EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 2, 2014, 166-178
ISSN 1307-5543 – www.ejpam.com

Categorical Properties of Regular Monomorphisms of S-posets

Hamid Rasouli

Department of Mathematics, College of Basic Sciences, Tehran Science and Research Branch, Islamic Azad University, Tehran, Iran

Abstract. Recall that monomorphisms in some categories such as the category of posets, and the category of topological spaces, are not necessarily embeddings. The notion of regular monomorphism, solves this problem in these two categories. We have the same situation in the category S-**Pos** of S-posets; that is posets with an action of a pomonoid S which preserves the order. In this category, regular monomorphisms exactly determine sub S-posets.
In this paper, we study some categorical properties of regular monomorphisms in the category of S-posets with action-preserving monotone maps.

2010 Mathematics Subject Classifications: 06F05, 18A20, 18A30, 20M30
Key Words and Phrases: S-poset, regular monomorphism

1. Introduction and Preliminaries

One of the very useful notions in many branches of mathematics as well as in computer science is the action of a semigroup or a monoid on a set. Such acts corresponds to representation of monoids. Also the action of a pomonoid S on partially ordered sets, namely S-posets, appears as representations of mappings between pomonoids (cf. [6]). Brazegar *et al.* [5], inspired by the work of Banaschewski [2] on $\mathcal{M}$-injectivity, introduced three different kinds of essentiality for a subclass $\mathcal{M}$ of monomorphisms of a category, and considered some category-theoretic conditions on $\mathcal{M}$ to study well-behaviour of $\mathcal{M}$-injectivity via these essential monomorphisms. In the present paper, considering $\mathcal{M}$ to be the class of regular monomorphisms (order-embeddings) in the category S-**Pos** of S-posets with action-preserving monotone maps, we investigate some categorical properties of $\mathcal{M}$ which mostly are useful in the study of well-behaviour of regular injectivity of S-posets (see also [4, 13]). Then we compare some categorical properties of monomorphisms and regular monomorphims in the categories of posets and S-posets.

A study of S-posets from a category-theoretic standpoint forms the content of [11], and extends the results found in [8]. For more information on various properties of S-posets, see also [7, 9, 10, 15].

Email address: hrasouli@srbiau.ac.ir

In the rest of this section we give some preliminaries about S-acts, posets, and S-posets needed in the sequel.

Let S be a monoid with identity 1. Recall that a (*left*) *S-act* A is a set equipped with a map $\lambda : S \times A \to A$, called its *action*, such that, denoting $\lambda(s,a)$ by sa, we have $1a = a$ and $(st)a = s(ta)$, for all $a \in A$, and $s,t \in S$. The category of all S-acts, with action-preserving maps between them, is denoted by S-**Act**. An *S-act congruence* θ on A is an equivalence relation with the property that $a\theta a'$, $a,a' \in A$, implies that $sa\theta sa'$, for all $s \in S$. A quotient S-act is the set A/θ with the natural action, $s[a] = [sa]$, which makes the canonical map $\gamma : A \to A/\theta$, $a \mapsto [a]$, an S-act map. For more information about S-acts, see [12].

A monoid S is said to be a *pomonoid* if it is also a poset whose partial order is compatible with the binary operation.

For a pomonoid S, a (*left*) *S-poset* is a poset A which is also an S-act whose action is monotone in both arguments. An *S-poset map* (*morphism*) is an action preserving monotone map between S-posets. Note that each poset P can be made into an S-poset with *trivial action*: $sp = p$, for every $p \in P, s \in S$.

Let A be an S-poset. An *S-poset congruence* on A is an S-act congruence θ with the property that the S-act A/θ can be made into an S-poset in such a way that the canonical S-act map $A \to A/\theta$ is an S-poset map. For a binary relation R on A, define the relation $\leq_R$ on A by $a \leq_R a'$ if and only if

$$a \leq a_1 R a'_1 \leq \ldots \leq a_n R a'_n \leq a',$$

for some $a_1, a'_1, \ldots, a_n, a'_n \in A$. Then an S-act congruence θ on A is an S-poset congruence if and only if $a\theta a'$ whenever $a \leq_\theta a' \leq_\theta a$. The S-poset quotient is then the S-act quotient A/θ with the partial order given by $[a] \leq [b]$ if and only if $a \leq_\theta b$. Also the S-poset congruence $\theta(H)$ on A generated by $H \subseteq A \times A$ can be characterized as follows (see [15, Proposition 3.3]):

$a\theta(H)a'$ if and only if $a = a'$, or there exist $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_m \in S$ such that

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, s_2d_2 \leq s_3c_3, \ldots, s_nd_n \leq a';$$
$$a' \leq t_1p_1, t_1q_1 \leq t_2p_2, t_2q_2 \leq t_3p_3, \ldots, t_mq_m \leq a,$$

where $(c_i, d_i), (p_j, q_j) \in H \cup H^{-1}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Moreover, the order relation on $A/\theta(H)$ can be defined by: $[a] \leq [a']$ if and only if $a \leq a'$, or there exist $s_1, s_2, \ldots, s_n \in S$ such that

$$a \leq s_1c_1, s_1d_1 \leq s_2c_2, s_2d_2 \leq s_3c_3, \ldots, s_nd_n \leq a',$$

where $(c_i, d_i) \in H \cup H^{-1}$ for $i = 1, 2, \ldots, n$.

