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Characterization of Prime Ideals in $(\mathscr{Z}^+, \leq_{\mathscr{D}})$

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Abstract. A convolution is a mapping $\mathscr{C}$ of the set $\mathscr{Z}^+$ of positive integers into the set $\mathscr{P}(\mathscr{Z}^+)$ of all subsets of $\mathscr{Z}^+$ such that, for any $n \in \mathscr{Z}^+$, each member of $\mathscr{C}(n)$ is a divisor of n. If $\mathscr{D}(n)$ is the set of all divisors of n, for any n, then $\mathscr{D}$ is called the Dirichlet's convolution. Corresponding to any general convolution $\mathscr{C}$, we can define a binary relation $\leq_{\mathscr{C}}$ on $\mathscr{Z}^+$ by "$m \leq_{\mathscr{C}} n$ if and only if $m \in \mathscr{C}(n)$". It is well known that $\mathscr{Z}^+$ has the structure of a distributive lattice with respect to the division order. The division ordering is precisely the partial ordering $\leq_{\mathscr{D}}$ induced by the Dirichlet's convolution $\mathscr{D}$. In this paper, we present a characterization for the prime ideals in $(\mathscr{Z}^+, \leq_{\mathscr{D}})$, where $\mathscr{D}$ is the Dirichlet's convolution.

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1. Introduction

A Convolution is a mapping $\mathscr{C} : \mathscr{Z}^+ \longrightarrow \mathscr{P}(\mathscr{Z}^+)$ such that $\mathscr{C}(n)$ is a set of positive divisors on n, $n \in \mathscr{C}(n)$ and $\mathscr{C}(n) = \bigcup_{m \in \mathscr{C}(n)} \mathscr{C}(m)$, for any $n \in \mathscr{Z}^+$. Popular examples are the Dirichlet's convolution $\mathscr{D}$ and the Unitary convolution $\mathscr{U}$ defined respectively by

$$\mathscr{D}(n) = \text{ The set of all positive divisors of } n$$

and

$$\mathscr{U}(n) = \text{ The set of Unitary divisors of } n$$

for any $n \in \mathscr{Z}^+$. If $\mathscr{C}$ is a convolution, then the binary relation $\leq_{\mathscr{C}}$ on $\mathscr{Z}^+$, defined by,

$$m \leq_{\mathscr{C}} n \text{ if and only if } m \in \mathscr{C}(n),$$

is a partial order on $\mathscr{Z}^+$ and is called the partial order induced by $\mathscr{C}$ [2]. It is well known that the Dirichlet's convolution induces the division order on $\mathscr{Z}^+$ with respect to which $\mathscr{Z}^+$ becomes a distributive lattice, where, for any $a, b \in \mathscr{Z}+$, the greatest common divisor(GCD) and the

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least common multiple(LCM) of a and b are respectively the greatest lower bound(glb) and the least upper bound(lub) of a and b . In fact, with respect to the division order, the lattice $\mathscr{Z}^+$ satisfies the infinite join distributive law given by

$$\left(a \vee (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I}(a \vee b_i)\right)$$

for any $a \in \mathscr{Z}^+$ and $\{b_i\}_{i \in I} \subseteq \mathscr{Z}^+$. In this paper, we discuss various aspects of ideals and filters in $(\mathscr{Z}^+, \leq_C)$ and eventually present a characterization of prime ideals in $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ where $\mathscr{D}$ is the Dirichlet's convolution Actually a general convolution may not induce a lattice structure on $\mathscr{Z}^+$. However , most of the convolutions we are considering induce a meet semi lattice structure on $\mathscr{Z}^+$. For this reason, we first consider a general semi lattice and study it's ideals and later extend these to $(\mathscr{Z}^+, \leq_D)$.

