EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 4, 2014, 395-404
ISSN 1307-5543 – www.ejpam.com

Low Dimensional Homology Groups of the Orthosymplectic Lie Superalgebra $\mathfrak{osp}(1,2)$

Guy Roger Biyogmam

Department of Mathematics, Southwestern Oklahoma State University, Weatherford, OK 73096, USA

Abstract. We realize the Lie superalgebra $\mathfrak{osp}(1,2)$ in terms of first order differential operators and endow it with the Lie superbracket of vector fields to determine the basis (co)cycles of low dimensional (co)homology groups of $\mathfrak{osp}(1,2)$ with trivial coefficients, using the complex introduced by Tanaka [7]. Our calculations agree with the result obtained by Fuks and Leites in [2].

2010 Mathematics Subject Classifications: 17B56, 17B66.

Key Words and Phrases: Lie superalgebras, Homology of Lie superalgebras.

1. Introduction and Generalities

Given a Lie superalgebra $\mathfrak{g}$ over a field k of characteristic 0, D. Fuks [1] introduced a Koszul complex associated to $\mathfrak{g}$. Using this complex, Fuks and Leites [2] calculated the cohomology groups with trivial coefficients of the classical Lie superalgebras. In particular, they found that

$$H^*(\mathfrak{osp}(1,2)) \cong H^*(\mathfrak{sp}(2)). \tag{1}$$

In [7], J. Tanaka introduced another Koszul complex for $\mathfrak{g}$. In this work, we use this complex and take advantage of the small basis of the superalgebra $\mathfrak{osp}(1,2)$ to calculate low dimensional (co)homology groups of $\mathfrak{osp}(1,2)$ with coefficients in $\mathbb{R}$. The result obtained agrees with (1). In particular, our calculations provide explicitly three generators of the group $H^3(\mathfrak{osp}(1,2);\ \mathbb{R})$ in terms of the basis of $\mathfrak{osp}(1,2)$. Note that in the non trivial case where these (co)homology groups are non zero, results providing the generators have been limited to the second degree [3, 4, 6, 8, 9].

Let us recall a few definitions. A Lie superalgebra [5] $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded algebra over a commutative ring or field such as $\mathbb{R}$ or $\mathbb{C}$ with a direct sum decomposition

Email address: guy.biyogmam@swosu.edu

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

together with a bilinear operation $[-,\,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+j}$, and satisfying

i) $[X, Y] + (-1)^{|X||Y|}[Y, X] = 0$ (super antisymmetry)

ii) $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|Y||Z|}[Y[X, Z]]$ (super Jacobi identity).

The elements X and Y are said to be homogeneous and the parity $|X|$ of an homogeneous element X is 0 or 1 according to whether it is in $\mathfrak{g}_0$ or $\mathfrak{g}_1$, in which case X is said to be *even* or *odd* respectively. The Grassmann algebra $\bigwedge^*(\mathfrak{g})$ is defined as the quotient of the tensor algebra $\bigotimes^*(\mathfrak{g})$ by the two-sided ideal of $\mathfrak{g}^{\otimes 2}$ generated by

$$\left\{X \otimes Y + (-1)^{|X||Y|} Y \otimes X,\ X, Y \in \mathfrak{g}\right\}.$$

Let $Vect(\mathfrak{g})$ be the superspace of vector fields on $\mathfrak{g}$. The superbracket of two vector fields X and Y is bilinear and defined for two homogeneous vector fields by:

$$[X, Y] = X \circ Y - (-1)^{|X||Y|} Y \circ X. \tag{2}$$

2. Lie Superalgebra Homology

For any Lie superalgebra $\mathfrak{g}$ over a ring k and V any $\mathfrak{g}$-module, J. Tanaka [7] defined the Lie algebra homology of $\mathfrak{g}$ with coefficients in V, written $H_*(\mathfrak{g};\ V)$, as the homology of the complex $\wedge^*(\mathfrak{g}) \otimes V$, namely

