $V(e_0)$ is a $\{e_0\}-$subgroupoid. Then, $V(e_0)$ is a vector space$-\{e_0\}-$subgroupoid of V.

For to prove the following assertions, we apply the Theorem 1.

(iii) For $x, y \in \varepsilon(V_0)$ there exist $u, v \in V_0$ such that $\varepsilon(u) = x$ and $\varepsilon(v) = y$. It follows $\beta(x) = \beta(\varepsilon(u)) = u$ and $\alpha(y) = \alpha(\varepsilon(v)) = v$. We suppose that the product $x \cdot y$ is defined. From $\beta(x) = \alpha(y)$ it follows $x = y$. Then $x \cdot y = \varepsilon(u) \cdot \varepsilon(u) = \varepsilon(u) \in \varepsilon(V_0)$. Also, for $x \in \varepsilon(V_0)$, we have $x^{-1} = i(x) = i(\varepsilon(u)) = \varepsilon(u) \in \varepsilon(V_0)$. Let now $a \in V$ and $x \in \varepsilon(V_0)$ such that $a \cdot x \cdot a^{-1}$ is defined. From $x = \varepsilon(u)$ and $\beta(a) = \alpha(x)$ it follows $\beta(a) = \alpha(\varepsilon(u)) = u$ and $x = \varepsilon(\beta(a))$. Then

$$a \cdot x \cdot a^{-1} = (a \cdot \varepsilon(\beta(a))) \cdot a^{-1} = a \cdot a^{-1} = \varepsilon(\alpha(a)) \in \varepsilon(V_0).$$

Hence, $\varepsilon(V_0)$ is a normal subgroupoid. Also, $\varepsilon(V_0)$ is a vector subspace in V, since $\varepsilon : V_0 \to V$ is a linear map. Therefore, $\varepsilon(V_0)$ is a vector space-normal subgroupoid.

(iv) Clearly, $\alpha(Is(V)) = \beta(Is(V)) = V_0$. Let $x, y \in Is(V)$ with $(x, y) \in V_{(2)}$. Then $\alpha(x) = \beta(x) = \alpha(y) = \beta(y)$. We have $\alpha(xy) = \beta(xy)$ and $\alpha(x^{-1}) = \beta(x^{-1})$. It follows that $xy, x^{-1} \in Is(V)$. Let now $a \in V$ and $x \in Is(V)$ such that $a \cdot x \cdot a^{-1}$ is defined. From $\alpha(a \cdot x \cdot a^{-1}) = \alpha(a)$ and $\beta(a \cdot x \cdot a^{-1}) = \beta(a^{-1}) = \alpha(a)$ it follows $\alpha(a \cdot x \cdot a^{-1}) = \beta(a \cdot x \cdot a^{-1})$. Then $a \cdot x \cdot a^{-1} \in Is(V)$ and $Is(V)$ is a normal subgroupoid.

Using the linearity of α and β, we have $\alpha(x-y) = \beta(x-y)$ and $\alpha(kx) = \beta(kx)$ for all $x, y \in Is(V)$ and $k \in K$. Therefore, $Is(V)$ is a vector subspace. Hence, $Is(V)$ is a vector space$-$ normal subgroupoid.

□

The group-subgroupoid $Is(V)$ is the union of all isotropy groups of V and it is called the *isotropy bundle* of the vector space-groupoid (V, V_0).

Example 3. *Let $a \in \mathbb{R}, a \neq 1$. Consider the vector spaces groups $V := \mathbb{R}^3$ and $V_0 := \mathbb{R}$. For (V, V_0), we define the structure functions $\alpha, \beta : \mathbb{R}^3 \to \mathbb{R}$, $\varepsilon : \mathbb{R} \to \mathbb{R}^3$ and*

$i:\mathbb{R}^3\to\mathbb{R}^3$ as follows: $\alpha(x_1,x_2,x_3):=ax_1+x_2$, $\beta(x_1,x_2,x_3):=x_1+x_2$, $\varepsilon(x_2):=(0,x_2,0)$ and $i(x_1,x_2,x_3):=(-x_1,(a+1)x_1+x_2,-x_3)$, for all $x_1,x_2,x_3\in\mathbb{R}$.