2. Preliminaries

Let us recall that a partial order on a non-empty set X is defined as a binary relation $\leq$ on X which is reflexive $(a \leq a)$, transitive $(a \leq b, b \leq c \Longrightarrow a \leq c)$ and antisymmetric $(a \leq b, b \leq a \Longrightarrow a = b)$ and that a pair $(X, \leq)$ is called a partially ordered set(poset) if X is a non-empty set and $\leq$ is a partial order on X. For any $A \subseteq X$ and $x \in X$, x is called a lower(upper) bound of A if $x \leq a$(respectively $a \leq x$) for all $a \in A$. We have the usual notations of the greatest lower bound(glb) and least upper bound(lub) of A in X. If A is a finite subset $\{a_1, a_2, \cdots, a_n\}$, the glb of A(lub of A) is denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_n$ or $\bigwedge_{i=1}^{n} a_i$ (respectively by $a_1 \vee a_2 \vee \cdots \vee a_n$ or $\bigvee_{i=1}^{n} a_i$). A partially ordered set $(X, \leq)$ is called a meet semi lattice if $a \wedge b$ (=glb$\{a, b\}$) exists for all a and $b \in X$. $(X, \leq)$ is called a join semi lattice if $a \vee b$ (=lub$\{a, b\}$) exists for all a and $b \in X$. A poset $(X, \leq)$ is called a lattice if it is both a meet and join semi lattice. Equivalently, lattice can also be defined as an algebraic system $(X, \wedge, \vee)$, where $\wedge$ and $\vee$ are binary operations which are associative, commutative and idempotent and satisfying the absorption laws, namely $a \wedge (a \vee b) = a = a \vee (a \wedge b)$ for all $a, b \in X$; in this case the partial order $\leq$ on X is such that $a \wedge b$ and $a \vee b$ are respectively the glb and lub of $\{a, b\}$. The algebraic operations $\wedge$ and $\vee$ and the partial order $\leq$ are related by

$$(a = a \wedge b \Longleftrightarrow a \leq b \Longleftrightarrow a \vee b = b).$$

Throughout the paper, $\mathscr{Z}^+$ and $\mathscr{N}$ denote the set of positive integers and the set of non-negative integers respectively.

Definition 1. *A mapping $\mathscr{C} : \mathscr{Z}^+ \longrightarrow \mathscr{P}(\mathscr{Z}^+)$ is called a convolution if the following are satisfied for any $n \in \mathscr{Z}^+$.*

(1). $\mathscr{C}(n)$ is a set of positive divisors of n

(2). $n \in \mathscr{C}(n)$

(3). $\mathscr{C}(n) = \bigcup_{m \in \mathscr{C}(n)} \mathscr{C}(m)$.

Definition 2. *For any convolution $\mathscr{C}$ and m and $n \in \mathscr{Z}^+$, we define*

$$\left(m \leq n \text{ if and only if } m \in \mathscr{C}(n)\right)$$

Then $\leq_{\mathscr{C}}$ is a partial order on $\mathscr{Z}^+$ and is called the partial order induced by $\mathscr{C}$ on $\mathscr{Z}^+$.

In fact, for any mapping $\mathscr{C} : \mathscr{Z}^+ \longrightarrow \mathscr{P}(\mathscr{Z}^+)$ such that each member of $\mathscr{C}(n)$ is a divisor of n, $\leq_{\mathscr{C}}$ is a partial order on $\mathscr{Z}^+$ if and only if $\mathscr{C}$ is a convolution, as defined above [1, 4].

Definition 3. *Let $\mathscr{C}$ be a convolution and p a prime number. Define a relation $\leq^p_{\mathscr{C}}$ on the set $\mathscr{N}$ of non-negative integers by*

$$\left(a \leq^p_{\mathscr{C}} b \text{ if and only if } p^a \in \mathscr{C}(p^b)\right)$$

for any a and $b \in \mathscr{N}$.

It can be easily verified that $\leq^p_{\mathscr{C}}$ is a partial order on $\mathscr{N}$, for each prime p. The following is a direct verification.

Theorem 1. *Let $\mathscr{C}$ be a convolution.*

(1). If $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a meet(join) semilattice, then so is $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for each prime p.

(2). If $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice, then so is $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for each prime p.

3. Ideals in $(\mathscr{Z}^+, \leq_D)$

Recall that most of the convolutions like Dirichlet's convolution, Unitary convolution and k-free convolution induce meet semi lattice structure on $\mathscr{Z}^+$ [3]. For this reason we study ideals in a general meet semi lattice and later study ideals in the lattice structure $\mathscr{Z}^+$ induced by the division ordering /. The division ordering / is precisely the partial ordering $\leq_D$ induced by the Dirichlet's convolution D. Throughout this section, unless otherwise stated, by a semi lattice we mean a meet semi lattice only.

Definition 4. *Let $(X, \leq)$ be a poset. A non-empty subset I of X is called an* ***initial segment*** *if*

$$a \in I, x \in X \text{ and } x \leq a \Longrightarrow x \in I.$$

Definition 5. *Let $(S, \wedge)$ be a semi lattice. A non-empty subset I of S is called an ideal of S if the following are satisfied*

(1). $x \in S$ and $x \leq a \in I \Longrightarrow x \in I$

(2). For any a and $b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$

Definition 6. *Let $(S, \wedge)$ be a semi lattice and $a \in S$. Then the set*

$$(a] := \{x \in S | x \leq a\} = \{y \wedge a | y \in S\}$$

*is an ideal of S and is called the **Principal ideal** generated by a in S. Note that $(a]$ is the smallest ideal of S containing a.*

Now, we present the following

Theorem 2. *Let a and b be elements of a meet semi lattice $(S, \wedge)$. Then the following are equivalent to each other.*

(1). There exists smallest ideal of S containing a and b.