Recall that the *product* of a family of S-posets is their cartesian product, with componentwise action and order. The *coproduct* is their disjoint union, with natural action and componentwise order. As usual, we use the symbols $\prod$ and $\coprod$ for product and coproduct, respectively. Also for a family $(A_\alpha)_{\alpha \in I}$ of S-posets each with a unique fixed element 0, the *direct sum* $\bigoplus A_\alpha$ is defined to be the sub S-poset of the product $\prod A_\alpha$ consisting of all $(a_\alpha)_{\alpha \in I}$ such that $a_\alpha = 0$ for all $\alpha \in I$ except a finite number of indices.

Throughout S denotes a pomonoid unless otherwise stated, and $\mathscr{M}$ stands for the class of regular monomorphisms of S-posets.

2. Categorical Properties of Regular Monomorphisms in S-Pos

In this section we investigate the categorical and algebraic properties, regarding composition, limits and colimits of the category S-**Pos** with respect to the class $\mathscr{M}$ of regular monomorphisms which are exactly order-embeddings.

We have divided the section into three subsections as follows:

2.1. Composition Properties of Regular Monomorphisms

In this subsection we study the composition properties of regular monomorphisms of S-posets. To find the importance of these properties, the reader is referred to [1, 2].

Lemma 1. *The class $\mathscr{M}$ is:*

i) Composition closed; that is, if $f : A \to B$ and $g : B \to C$ belong to $\mathscr{M}$, then gf also belongs to $\mathscr{M}$.

ii) Isomorphism closed; that is, it contains all isomorphisms and is closed under composition with isomorphisms.

iii) Left cancellable; that is, if $gf \in \mathscr{M}$, then $f \in \mathscr{M}$.

Proof. Since $\mathscr{M}$ is in fact the class of all order-embeddings, the proof is obvious. □

Remark 1. *For every pomonoid S, the class $\mathscr{M}$ is not right cancellable. For example, consider the inclusions $\mathbf{2} \overset{\mathbf{f}}{\hookrightarrow} \mathbf{2}\dot{\cup}\mathbf{1} \overset{\mathbf{g}}{\hookrightarrow} \mathbf{3}$ with trivial actions of a pomonoid S on $\mathbf{2}$, $\mathbf{2}\dot{\cup}\mathbf{1}$ and $\mathbf{3}$. Then $gf \in \mathscr{M}$ but g is not in $\mathscr{M}$. Also, for every pomonoid S, there always exists a monomorphism that is not regular. To see this, it suffices to take the inclusion $i : \mathbf{1}\dot{\cup}\mathbf{1} \hookrightarrow \mathbf{2}$ with trivial actions of a pomonoid S on $\mathbf{1}\dot{\cup}\mathbf{1}$ and $\mathbf{2}$.*

Here we investigate the factorization property in S-**Pos**. Recall that for two morphisms $f : A \to B$ and $g : C \to D$ in a category $\mathscr{C}$, f is called *vertical* on g if for morphisms $u : A \to C$ and $v : B \to D$ which $v \circ f = g \circ u$, there exists a unique morphism $w : B \to C$ such that $w \circ f = u$ and $g \circ w = v$. Also a *factorization diagonalization system* for $\mathscr{C}$, is a pair $(\mathscr{E}, \mathscr{M})$ where $\mathscr{E}$ and $\mathscr{M}$ are some classes of morphisms, with the following properties:

- $\mathscr{E}$ and $\mathscr{M}$ are composition and isomorphism closed.
- For every $e \in \mathscr{E}$ and $m \in \mathscr{M}$, e is vertical on m.
- Every morphism $f \in \mathscr{C}$ has a factorization of the form $f = m \circ e$ in which $m \in \mathscr{M}$ and $e \in \mathscr{E}$.

In this case, we say that $\mathscr{C}$ has $(\mathscr{E}, \mathscr{M})$-*factorization diagonalization* property.