Let $V_{(2)}:=\{((x_1,x_2,x_3),(y_1,y_2,y_3))\in\mathbb{R}^3\times\mathbb{R}^3 \mid y_2=x_1+x_2-ay_1\}$ be the set of composable pairs. The multiplication $m:V_{(2)}\to V$ is given by:

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3):=(x_1+y_1,x_2-ay_1,x_3+y_3),\ \text{if } y_2=x_1+x_2-ay_1.$$

It is easy to check that the above structure functions determine on V a structure of a groupoid over V_0. Also, the maps α,β,ε and i are linear maps. Therefore, the conditions (i) and (ii) from the Definition 6 hold.

Let $x,y,z,t\in\mathbb{R}^3$ such that $x\cdot y$ and $z\cdot t$ are defined. Then $x=(x_1,x_2,x_3)$, $y=(y_1,y_2,y_3)$, $z=(z_1,z_2,z_3)$, $t=(t_1,t_2,t_3)$ such that $y_2=x_1+x_1-ay_1$ and $t_2=z_1+z_2-at_1$. We have $x\cdot y=(x_1+y_1,x_2-ay_1,x_3+y_3)$, $z\cdot t=(z_1+t_1,z_2-at_1,z_3+t_3)$. Then

$$(x\cdot y)+(z\cdot t)=(x_1+y_1+z_1+t_1,x_2-ay_1+z_2-at_1,x_3+y_3+z_3+t_3)$$

and

$$(x+z)\cdot(y+t)=(x_1+z_1+y_1+t_1,x_2+z_2-a(y_1+t_1),x_3+z_3+y_3+t_3).$$

Hence, $(x\cdot y)+(z\cdot t)=(x+y)\cdot(z+t)$ and the interchange law (10) holds. Therefore, $(\mathbb{R}^3,\alpha,\beta,m,\varepsilon,i,+,\varphi,\mathbb{R})$ is a vector space-groupoid.

We have $\varepsilon(V_0)=\{(0,u,0)|u\in\mathbb{R}\}$ and $Is(V)=\{(0,u,v)|u,v\in\mathbb{R}\}$ are vector space-normal subgroupoid. The isotropy group at $e_0=0$ is $V(0)=\{(0,0,v)|v\in\mathbb{R}\}$.

Let us we consider the Euclidean space $\mathbb{R}^3$ with the Cartesian coordinate system $Ox_1x_2x_3$. The $\alpha-$fibres $\alpha^{-1}(u)$ for $u\in\mathbb{R}$ are represented by parallel planes of equation $x_1+2x_2-u=0$. Also, the $\beta-$fibres $\beta^{-1}(v)$ for $v\in\mathbb{R}$ are represented by parallel planes of equation $x_1+x_2-v=0$.

Let be the points A_1,A_2,A_3,A_4 associated to elements $\varepsilon(\beta(x)),x,x\cdot y,y\in V$, for $\beta(x)=\alpha(y)$. We have $A_1(0,b_1+b_2,0)$, $A_2(b_1,b_2,b_3)$, $A_3(b_1+c_1,b_2-ac_1,b_3+c_3)$ and $A_4(c_1,b_1+b_2-ac_1,c_3)$. Then: the simple quadrilateral $A_1A_2A_3A_4$ is a parallelogram.

Indeed, the straight line through A_1 and A_4 has the equation: $\frac{x_1}{c_1}=\frac{x_2-(b_1+b_2)}{-ac_1}=\frac{x_3}{c_3}$ and the distance from A_1 and A_4 is $d(A_1,A_4)=\sqrt{(1+a^2)c_1^2+c_3^2}$. Also, the straight line through A_2 and A_3 has the equation: $\frac{x_1-b_1}{c_1}=\frac{x_2-b_2}{-ac_1}=\frac{x_3-b_3}{c_3}$ and the distance from A_2 and A_3 is $d(A_2,A_3)=\sqrt{(1+a^2)c_1^2+c_3^2}$.