(2). The intersection of all ideals of S containing a and b is again an ideal of S.

(3). a and b have least upper bound in S.

Proof. $(1) \Longleftrightarrow (2)$: is trivial.
$(1) \Longrightarrow (3)$: Let I be the smallest ideal of S containing a and b. Then, there exists $x \in I$ such that

$$a \leq x \text{ and } b \leq x$$

Therefore x is an upper bound of a and b. If y is any other upper bound of a and b, then $(y]$ is an ideal of S containing a and b and hence $I \subseteq (y]$. Since $x \in I$, we get that $x \in (y]$ and therefore $x \leq y$. Thus x is the least upper bound of a and b.
$(3) \Longrightarrow (1)$: Let $a \vee b$ be the least upper bound of a and b. Then $a \leq a \vee b$ and $b \leq a \vee b$ and hence $(a \vee b]$ is an ideal containing a and b. If I is any ideal containing a and b, then there exists $x \in I$ such that

$$a \leq x \text{ and } b \leq x \text{ and hence } a \vee b \leq x$$

so that $a \vee b \in I$ and $(a \vee b] \subseteq I$. Thus $(a \vee b]$ is the smallest ideal of S containing a and b. □

Although the intersection of an arbitrary class of ideals need not be an ideal, a finite intersection is always an ideal.

Theorem 3. *Let $(S, \wedge)$ be a semi lattice and $\mathscr{I}(S)$ the set of all ideals of S. Then $(\mathscr{I}(S), \cap)$ is a semilattice and $a \mapsto (a]$ is an embedding of $(S, \wedge)$ onto $(\mathscr{I}(S), \cap)$.*

Proof. By the above theorem, it follows that $(\mathscr{I}(S), \cap)$ is a semi lattice.Also, for any a and b in S, we have

$$(a] \cap (b] = (a \wedge b]$$

and

$$(a] \subseteq (b] \Longleftrightarrow a \in (b] \Longleftrightarrow a \leq b$$

Therefore $a \mapsto (a]$ is an embedding of S into $\mathscr{I}(S)$. □

Theorem 4. *A semi lattice $(S, \wedge)$ is a lattice if and only if $\mathscr{I}(S)$ is a lattice and, in this case, $a \mapsto (a]$ is an embedding of the lattice S into the lattice $\mathscr{I}(S)$.*

Proof. It is well known that the set $\mathscr{I}(S)$ of ideals of a lattice $(S, \wedge, \vee)$ is again a lattice in which,

$$I \wedge J = I \cap J$$

and

$$I \vee J = \{x \in S | x \leq a \wedge b, \text{ for some } a \in I \text{ and } b \in J\}$$

for any ideals I and J, in this case,

$$(a] \vee (b] = (a \vee b]$$

for any a and b in S, so that $a \mapsto (a]$ is an embedding of lattices.
Conversely, suppose that $\mathscr{I}(S)$ is a lattice. Let a and $b \in S$ and I be the least upper bound of $(a]$ and $(b]$ in $\mathscr{I}(S)$. Then I is the smallest ideal containing a and b and hence by Theorem 2, $a \vee b$ exists in S. Therefore S is a lattice. □

For a lattice $(L, \wedge, \vee)$, any ideal of the semi lattice $(L, \wedge)$ turns out to be the usual ideal of the lattice $(L, \wedge, \vee)$.

Definition 7. *Let $(S, \wedge)$ be a semi lattice. A non-empty subset F of S is called filter of S if, for any $a, b \in S$,*

$$a \wedge b \in F \Leftrightarrow a \in F \text{ and } b \in F$$

Theorem 5. *Let $(S, \wedge)$ be a semi lattice and P a proper ideal of S. Then the following are equivalent to each other*

(1). For any elements a and b in S, $a \wedge b \in P \Longrightarrow a \in P$ or $b \in P$

(2). For any ideals I and J of S, $I \cap J \subseteq P \Longrightarrow I \subseteq P$ or $J \subseteq P$

(3). $S - P$ is a filter of S.