To find a factorization diagonalization system for S-posets, we use the decomposition and first isomorphism theorems (see [7, Proposition 2.3], [8, Theorem 1]). First recall that for every S-poset homomorphism $f : A \to B$, the *subkernel* of f is defined by

$$K_f = \{(a, a') \in A \times A | f(a) \leq f(a')\}.$$

Proposition 1. *Let $\mathscr{E}$ be the class of all S-poset epimorphisms. Then S-**Pos** has $(\mathscr{E},\mathscr{M})$-factorization diagonalization property.*

Proof. First, notice that in view of Lemma 1, $\mathscr{E}$ and $\mathscr{M}$ are composition and isomorphism closed. Also [7, Proposition 2.3] implies that each S-poset homomorphism $f : A \to B$ can be decomposed as $f = me$, where

$$e = \overline{f}\pi : A \xrightarrow{\pi} A/kerf \overset{\overline{f}}{\simeq} f(A),$$

and $m = i : f(A) \hookrightarrow B$. Clearly, $m \in \mathscr{M}$ and $e \in \mathscr{E}$. Now, we show that every $e \in \mathscr{E}$ is vertical on every $m \in \mathscr{M}$. Let $u : A \to C$ and $v : B \to D$ be some S-poset homomorphisms such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "m"] & D \end{tikzcd}$$

We claim that $K_e \subseteq K_u$. Suppose $e(a) \leq e(a')$, for $a, a' \in A$. Then

$$mu(a) = ve(a) \leq ve(a') = mu(a').$$

Since m is a regular monomorphism, we get $u(a) \leq u(a')$, as claimed. Using [8, Theorem 1], there exists a unique $d : B \to C$ such that $de = u$. Also we have $mde = mu = ve$ and hence $md = v$ because e is an epimorphism. This completes the proof. □

2.2. Limits of regular monomorphisms

In this subsection some of the categorical properties of regular monomorphisms related to limits such as products and pullbacks are studied.

Proposition 2. *i) The class $\mathscr{M}$ is closed under products.*

ii) Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a family of regular monomorphisms. Then their product homomorphism $f : A \to \prod B_\alpha$ is also a regular monomorphism.

Proof. It is straightforward. □

Proposition 3. *Let $\{f_\alpha : A \to B_\alpha | \alpha \in I\}$ be a source of regular monomorphisms. Then the homomorphism $f : A \to \varprojlim B_\alpha$ (existing by the universal property of limits) is also a regular monomorphism.*

Proof. Let $f(a) \leq f(a')$ for some $a, a' \in A$. For every $\alpha \in I$, we have

$$f_\alpha(a) = \pi_\alpha f(a) \leq \pi_\alpha f(a') = f_\alpha(a'),$$

where $\pi_\alpha : \varprojlim B_\alpha \to B_\alpha$ is a limit morphism. Since f_α is a regular monomorphism by the assumption, $a \leq a'$. Thus f is a regular monomorphism. □

The above result as well as Proposition 2 (ii) are also true whenever for some (not necessarily all) $\alpha \in I$, f_α is a regular monomorphism.

Recall that a class of morphisms of a category is called *pullback stable* if pullbacks transfer those morphisms. In the next result, we study this property for regular monomorphisms of S-posets.

Proposition 4. *The class $\mathscr{M}$ is pullback stable.*

Proof. Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r, "p_B"] \arrow[d, "p_A"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

where P is the sub S-poset $\{(a,b) : f(a) = g(b)\}$ of $A \times B$, and pullback maps $p_A : P \to A$, $p_B : P \to B$ are restrictions of the projection maps. Assume that $f \in \mathscr{M}$. We show that $p_B \in \mathscr{M}$. Let $p_B(a,b) \leq p_B(a',b')$, for $a, a' \in A, b, b' \in B$. Then $b \leq b'$ and hence

$$f(a) = g(b) \leq g(b') = f(a').$$

This implies $a \leq a'$, because f is a regular monomorphism. Therefore, $(a,b) \leq (a',b')$, as required. □

2.3. Colimits of Regular Monomorphisms

In this subsection we investigate the colimit properties, such as coproducts, direct sums, pushouts and directed colimits (direct limits) of regular monomorphisms.

Proposition 5. *The class $\mathscr{M}$ is closed under coproducts and direct sums.*

Proof. Assume that $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ is a family of regular monomorphisms and $\coprod f_\alpha : \coprod A_\alpha \to \coprod B_\alpha$ is the coproduct morphism (which uniquely exists by the universal property of coproducts):

$$\begin{tikzcd} A_\alpha \arrow[r, "f_\alpha"] \arrow[d, "\iota_\alpha"'] & B_\alpha \arrow[d, "\iota'_\alpha"] \\ \coprod A_\alpha \arrow[r, "\coprod f_\alpha"] & \coprod B_\alpha \end{tikzcd}$$

We show that $\coprod f_\alpha$ is a regular monomorphism.

Let $(\coprod f_\alpha)(a,\alpha) \leq (\coprod f_\alpha)(a',\alpha')$, where $a \in A_\alpha, a' \in A_{\alpha'}, \alpha, \alpha' \in I$. It can be written as $(\coprod f_\alpha)\iota_\alpha(a) \leq (\coprod f_\alpha)\iota_{\alpha'}(a')$ and by the commutativity of the diagram, we get

$$(f_\alpha(a), \alpha) = \iota'_\alpha f_\alpha(a) = (\coprod f_\alpha)\iota_\alpha(a) \leq (\coprod f_\alpha)\iota_{\alpha'}(a') = \iota'_{\alpha'} f_{\alpha'}(a') = (f_{\alpha'}(a'), \alpha');$$

But this is impossible except $\alpha = \alpha'$ and then $f_\alpha(a) \leq f_\alpha(a')$. Since f_α is order-embedding, $a \leq a'$. Consequently, $(a,\alpha) = (a,\alpha') \leq (a',\alpha')$, as claimed.