Let be the points B_1,B_2,B_3,B_4 associated to $\varepsilon(\alpha(x)),x,\varepsilon(\beta(x)),x^{-1}\in V$. We have $B_1(0,ab_1+b_2,0)$, $B_2(b_1,b_2,b_3)$, $B_3(0,b_1+b_2,0)$ and $B_4(-b_1,(a+1)b_1+b_2,-b_3)$. Then: the simple quadrilateral $B_1B_2B_3B_4$ is a parallelogram.

Indeed, the straight line through B_1 and B_2 has the equation: $\frac{x_1}{b_1}=\frac{x_2-(ab_1+b_2)}{-ab_1}=\frac{x_3}{b_3}$ and $d(B_1,B_2)=\sqrt{(1+a^2)b_1^2+b_3^2}$. Also, the straight line through B_3 and B_4 has the equation: $\frac{x_1}{-b_1}=\frac{x_2-(b_1+b_2)}{ab_1}=\frac{x_3}{-b_3}$ and $d(B_3,B_4)=d(B_1,B_2)$.

Definition 8. *Let $(V_j,\alpha_j,\beta_j,m_j,\varepsilon_j,i_j,+_j,\varphi_j,V_{j,0})$, $j=1,2$ be two vector space-groupoids. A groupoid morphism $(f,f_0):(V_1,V_{1,0})\to(V_2,V_{2,0})$ with property that $f:V_1\to V_2$ and*

$f_0 : V_{1,0} \to V_{2,0}$ are linear maps, is called **vector space-groupoid morphism** *or* **morphism of vector space-groupoids**.

*If $V_{2,0} = V_{1,0}$ and $f_0 = Id_{V_{1,0}}$, then we say that $(f, Id_{V_{1,0}}) : (V_1, V_{1,0}) \to (V_2, V_{1,0})$ is a $V_{1,0}-$***morphism of vector space-groupoids**. *It is denoted by $f : V_1 \to V_2$.*

The category of vector space-groupoids, denoted by $\mathcal{VSG}pd$, has its objects all vector space-groupoids (V, V_0) and as morphisms from (V, V_0) to (V', V_0') the set of all morphisms of vector space-groupoids.

Finally we will present the concept of vector groupoid defined by V. Popuţa and Gh. Ivan [13, 14].

Definition 9 ([13]). *By vector groupoid, we mean a groupoid $(V, \alpha, \beta, m, \varepsilon, i, V_0)$ which verifies the following conditions:*

(9.1) *V and V_0 are vector spaces;*

(9.2) *$\alpha, \beta : V \to V_0$ are linear maps;*

(9.3) *the inclusion $\varepsilon : V_0 \to V$ and the inversion $i : V \to V$ are linear maps and the following condition is verified:*

(9.3.1) $x + i(x) = \varepsilon(\alpha(x)) + \varepsilon(\beta(x))$ *for all* $x \in V$;

(9.4) *the multiplication $m : V_{(2)} \to V$ satisfy the following relations:*

(9.4.1) $x \cdot (y + z - \varepsilon(\beta(x))) = x \cdot y + x \cdot z - x$, $(\forall)\ x, y, z \in V$ *such that* $\alpha(y) = \beta(x) = \alpha(z)$;

(9.4.2) $x \cdot (ky + (1-k)\varepsilon(\beta(x))) = k(x \cdot y) + (1-k)x$, $(\forall)\ (x, y) \in V_{(2)}$;

(9.4.3) $(y + z - \varepsilon(\alpha(x))) \cdot x = y \cdot x + z \cdot x - x$, $(\forall)\ x, y, z \in V$ *such that* $\alpha(x) = \beta(y) = \beta(z)$;

(9.4.4) $(ky + (1-k)\varepsilon(\alpha(x))) \cdot x = k(y \cdot x) + (1-k)x$, $(\forall)\ (y, x) \in V_{(2)}$.

Theorem 8. *Each vector space-groupoid is a vector groupoid in the sense of Definition 9.*

Proof. We suppose that $(V, \alpha, \beta, m, \varepsilon, i, +, \varphi, V_0)$ is a vector space-groupoid. From Definition 6 and (17) it follows that the conditions (9.1) – (9.3) are satisfied.