Proof. (1) $\Longrightarrow$ (2): Let I and J be ideals of S. Suppose that $I \nsubseteq P$ and $J \nsubseteq P$. Then there exist $a \in I$ and $b \in J$ such that $a \notin P$ and $b \notin P$. Then,by (1), $a \wedge b \notin P$. But $a \wedge b \leq a \in I$ and $a \wedge b \leq b \in J$ and hence $a \wedge b \in I \cap J$. Therefore $I \cap J \nsubseteq P$.
(2) $\Longrightarrow$ (3): If $a \leq b$ and $a \in S - P$, then clearly $b \in S - P$. Also,

$$\begin{aligned} a \text{ and } b \in S - P &\Longrightarrow a \notin P \text{ and } b \notin P \\ &\Longrightarrow (a] \nsubseteq P \text{ and } (b] \nsubseteq P \\ &\Longrightarrow (a \wedge b] = (a] \cap (b] \nsubseteq P \\ &\Longrightarrow x \notin P \text{ for some } x \leq a \wedge b \\ &\Longrightarrow x \leq a \wedge b \text{ and } x \in S - P \\ &\Longrightarrow a \wedge b \in S - P \end{aligned}$$

Thus $S - P$ is a filter of S.
(3) $\Longrightarrow$ (1): For any a and $b \in S$,

$$\begin{aligned} a \notin P \text{ and } b \notin P &\Longrightarrow a \text{ and } b \in S - P \\ &\Longrightarrow a \wedge b \in S - P \\ &\Longrightarrow a \wedge b \notin P \end{aligned}$$

□

Definition 8. *Any proper ideal P of a semi lattice $(S, \wedge)$ is said to be a* ***prime ideal*** *if any one (and hence all) of the conditions in Theorem 5 is satisfied.*

4. Prime Ideals in $(\mathscr{Z}^+, \leq_D)$

Now we shall turn our attention to the particular case of the lattice structure on $\mathscr{Z}^+$ induced by the division ordering / and study the ideals and prime ideals of $\mathscr{Z}^+$. The division ordering is precisely the partial ordering $\leq_D$ induced by the Dirichlet's convolution D.

First we observe that $\left(\theta : (\mathscr{Z}^+, /) \longrightarrow (\sum\limits_P \mathscr{N}, \leq)\right)$ is an order isomorphism where θ is defined by

$$\left(\theta(a)(p) = \text{ The largest } n \in \mathscr{N} \text{ such that } p^n \text{ divides } a \text{ , for any } a \in \mathscr{Z}^+ \text{ and } p \in \mathscr{P}\right)$$

and

$$\left(\sum_P \mathscr{N}\right) = \{f : \mathscr{P} \longrightarrow \mathscr{N} | f(p) = 0 \text{ for all but finite } p\}.$$

Here $\mathscr{P}$ stands for the set of primes and $\mathscr{N}$ stands for the set of non-negative integers.

Definition 9. *Adjoin an external element ∞ to $\mathscr{N}$ and extend the usual ordering $\leq$ on $\mathscr{N}$ to $\mathscr{N} \cup \{\infty\}$ by defining $a < \infty$ for all $a \in \mathscr{N}$. We shall denote $\mathscr{N} \cup \{\infty\}$ together with this extended usual order by $\mathscr{N}^\infty$.*

Theorem 6. *Let $\alpha : \mathscr{P} \longrightarrow \mathscr{N}^\infty$ be a mapping and define*

$$I_\alpha = \{n \in \mathscr{Z}^+ | \theta(n)(p) \leq \alpha(p) \textit{ for all } p \in \mathscr{P}\}$$

Then I_α is an ideal of $(\mathscr{Z}^+, /)$ and every ideal of $(\mathscr{Z}^+, /)$ is of the form I_α for some mapping $\alpha : \mathscr{P} \longrightarrow \mathscr{N}^\infty$

Proof. Since no prime divides the integer 1, we get that $\theta(1)(p) = 0 \leq \alpha(p)$ for all $p \in \mathscr{P}$ and hence $1 \in I_\alpha$. Therefore I_α is a non-empty subset of $\mathscr{Z}^+$.

$$\begin{aligned} m \text{ and } n \in I_\alpha &\Longrightarrow \theta(m)(p) \leq \alpha(p) \text{ and } \theta(n)(p) \leq \alpha(p) \text{ for all } p \in \mathscr{P} \\ &\Longrightarrow \theta(m \vee n)(p) = Max\{\theta(m)(p), \theta(n)(p)\} \leq \alpha(p) \text{ for all } p \in \mathscr{P} \end{aligned}$$

$$\Longrightarrow m \vee n \in I_\alpha$$

and

$$\begin{aligned} m \leq_D n \in I_\alpha &\Longrightarrow \theta(m)(p) \leq \theta(n)(p) \leq \alpha(p) \text{ for all } p \in \mathscr{P} \\ &\Longrightarrow \theta(m)(p) \leq \alpha(p) \text{ for all } p \in \mathscr{P} \\ &\Longrightarrow m \in I_\alpha. \end{aligned}$$