For the second part, let $\{f_\alpha : A_\alpha \to B_\alpha | \alpha \in I\}$ be a family of regular monomorphisms such that A_α and B_α have a unique zero element, and $f : \bigoplus A_\alpha \to \bigoplus B_\alpha$ be the homomorphism induced by the product of f_α's. In fact, $f = \prod f_\alpha \Big|_{\bigoplus A_\alpha}$. Since $\prod f_\alpha$ is a regular monomorphism by Propoition 2, this clearly implies that so is f. $\square$

Here we show that pushouts transfer regular monomorphisms in S-**Pos**.

Theorem 1. S-**Pos** *has* $\mathscr{M}$*-transferring pushouts.*

Proof. Consider the pushout diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "q_B"] \\ C \arrow[r, "q_C"] & Q \end{tikzcd}$$

Recall that $Q = (B \sqcup C)/\theta(H)$ and $\theta(H)$ is the S-poset congruence on $B \sqcup C$ generated by $H = \{((1, f(a)), (2, g(a))) : a \in A\}$, where $i_B : B \to B \sqcup C, i_C : C \to B \sqcup C$ are the coproduct injections given by $i_B(b) = (1, b)$ and $i_C(c) = (2, c)$, for every $b \in B, c \in C$. Also the pushout maps are given as $q_C = \pi i_C : C \to Q$, $q_B = \pi i_B : B \to Q$, where $\pi : B \sqcup C \to Q$ is the canonical epimorphism. Suppose f is a regular monomorphism. To show that q_C is a regular monomorphism, let $q_C(c) \leq q_C(c')$, for $c, c' \in C$. Thus we have

$$[(2,c)] = \pi i_C(c) = q_C(c) \leq q_C(c') = \pi i_C(c') = [(2,c')].$$

Then, by [15, Proposition 3.3], we get $(2,c) \leq (2,c')$ (and hence $c \leq c'$) or there exist $s_1, s_2, \ldots, s_n \in S$ such that

$$(2,c) \leq s_1c_1, s_1d_1 \leq s_2c_2, s_2d_2 \leq s_3c_3, \ldots, s_nd_n \leq (2,c'),$$

where $(c_i, d_i) \in H \cup H^{-1}$ for $i = 1, 2, \ldots, n$. It follows that there exist $a_1, \ldots, a_n \in A$ such that

$$\begin{gathered}(2,c) \leq s_1(2, g(a_1)), s_1(1, f(a_1)) \leq s_2(1, f(a_2)), s_2(2, g(a_2)) \leq \\ s_3(2, g(a_3)), \ldots, s_{n-1}(1, f(a_{n-1})) \leq s_n(1, f(a_n)), s_n(2, g(a_n)) \leq (2,c').\end{gathered}$$

This gives the following:

$$\begin{gathered}c \leq g(s_1a_1), f(s_1a_1) \leq f(s_2a_2), g(s_2a_2) \leq \\ g(s_3a_3), \ldots, f(s_{n-1}a_{n-1}) \leq f(s_na_n), g(s_na_n) \leq c'.\end{gathered}$$

Since f is a regular monomorphism,

$$s_1a_1 \leq s_2a_2, s_3a_3 \leq s_4a_4, \ldots, s_{n-1}a_{n-1} \leq s_na_n.$$

Hence, we get

$$c \le g(s_1a_1) \le g(s_2a_2) \le g(s_3a_3) \le \ldots \le g(s_{n-1}a_{n-1}) \le g(s_na_n) \le c'.$$

□

For a class $\mathscr{E}$ of morphisms of a category, we say that multiple pushouts transfer $\mathscr{E}$-morphisms if in the multiple pushout $(Q, (A_\alpha \xrightarrow{q_\alpha} Q)_{\alpha\in I})$ of a family $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$ of $\mathscr{E}$-morphisms, $q_\alpha \in \mathscr{E}$, for every $\alpha \in I$.

Analogously to the pushouts, the following result is obtained.