Let $x, y, z \in V$ such that $\alpha(y) = \beta(x) = \alpha(z)$. Denote $t := y + z - \varepsilon(\beta(x))$. Using the linearity of α, we have $\alpha(t) = \alpha(y) + \alpha(z) - \alpha(\varepsilon(\beta(x))) = \beta(x)$ and $(x, t) \in V_{(2)}$.

Applying (16), we have $x \cdot t = x + t - \varepsilon(\beta(x)) = x + y + z - 2\varepsilon(\beta(x))$ and

$$x \cdot y + x \cdot z - x = (x + y - \varepsilon(\beta(x))) + (x + z - \varepsilon(\beta(x))) - x = x + y + z - 2\varepsilon(\beta(x)).$$

Then, $x \cdot t = x \cdot y + x \cdot z - x$. Hence, the relation (9.4.1) holds.

Let $y, x) \in V_{(2)}$. Denote $v := ky + (1-k)\varepsilon(\alpha(x))$. Using the linearity of β, we have $\beta(v) = k\beta(y) + (1-k)\beta(\varepsilon(\alpha(x))) = \alpha(x)$, since $\beta(y) = \alpha(x)$. Then $(v, x) \in V_{(2)}$.

Applying (16), we have $v \cdot x = v + x - \varepsilon(\beta(v)) = x + ky - k\varepsilon(\alpha(x))$ and

$$k(y \cdot x) + (1-k)x = k(y + x - \varepsilon(\beta(y))) + (1-k)x = x + ky - k\varepsilon(\alpha(x)).$$

Then, $v \cdot x = k(y \cdot x) + (1-k)x$. Hence, (9.4.4) holds. Similarly, we prove that (9.4.2) and (9.4.3) are verified. Therefore, (V, V_0) is a vector groupoid. □

References

[1] H. Brandt. Über eine Verallgemeinerung des Gruppenbegriffes. *Mathematische Annalen*, 96(1): 360-366, 1926.

[2] R. Brown. *Topology and Groupoids*. BookSurge LLC, U.K., 2006.

[3] R. Brown and O. Mucuk. Covering groups of non-connected topological groups revisited. *Mathematical Proceedings of the Cambridge Philosophical Society*, 115: 97-110, 1994.

[4] R. Brown and C.B. Spencer. $G-$groupoids, crossed modules and the fundamental groupoid of a topological group. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Serie A: Mathematical Sciences*, 79: 296-302, 1976.

[5] C. Ehresmann. *Oèuvres complètes*. Dunod, Paris, 1950.

[6] P. J. Higgins. *Notes on Categories and Groupoids*. Von Nostrand Reinhold Mathematical Studies 32, London,1971. Reprints in Theory and Applications of Categories, no. 7: 1-195, 2005.

[7] I. İcen, A. F. Özcan, and M. H. Gürsoy. Topological group-groupoids and their coverings. *Indian Journal of Pure and Applied Mathematics*, 36(9): 493-502, 2005.

[8] Gh. Ivan. Strong morphisms of groupoids. *Balkan Journal of Geometry and Its Applications* (BJGA), 4(1): 91-102, 1999.

[9] K. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry.* London Mathematical Society, Lecture Notes Series, 213, Cambridge University Press, 2005.

[10] O. Mucuk. Covering and ring-groupoids. *Georgian Mathematical Journal*, 5(5): 475-482, 1998.

[11] A.F. Özcan, I. İcen, and M. H. Gürsoy. Topological ring-groupoids and liftings. *Iranian Journal of Science and Technology Transactions. A: Science*, 30(3): 305-362, 2007.

[12] V. Popuţa. Some classes of Brandt Groupoids. *Scientific Bulletin of "Politehnica" University of Timişoara*, 52(66), no. 1:50-54, 2007.

[13] V. Popuţa and Gh. Ivan. A groupoid structure on a vector space. *Bul. Ştiinţ. Univ. Politeh. Timişoara, Ser. Mat. Fiz.*, 56(70), no.1: 55-64, 2011.

[14] V. Popuţa and Gh. Ivan. Vector groupoids. *Theoretical Mathematics & Applications*, 2(2):1-12, 2012.

[15] A. Ramsey and J. Renault. *Groupoids in Analysis, Geometry and Physics.* Contemporary Mathematics, 282, AMS Providence, RI, 2001.