Thus I_α is an ideal of $(\mathscr{Z}^+, /)$. Conversely suppose that I is any ideal of $(\mathscr{Z}^+, /)$. Define $\alpha : \mathscr{P} \longrightarrow \mathscr{N}^\infty$ by

$$\alpha(p) = Sup\{\theta(n)(p) | n \in I\} \text{ for any } p \in \mathscr{P}$$

Note that $\alpha(p)$ is either a non-negative integer or ∞, for any $p \in \mathscr{P}$. Therefore α is a mapping of $\mathscr{P}$ into $\mathscr{N}^\infty$.

$$\begin{aligned} n \in I &\Longrightarrow \theta(n)(p) \leq \alpha(p) \text{ for all } p \in \mathscr{P} \\ &\Longrightarrow n \in I_\alpha \end{aligned}$$

Therefore $I \subseteq I_\alpha$. On the other hand, suppose $n \in I_\alpha$. Then $\theta(n)(p) \leq \alpha(p)$ for all $p \in \mathscr{P}$. Since $\theta(n) \in \sum_P \mathscr{N}$, $|\theta(n)|$ is finite. If $|\theta(n)| = \phi$, then $n = 1 \in I$. Suppose $|\theta(n)|$ is non-empty. Let $|\theta(n)| = \{p_1, p_2 \cdots, p_r\}$. Then $\theta(n)(p) = 0$ for all $p \neq p_i$, $1 \leq i \leq r$ and $\theta(n)(p_i) \in \mathscr{N}$. Now, for each $1 \leq i \leq r$, $\theta(n)(p_i) \leq \alpha(p_i) = Sup\{\theta(m)(p_i) | m \in I\}$ and hence there exists $m_i \in I$ such that $\theta(n)(p_i) \leq \theta(m)(p_i)$. Now, put $m = m_1 \vee m_2 \vee \cdots \vee m_r$, then $m \in I$ and

$$\theta(n)(p_i) \leq Max\{\theta(m_1)(p_i), \ldots, \theta(m_i)(p_i)\} = \theta(m)(p_i)$$

for all $1 \leq i \leq r$. Also, since $\theta(n)(p) = 0$ for all $p \neq p_i$, we get that $\theta(n)(p) \leq \theta(m)(p)$ for all $p \in \mathscr{P}$ so that $n \leq_D m \in I$ and therefore $n \in I$. Therefore $I_\alpha \subseteq I$. Thus $I = I_\alpha$. $\square$

Note that, if α is the constant map $\overline{0}$ defined by $\alpha(p) = 0$ for all $p \in \mathscr{P}$, then $I_\alpha = \{1\}$ and that, if α is the constant map $\overline{\infty}$, then $I_\alpha = \mathscr{Z}^+$.

Definition 10. *For any mappings α and β from $\mathscr{P}$ into $\mathscr{N}^\infty$, define*

$$\alpha \leq \beta \text{ if and only if } \alpha(p) \leq \beta(p) \text{ for all } p \in \mathscr{P}.$$

Thus $\leq$ is a partial order on $(\mathscr{N}^\infty)^{\mathscr{P}}$.

Theorem 7. *The map $\alpha \mapsto I_\alpha$ is an order isomorphism of the poset $((\mathscr{N}^\infty)^{\mathscr{P}}, \leq)$, onto the poset $(\mathscr{I}(\mathscr{Z}^+), \subseteq)$ of all ideals of $(\mathscr{Z}^+, /)$.*