Theorem 2. *Multiple pushouts transfer regular monomorphisms.*

Proof. Assume that $(Q, (A_\alpha \xrightarrow{q_\alpha} Q)_{\alpha\in I})$ is the multiple pushout of a family $\{f_\alpha : A \to A_\alpha | \alpha \in I\}$ of regular monomorphisms. Recall that $Q = (\coprod A_\alpha)/\theta(H)$, where $\theta(H)$ is the S-poset congruence on $\coprod A_\alpha$ generated by $H = \{(i_\alpha(f_\alpha(a)), i_\beta(f_\beta(a))) | a \in A, \alpha, \beta \in I\}$, and $q_\alpha = \pi i_\alpha$, where $\pi : \coprod A_\alpha \to Q$ and $i_\alpha : A_\alpha \to \coprod A_\alpha$ are the natural map and coproduct injection, respectively. We take $\alpha \in I$ and prove that q_α is a regular monomorphism. Let $q_\alpha(a_\alpha) \le q_\alpha(a'_\alpha)$, for $a_\alpha, a'_\alpha \in A_\alpha$. Then we have

$$[(\alpha, a_\alpha)] = \pi i_\alpha(a_\alpha) = q_\alpha(a_\alpha) \le q_\alpha(a'_\alpha) = \pi i_\alpha(a'_\alpha) = [(\alpha, a'_\alpha)].$$

Using [15, Proposition 3.3], this implies that $(\alpha, a_\alpha) \le (\alpha, a'_\alpha)$ (and then $a_\alpha \le a'_\alpha$) or there exist $s_1, s_2, \ldots, s_n \in S$ such that

$$(\alpha, a_\alpha) \le s_1c_1, s_1d_1 \le s_2c_2, s_2d_2 \le s_3c_3, \ldots, s_nd_n \le (\alpha, a'_\alpha),$$

where $(c_i, d_i) \in H \cup H^{-1}$ for $i = 1, 2, \ldots, n$. Thus there exist $a_1, \ldots, a_n \in A$ such that

$$\begin{gathered}(\alpha, a_\alpha) \le s_1(\alpha, f_\alpha(a_1)), s_1(\alpha_1, f_{\alpha_1}(a_1)) \le s_2(\alpha_1, f_{\alpha_1}(a_2)),\\ s_2(\alpha_2, f_{\alpha_2}(a_2)) \le s_3(\alpha_2, f_{\alpha_2}(a_3)), \ldots, s_{n-1}(\alpha_{n-1}, f_{\alpha_{n-1}}(a_{n-1}))\\ \le s_n(\alpha_{n-1}, f_{\alpha_{n-1}}(a_n)), s_n(\alpha, f_\alpha(a_n)) \le (\alpha, a'_\alpha).\end{gathered}$$

Then we get

$$\begin{gathered}a_\alpha \le f_\alpha(s_1a_1), f_{\alpha_1}(s_1a_1) \le f_{\alpha_1}(s_2a_2), f_{\alpha_2}(s_2a_2) \le\\ f_{\alpha_2}(s_3a_3), \ldots, f_{\alpha_{n-1}}(s_{n-1}a_{n-1}) \le f_{\alpha_{n-1}}(s_na_n), f_\alpha(s_na_n) \le a'_\alpha.\end{gathered}$$

Since $f_{\alpha_1}, \ldots, f_{\alpha_{n-1}}$ are regular monomorphisms,

$$s_1a_1 \le s_2a_2, \ldots, s_{n-1}a_{n-1} \le s_na_n,$$

whence

$$a_\alpha \le f_\alpha(s_1a_1) \le f_\alpha(s_2a_2) \le f_\alpha(s_3a_3) \le \ldots \le f_\alpha(s_{n-1}a_{n-1}) \le f_\alpha(s_na_n) \le a'_\alpha,$$

as required. □

Corollary 1. *Every multiple pushout of regular monomorphisms (the diagonal maps on the multiple pushout diagram) is a regular monomorphism.*

Proof. Apply Lemma 1(i) and Theorem 2. □

Definition 1. *Let $\mathscr{E}$ be a class of morphisms of a category $\mathscr{C}$. We say that $\mathscr{C}$ has:*

i) $\mathscr{E}$-bounds if for every small and non-empty family $\{h_\alpha : A \to B_\alpha\}_{\alpha\in I}$ of $\mathscr{E}$-morphisms, there is an $\mathscr{E}$-morphism $h : A \to B$ which factorizes through all h_α's.

ii) $\mathscr{E}$-amalgamation property if in (i), h factorizes through all h_α's by $\mathscr{E}$-morphisms.

In view of Corollary 1, the following is immediate:

Proposition 6. S-**Pos** *has $\mathscr{M}$-amalgamation property and so also has $\mathscr{M}$-bounds.*

Finally, we study directed colimit of regular monomorphisms in S-**Pos**. Recall that a *directed system* of S-posets and S-poset maps is a family $(A_i)_{i\in I}$ of S-posets indexed by an up-directed set I endowed by a family $(\psi_{ij} : A_i \to A_j)_{i\leq j\in I}$ of S-poset maps such that given $i \leq j \leq k \in I$, $\psi_{ik} = \psi_{jk}\psi_{ij}$, and $\psi_{ii} = id$. Also the pair $(\underrightarrow{lim}A_i, \{\alpha_i : A_i \to \underrightarrow{lim}A_i\})$ or in abbreviation, $\underrightarrow{lim}A_i$ is called the *directed colimit* (or *direct limit*) of the directed system $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ if for every $i \leq j \in I$, $\alpha_j\psi_{ij} = \alpha_i$, and for every $(B, f_i : A_i \to B)$ with $f_j\psi_{ij} = f_i$, $i \leq j \in I$, there exists a unique S-poset map $\nu : \underrightarrow{lim}A_i \to B$ such that $\nu\alpha_i = f_i$, for every $i \in I$.