Proof. Let α and $\beta : \mathscr{P} \mapsto \mathscr{N}^\infty$ be any mappings. Clearly, $\alpha \leq \beta \Rightarrow I_\alpha \subseteq I_\beta$. On the other hand, suppose that $I_\alpha \subseteq I_\beta$. We shall prove that $\alpha(p) \leq \beta(p)$ for all $p \in \mathscr{P}$ so that $\alpha \leq \beta$. To prove this, let us fix $p \in \mathscr{P}$. If $\beta(p) = \infty$ or $\alpha(p) = 0$, trivially $\alpha(p) \leq \beta(p)$. Therefore, we can assume that $\beta(p) < \infty$ and $\alpha(p) > 0$. Consider $n = p^{\beta(p)+1}$. Then

$$\theta(n)(p) = \beta(p) + 1 \nleq \beta(p).$$

and hence $n \notin I_\beta$. Since $I_\alpha \subseteq I_\beta$, $n \notin I_\alpha$ and therefore $\theta(n)(q) \nleq \alpha(q)$ for some $q \in \mathscr{P}$. But $\theta(n)(q) = 0$ for all $q \neq p$. Thus

$$\begin{aligned}\beta(p) + 1 =& \theta(n)(p) \nleq \alpha(p)\\ \alpha(p) <& \beta(p) + 1.\end{aligned}$$

Therefore $\alpha(p) \leq \beta(p)$. This is true for all $p \in \mathscr{P}$. Thus $\alpha \leq \beta$. Also $\alpha \mapsto I_\alpha$ is a surjection. Thus $\alpha \mapsto I_\alpha$ is an order isomorphism of $((\mathscr{N}^\infty)^{\mathscr{P}}, \leq)$, onto $(\mathscr{I}(\mathscr{Z}^+), \subseteq)$. □

Corollary 1. *For any α and $\beta : \mathscr{P} \to \mathscr{N}^\infty$,*

$$I_\alpha \cap I_\beta = I_{\alpha \wedge \beta}.$$

and

$$I_\alpha \cup I_\beta = I_{\alpha \vee \beta}.$$

where $\alpha \wedge \beta$ and $\alpha \vee \beta$ are point-wise g.l.b and l.u.b of α and β.

First we state the following two theorems from "Lattice Structures on $\mathscr{Z}^+$ induced by convolutions" [3].

Theorem 8. *Let $\mathscr{C}$ be a convolution which is closed under finite intersections and $\leq_{\mathscr{C}}$ be the partial order on $\mathscr{Z}^+$ induced by $\mathscr{C}$. Then $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice if and only if it is directed above.*

Theorem 9. *Let $\mathscr{C}$ be a convolution.*

(1). If $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a meet(join) semilattice, then so is $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for each prime p

(2). If $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice, then so is $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for each prime p.

Theorem 10. *Let $\mathscr{C}$ be a multiplicative convolution such that $(\mathscr{Z}^+, /)$ is a meet semi lattice. For any $\alpha : \mathscr{P} \to \mathscr{N}^\infty$, let*

$$I_\alpha = \{n \in \mathscr{Z}^+ | \theta(n)(p) \leq^{\mathscr{P}}_{\mathscr{C}} \alpha(p) \text{ for all } p \in \mathscr{P}\}.$$

Then the following are equivalent to each other.

(1). I_α is an ideal of $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ for any $\alpha : \mathscr{P} \to \mathscr{N}^\infty$.

(2). $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is directed below

(3). $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is a lattice.

Proof. (2) $\Leftrightarrow$ (3) follows from Theorem 8
(1) $\Rightarrow$ (2) : Let $\alpha : \mathscr{P} \to \mathscr{N}^\infty$ be defined by $\alpha(p) = \infty$ for all $p \in \mathscr{P}$. Then

$$I_\alpha = \{n \in \mathscr{Z}^+ | \theta(n)(p) \leq^{\mathscr{P}}_{\mathscr{C}} \alpha(p) = \infty \text{ for all } p \in \mathscr{P}\}$$

and hence, by (1), $\mathscr{Z}^+$ is an ideal of $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ which implies that $(\mathscr{Z}^+, \leq_{\mathscr{C}})$ is directed above.
(3) $\Rightarrow$ (1) : From (3) and Theorems 8 and 9, it follows that $(\mathscr{N}, \leq^p_{\mathscr{C}})$ is a lattice for each $p \in \mathscr{P}$

and $\theta(m \vee n)(p) = \theta(m)(p) \vee \theta(n)(p)$ in $(\mathscr{N}, \leq^p_{\mathscr{C}})$ for any m and $n \in \mathscr{Z}^+$ and $p \in \mathscr{P}$. Let $\alpha : \mathscr{P} \to \mathscr{N}^\infty$ be any mapping. Then, for any m and $n \in \mathscr{Z}^+$,

$$\begin{aligned} m \leq_{\mathscr{C}} n \in I_\alpha &\Longrightarrow \theta(m)(p) \leq^p_{\mathscr{C}} \theta(n)(p) \leq^p_{\mathscr{C}} \alpha(p) \text{ for all } p \in \mathscr{P}. \\ &\Longrightarrow \theta(m)(p) \leq^p_{\mathscr{C}} \text{ for all } p \in \mathscr{P}. \\ &\Longrightarrow m \in I_\alpha. \end{aligned}$$