Recall from [7] that the directed colimit of a directed system $((A_i)_{i\in I}, (\psi_{ij})_{i\leq j})$ of S-posets exists, and may be represented as $(A/\theta, (\psi_i : A_i \to A/\theta)_{i\in I})$, where

i) $A = \coprod A_i$;

ii) $a\theta a' (a \in A_i, a' \in A_j)$ if and only if $\exists k \geq i, j : \psi_{ik}(a) = \psi_{jk}(a')$;

iii) $[a]_\theta \leq [a']_\theta (a \in A_i, a' \in A_j)$ if and only if $\exists k \geq i, j : \psi_{ik}(a) \leq \psi_{jk}(a')$;

iv) for each $i \in I$ and $a \in A_i$, $\psi_i(a) = [a]_\theta$.

Theorem 3. *Let I be an up-directed set and $\{h_\alpha : A_\alpha \to B_\alpha \mid \alpha \in I\}$ be a directed family of regular monomorphisms. Then the directed colimit homomorphism induced by $h : \underrightarrow{lim}A_\alpha \to \underrightarrow{lim}B_\alpha$ is a regular monomorphism.*

Proof. Let $(\underrightarrow{lim}A_\alpha, f_\alpha), (\underrightarrow{lim}B_\alpha, g_\alpha)$ be directed colimits of the directed systems $((A_\alpha)_{\alpha\in I}, (\psi_{\alpha\beta})_{\alpha\leq\beta})$ and $((B_\alpha)_{\alpha\in I}, (\varphi_{\alpha\beta})_{\alpha\leq\beta})$, respectively. Suppose $\{h_\alpha : A_\alpha \to B_\alpha \mid \alpha \in I\}$ is a directed family of regular monomorphisms such that for every $\alpha \leq \beta$, $f_\beta\psi_{\alpha\beta} = f_\alpha$ and $g_\beta\varphi_{\alpha\beta} = g_\alpha$. Then

$$g_\beta h_\beta \psi_{\alpha\beta} = g_\beta\varphi_{\alpha\beta}h_\alpha = g_\alpha h_\alpha.$$

Thus $h = \underrightarrow{lim}h_\alpha$ exists by the universal property of colimits. Consider $\underrightarrow{lim}A_\alpha = (\coprod_\alpha A_\alpha)/\rho$ and $\underrightarrow{lim}B_\alpha = (\coprod_\alpha B_\alpha)/\rho'$. Let $h[a_\alpha]_\rho \leq h[a_\beta]_\rho$. Then we have

$$[h_\alpha(a_\alpha)]_{\rho'} = g_\alpha h_\alpha(a_\alpha) = hf_\alpha(a_\alpha) = h[a_\alpha]_\rho \leq$$

$$h[a_\beta]_\rho = hf_\beta(a_\beta) = g_\beta h_\beta(a_\beta) = [h_\beta(a_\beta)]_{\rho'}.$$

Therefore, $h_\alpha(a_\alpha) \leq_{\rho'} h_\beta(a_\beta)$ and hence there exists $\gamma \in I$ such that $\gamma \geq \alpha, \beta$ and $\varphi_{\alpha\gamma} h_\alpha(a_\alpha) \leq \varphi_{\beta\gamma} h_\beta(a_\beta)$. This implies that $h_\gamma \psi_{\alpha\gamma}(a_\alpha) \leq h_\gamma \psi_{\beta\gamma}(a_\beta)$. Now, since h_γ is a regular monomorphism by hypothesis, we get $\psi_{\alpha\gamma}(a_\alpha) \leq \psi_{\beta\gamma}(a_\beta)$ which gives that $a_\alpha \leq_\rho a_\beta$. Consequently, $[a_\alpha]_\rho \leq [a_\beta]_\rho$ and hence h is a regular monomorphism. □

Corollary 2. S-**Pos** *has* $\mathscr{M}$*-directed colimits.*

Proof. Assume that $(\underrightarrow{lim} B_\alpha, g_\alpha)$ is the directed colimit of the directed system

$$((B_\alpha)_{\alpha \in I}, (\varphi_{\alpha\beta})_{\alpha \leq \beta}),$$

and $\{h_\alpha : A \to B_\alpha \mid \alpha \in I\}$ is a directed family of regular monomorphisms such that $g_\beta \varphi_{\alpha\beta} = g_\alpha$, for every $\alpha \leq \beta$. Let $h : A \to \underrightarrow{lim} B_\alpha$ be the directed colimit of regular monomorphisms $h_\alpha : A \to B_\alpha$, $\alpha \in I$. Recall that

$$h = \underrightarrow{lim} h_\alpha = g_\alpha h_\alpha = g_\beta h_\beta = g_\gamma h_\gamma = \dots .$$

Now, take

$$\Sigma =: \{id : A_\alpha \to A_\gamma \mid \alpha \in I - \{\gamma\}, A_\alpha = A = A_\gamma\}.$$

It is clear that Σ is a directed system and $\underrightarrow{lim} A_\alpha = \coprod A_\alpha$. By Theorem 3, the induced directed colimit homomorphism $h' : \coprod A_\alpha \to \underrightarrow{lim} B_\alpha$ is a regular monomorphism. On the other hand, obviously the canonical map $i_\alpha : A_\alpha \to \coprod A_\alpha$ is also a regular monomorphism. Hence, in view of Lemma 1 (i), $h = h' i_\alpha : A_\alpha = A \to \underrightarrow{lim} B_\alpha$ is a regular monomorphism. □

Definition 2. *Let* $\mathscr{E}$ *be a class of morphisms of a category* $\mathscr{C}$*. We say that* $\mathscr{C}$ *fulfills the* $\mathscr{E}$*-chain condition if for every directed system* $((A_\alpha)_{\alpha \in I}, (\varphi_{\alpha\beta})_{\alpha \leq \beta \in I})$ *whose index set* I *is a well-ordered chain with the least element 0, and* $\varphi_{0\alpha} \in \mathscr{E}$ *for all α, there is a (so called "upper bound") family* $(g_\alpha : A_\alpha \to A)_{\alpha \in I}$ *with* $g_0 \in \mathscr{E}$ *and* $g_\beta \varphi_{\alpha\beta} = g_\alpha$*.*

Proposition 7. S-**Pos** *fulfills the* $\mathscr{M}$*-chain condition.*

Proof. Take $A = \underrightarrow{lim}_\alpha A_\alpha$ and let $g_\alpha : A_\alpha \to A$ be the colimit maps. Then, applying Corollary 2, we get the result. □

3. Comparison of Categorical Properties of Monomorphisms and Regular Monomorphisms

In this section some category-theoretic notions relative to monomorphisms and regular monomorphisms in **Pos** and S-**Pos** is studied. We show that these morphisms have a different behaviour with some categorical properties.

First notice that using similar arguments in proofs of results in Section 2 for the class of monomorphisms, some properties regarding composition closed, limits (products and pullbacks), colimits (coproducts and direct sums) and directed colimits hold in **Pos** and S-**Pos**.

About factorization diagonalization property, note the following:

Remark 2. *S-**Pos** does not generally have $(\mathscr{E}, \mathscr{M}ono)$-factorization diagonalization property, where $\mathscr{E}$ and $\mathscr{M}ono$ are the classes of epimorphisms and monomorphisms, respectively. In fact, although each S-poset morphism f has a factorization as $f = me$, where $m \in \mathscr{M}ono, e \in \mathscr{E}$ (see the proof of Proposition 1), there exist $e \in \mathscr{E}$ and $m \in \mathscr{M}ono$ such that e is not vertical on m. To see this, for a pomonoid S, consider the S-posets $A = \mathbf{1} \sqcup \mathbf{1} = \{a, b\}$, $B = \mathbf{2} = (\{a, b\}, a \leq b)$, $C = \mathbf{2} \sqcup \mathbf{1} = (\{a, b, c\}, a \leq b)$ and $D = \mathbf{3} = (\{a, b, c\}, a \leq b \leq c)$ with trivial actions of S. Take the inclusion maps $e : A \hookrightarrow B$ and $m : C \hookrightarrow D$. Clearly, $m \in \mathscr{M}ono$ and $e \in \mathscr{E}$. We show that e is not vertical on m. For this, define the morphisms $u : A \to C$ and $v : B \to D$ by $u(a) = v(a) = b, u(b) = v(b) = c$. Consider the following commutative diagram*

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ C \arrow[r, "m"] & D \end{tikzcd}$$

One can easily check that there is no morphism $d : B \to C$ such that $de = u$.

The following example shows that, in contrast to some categories such as S-**Act**, for a monoid S, pushouts do not transfer monomorphisms in S-**Pos** (see [4, Theorem 3.2(1)]).