and

$$\begin{aligned} m \text{ and } n \in I_\alpha &\Longrightarrow \theta(m)(p) \leq^p_{\mathscr{C}} \alpha(p) \text{ and } \theta(n)(p) \leq^p_{\mathscr{C}} \alpha(p) \text{ for all } p \in \mathscr{P}. \\ &\Longrightarrow \theta(m)(p) \vee \theta(n)(p) \leq^p_{\mathscr{C}} \alpha(p) \text{ for all } p \in \mathscr{P}. \\ &\Longrightarrow m \vee n \in I_\alpha. \end{aligned}$$

Therefore I_α is an ideal of $(\mathscr{Z}^+, \leq_{\mathscr{C}})$. □

Now, we have the following Theorems which characterize the prime ideals of the lattice $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ where $\mathscr{D}$ is the Dirichlet's convolution.

Theorem 11. *Let $\alpha : \mathscr{P} \to \mathscr{N}^\infty$ be a mapping and I_α is an ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ defined by $I_\alpha = \{n \in \mathscr{Z}^+ | \theta(n)(p) \leq \alpha(p)$ for all $p \in \mathscr{P}\}$. Then the following are equivalent to each other.*

(1). I_α is a prime ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$.

(2). $\alpha(p) \neq \infty$ for some $p \in \mathscr{P}$ and for any β and $\gamma : \mathscr{P} \longrightarrow \mathscr{N}^\infty$,

$$\beta \wedge \gamma \leq \alpha \Longrightarrow \beta \leq \alpha \text{ or } \gamma \leq \alpha.$$

(3). There exists unique $p \in \mathscr{P}$ such that

$$\alpha(p) \neq \infty \text{ and } \alpha(q) = \infty \text{ for all } q \neq p \in \mathscr{P}.$$

Proof. (1) $\Longrightarrow$ (2) follows from Theorem 7, in which we have proved that $\beta \mapsto I_\beta$ is an isomorphism of the lattice $((\mathscr{N}^\infty)^{\mathscr{P}}, \leq)$ onto the lattice of ideals of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ from the fact that $I_\beta \cap I_\gamma = I_{\beta \wedge \gamma}$ for any β and $\gamma : \mathscr{P} \longrightarrow \mathscr{N}^\infty$. If $\alpha(p) = \infty$ for all $p \in \mathscr{P}$, then, since $\theta(n)(p) \in \mathscr{N}$ for all $n \in \mathscr{Z}^+$ and $p \in \mathscr{P}$,

$$I_\alpha = \{n \in \mathscr{Z}^+ | \theta(n)(p) < \infty\} = \mathscr{Z}^+$$

which is a contradiction to the fact that every prime ideal is a proper ideal. Thus $\alpha(p) \neq \infty$ for some $p \in \mathscr{P}$.

(2) $\Longrightarrow$ (3): Suppose that α satisfies (2). Fix $p \in \mathscr{P}$ such that $\alpha(p) \neq \infty$. Then $\alpha(p) \in \mathscr{N}$. Now, define β and $\gamma : \mathscr{P} \longrightarrow \mathscr{N}^\infty$ by

$$\beta(q) = \begin{cases} 0 & \text{if } q = p \\ \infty & \text{if } q \neq p \end{cases}$$

and

$$\gamma(q) = \begin{cases} \infty & \text{if } q = p \\ 0 & \text{if } q \neq p \end{cases}$$

for any $q \in \mathscr{P}$. Then,

$$(\beta \wedge \gamma)(q) = \beta(q) \wedge \gamma(q) = 0 \leq \alpha(q)$$

for all $q \in \mathscr{P}$ and hence $\beta \wedge \gamma \leq \alpha$. Since $\alpha(p) \neq \infty$ and $\gamma(p) = \infty$, $(\gamma)(p) \nleq \alpha(p)$ and hence $\gamma \nleq \alpha$. Therefore, by (2), $\beta \leq \alpha$ and hence

$$\infty = \beta(q) \leq \alpha(q) \text{ for all } q \neq p.$$

Therefore $q(p) = \infty$ for all $q \neq p$ in $\mathscr{P}$. This also implies the uniqueness of p.
(3) $\Longrightarrow$ (1): Let $p \in \mathscr{P}$ such that

$$\alpha(p) \neq \infty \text{ and } \alpha(q) = \infty \text{ for all } q \neq p \in \mathscr{P}.$$