Example 1. *For a pomonoid S, consider the S-posets $A = \mathbf{1} \sqcup \mathbf{1} = \{a, b\}$, $B = \mathbf{2} = (\{a, b\}, a \leq b)$ and $C = \mathbf{2} \sqcup \mathbf{1} = (\{a, b, c\}, a \leq c)$ with trivial actions of S, and also the inclusion $i : A \hookrightarrow B$ and the homomorphism $j : A \to C$ given by $j(a) = c, j(b) = a$. The pushout of i and j is described as*

$$\Big(Q = (B \sqcup C)/\theta(H), q_B = \pi i_B, q_C = \pi i_C\Big)$$

where $\theta(H)$ is the S-poset congruence on $B \sqcup C$ generated by

$$H = \{((1, a), (2, c)), ((1, b), (2, a))\},$$

$\pi : B \sqcup C \to Q$ is the natural map, and i_B, i_C are the coproduct injections. We claim that $[(2, a)] = [(2, c)]$. First note that since $a \leq c$ in C, $(2, a) \leq (2, c)$ in $B \sqcup C$ which implies that $[(2, a)] \leq [(2, c)]$. On the other hand, $(2, c) \leq_{H \cup H^{-1}} (2, a)$ because we have

$$(2, c) \leq (2, c)H^{-1}(1, a) \leq (1, b)H(2, a) \leq (2, a).$$

This gives that $[(2, c)] \leq [(2, a)]$. Therefore,

$$q_C(a) = [(2, a)] = [(2, c)] = q_C(c),$$

which shows that q_C is not a monomorphism.

Injectivity and absolute retractness relative to a class of morphisms of a category are two category-theoretic close notions, which usually coincide under some conditions (cf. [4]). Recall from [3] that in the category **Pos** of posets and order-preserving maps, these concepts with respect to regular monomorphisms (order-embeddings) are the same, which are exactly complete posets. Here we show that absolute retractness is actually a different notion to injectivity of posets relative to monomorphisms (one-one monotone maps). More precisely, there exists no non-trivial injective poset, but we show that absolute retract posets are exactly complete chains.

Let us first recall some definitions.

A poset P is called (*regular*) *absolute retract* if each (regular) monomorphism $f : P \to Q$ in **Pos** is a section.

Notice that, using Zorn's lemma, there exists a total order $\preceq$ on a poset $(P, \leq)$ which is *compatible* with the partial order $\leq$; this means $a \leq b$ implies that $a \preceq b$, for every $a, b \in P$ (see [14]).

In the following result, all absolute retract posets is characterized.

Theorem 4 (Characterization of absolute retract objects in **Pos**). *Let P be a poset. Then P is absolute retract if and only if it is a complete chain.*

Proof. Assume that $(P, \leq)$ is an absolute retract poset. This clearly implies that P is regular absolute retract and then complete by [3, Proposition 1]. To show that P is a chain, consider a compatible total order $\preceq$ on P. Thus the morphism $i : (P, \leq) \to (P, \preceq)$, mapping the elements of P identically, is a monomorphism and then has a left inverse by hypothesis. This obviously implies that P is a chain. For the converse, let P be a complete chain and $f : P \to Q$ be a monomorphism in **Pos**. Since P is a chain, f is order-embedding. On the other hand, since P is complete, it is regular absolute retract by [3, Proposition 1]. Consequently, f has a left inverse and hence P is absolute retract.

In what follows, we study enough absolute retractness of posets. To this end, we recall some required notions.

Let P be a poset. A poset E is said to be an *extension* of P if P is embedded into E. Also we say that E is a *monomorphic extension* of P if there exists a monomorphism from P to E. An extension E of P is called *join dense* if each element of E is the join of its predecessors in P, that is,

$$e = \bigvee \{p \in P : p \leq e\},$$

for every $e \in E$. *Meet density* is defined dually. The *Dedekind-MacNeille completion* of P, denoted by $\mathrm{DM}(P)$, is a complete extension of P which is both join and meet dense.

Lemma 2. *Let P be a chain. Then so is* $\mathrm{DM}(P)$.

Proof. Suppose P is a chain, and $x, y \in \mathrm{DM}(P), x \nleq y$. Then there exists an $s \in P$ such that $s \leq x$ and $s \nleq y$ (join density) and hence also a $t \in P$ such that $t \geq y$ and $s \nleq t$ (meet density). Since P is a chain, $t \leq s$. Thus we get $y \leq t \leq s \leq x$, showing that $\mathrm{DM}(P)$ is a chain. □

Finally, the following result is obtained:

Proposition 8. **Pos** *has enough absolute retracts: Each poset can be monomorphically extended to an absolute retract poset.*

Proof. Let P be a poset. In view of Theorem 4, it suffices to find a complete chain as a monomorphic extension of P. To see this, first note that P is embedded into the complete poset $\mathrm{DM}(P)$. Moreover, $\mathrm{DM}(P)$ is monomorphically extended to the chain $C = \mathrm{DM}(P)$ with a compatible total order. If C is complete, the assertion holds. Otherwise, using Lemma 2, it suffices to consider the complete chain $Q = \mathrm{DM}(C)$ in which C is embedded. Consequently, P is monomorphically extended to the complete chain Q, as required. $\square$

Remark 3. *In* **Pos** *poushouts do not transfer monomorphisms, otherwise, in view of [5, Theorem 3.6] and Proposition 8,* **Pos** *has enough injectives, a contradiction.*

ACKNOWLEDGEMENTS I would like to express my appreciation to the referee for carefully reading the paper.

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