Then I_α is a proper ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$. Let J and K be any ideals of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ such that $J \cap K \subseteq I_\alpha$. Then there exists β and $\gamma : \mathscr{P} \longrightarrow \mathscr{N}^\infty$ such that $J = I_\beta$ and $K = I_\gamma$. Now, $I_{\beta\wedge\gamma} = I_\beta \cap I_\gamma = J \cap K \subseteq I_\alpha$ and hence $\beta \wedge \gamma \leq \alpha$ so that

$$Min\{\beta(p), \gamma(p)\} = (\beta \wedge \gamma)(p) \leq \alpha(p).$$

Therefore $\beta(p) \leq \alpha(p)$ or $\gamma(p) \leq \alpha(p)$. Since $\alpha(q) = \infty$ for all $q \neq p$, it follows that $\beta \leq \alpha$ or $\gamma \leq \alpha$ and hence $I_\beta \subseteq I_\alpha$ or $I_\gamma \subseteq I_\alpha$. Therefore $J \subseteq I_\alpha$ or $K \subseteq I_\alpha$. Thus I_α is a prime ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$. □

Definition 11. *For any prime number p and $a \in \mathscr{N}$, define*

$$I_{p,a} = \{n \in \mathscr{Z}^+ | \theta(n)(p) \leq a\}.$$

Then $I_{p,a}$ is an ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$. In fact $I_{p,a} = I_\alpha$, where $\alpha : \mathscr{P} \longrightarrow \mathscr{N}^\infty$ is defined by

$$\alpha(q) = \begin{cases} a & \text{if } q = p \\ \infty & \text{if } q \neq p \end{cases}$$

Note that $I_{p,a} = \{n \in \mathscr{Z}^+ | p^{a+1}$ does not divide $n\}$.

Theorem 12. *An ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ is prime if and only if it is of the form $I_{p,a}$ for some $p \in \mathscr{P}$ and $a \in \mathscr{N}$.*

Proof. Let I be an ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$. Then $I = I_\alpha$ for some mapping $\alpha : \mathscr{P} \longrightarrow \mathscr{N}^\infty$. Now, by Theorem 11, I is prime $\Longleftrightarrow$ there exists $p \in \mathscr{P}$ such that $\alpha(p) \neq \infty$ and $\alpha(q) = \infty$ for all $q \neq p$ and $I = I_\alpha \Longleftrightarrow I = I_{p,a}$, where $a = \alpha(p)$. □

Theorem 13. *For any p and $q \in \mathscr{P}$ and a and $b \in \mathscr{N}$,*

$$I_{p,a} \subseteq I_{q,b} \Longleftrightarrow p = q \text{ and } a \leq b$$

Proof. If $p = q$ and $a \leq b$, then

$$\begin{aligned} n \in I_{p,q} &\Longrightarrow \theta(n)(p) \leq a \leq b \\ &\Longrightarrow \theta(n)(q) \leq b \\ &\Longrightarrow n \in I_{q,b} \end{aligned}$$

and hence $I_{p,a} \subseteq I_{q,b}$. Conversely suppose that $I_{p,a} \subseteq I_{q,b}$. If $p \neq q$, then

$$\theta(q^{b+1})(p) = 0 \leq a$$

and hence $q^{b+1} \in I_{p,a} \subseteq I_{q,b}$ so that $\theta(q^{b+1})(b) \leq b$, which is a contradiction. Therefore $p = q$. Now, since $\theta(p^a)(p) = a$, $p^a \in I_{p,a} \subseteq I_{q,b}$ and hence $a = \theta(p^a)(q) \leq b$. Thus $p = q$ and $a \leq b$. □

The following are immediate consequences of Theorems 11,12 and 13.

Corollary 2. *For each $p \in \mathscr{P}$, let $\mathscr{P}_p = \{I_{p,a} | a \in \mathscr{N}\}$. Then the following hold.*

(1). $\mathscr{P}_p$ is a chain of prime ideals of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ for each $p \in \mathscr{P}$.

(2). $\mathscr{P}_p \cap \mathscr{P}_q = \phi$ for all $p \neq q \in \mathscr{P}$.

(3). $\bigcup_{p \in \mathscr{P}} \mathscr{P}_p$ is the set of all prime ideals of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$.

Corollary 3. *I is a minimal prime ideal of $(\mathscr{Z}^+, \leq_{\mathscr{D}})$ if and only if*

$$I = I_{p,0} = \{n \in \mathscr{Z}^+ | p \text{ does not divide } n\}$$

for some $p \in \mathscr{P}$.

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