$$0 \xleftarrow{0} V \xleftarrow{d} \mathfrak{g}^{\wedge^1} \otimes V \xleftarrow{d} \mathfrak{g}^{\wedge^2} \otimes V \xleftarrow{d} \ldots \xleftarrow{d} \mathfrak{g}^{\wedge^{n-1}} \otimes V \xleftarrow{d} \mathfrak{g}^{\wedge^n} \otimes V \leftarrow \ldots$$

where $\mathfrak{g}^{\wedge^n}$ is the nth exterior power (as defined above) of $\mathfrak{g}$ over k, and where

$$d(g_1 \wedge \ldots \wedge g_n \otimes v) = \sum_{1 \le j \le n} (-1)^{j+\zeta_i'} g_1 \wedge \ldots \widehat{g_j} \ldots \wedge g_n \otimes [g_j,\ v]$$

$$+ \sum_{1 \le i < j \le n} (-1)^{i+j+\zeta_i+\zeta_j+\epsilon_i\epsilon_j} [g_i,\ g_j] \wedge g_1 \wedge \ldots \widehat{g_i} \ldots \widehat{g_j} \ldots \wedge g_n \otimes v,$$

where $\epsilon_i = |X_i|$, $\zeta_i' = \epsilon_i(\epsilon_{i+1} + \ldots + \epsilon_n)$, $\zeta_i = \epsilon_i(\epsilon_1 + \ldots + \epsilon_{i-1})$, and $\widehat{g_i}$ means that the variable g_i is deleted. In particular if $V = k$ the trivial module, we identify $g_1 \wedge \ldots \wedge g_n$ with $g_1 \wedge \ldots \wedge g_n \otimes 1$ and have

$$d(g_1 \wedge \ldots \wedge g_n) = \sum_{1 \le i < j \le n} (-1)^{i+j+\zeta_i+\zeta_j+\epsilon_i\epsilon_j} [g_i,\ g_j] \wedge g_1 \wedge \ldots \hat{g}_i \ldots \hat{g}_j \ldots \wedge g_n.$$

Note that this complex is infinite since for $g \in \mathfrak{g}_1$ (the odd part of $\mathfrak{g}$), $g \wedge g$ is not always 0 as in the Lie algebra case. The standard Koszul complex for homology of Lie superalgebras is the complex introduced by D. Fuks [1]. For trivial coefficients, it is defined as follows:

$$0 \xleftarrow{0} k \xleftarrow{d} C_1(\mathfrak{g}) \xleftarrow{d} C_2(\mathfrak{g}) \xleftarrow{d} \dots \xleftarrow{d} C_{n-1}(\mathfrak{g}) \xleftarrow{d} C_n(\mathfrak{g}) \leftarrow \dots$$

where $C_n(\mathfrak{g}) = \bigoplus_{p+q=n} \wedge^p(\mathfrak{g}_0) \otimes S^q(\mathfrak{g}_1)$ and where the differentials $C_n(\mathfrak{g}) \xrightarrow{d_n} C_{n-1}(\mathfrak{g})$ are given by

$$\begin{aligned} d_n\big((g_1 \wedge \dots \wedge g_n) \otimes (h_1 \dots h_q)\big) = & \sum_{1 \le i < j \le p} (-1)^{i+j}\big([g_i,\ g_j] \wedge g_1 \wedge \dots \widehat{g}_i \dots \widehat{g}_j \dots \wedge g_p\big) \otimes \big(h_1 \dots h_q\big) \\ & + \sum_{1 \le i \le n} (-1)^{i-1}\big(g_1 \wedge \dots \widehat{g}_i \dots \wedge g_p\big) \otimes \big(g_i.(h_1 \dots \dots h_q)\big) \\ & + \sum_{1 \le i < j \le q} \big(g_1 \wedge \dots \dots \wedge g_p\big) \otimes \big(h_1 \dots \widehat{h}_i \dots \widehat{h}_j \dots h_q\big), \end{aligned}$$

for $n \ge 2$, $x_i \in \mathfrak{g}_0$, $y_j \in \mathfrak{g}_1$. In the following subsection, we calculate $H_*(\mathfrak{osp}(1,2);\ \mathbb{R})$, using J. Tanaka's definition.

3. Lie Superalgebra Homology of $\mathfrak{osp}(1,2)$

Throughout this section, we assume that $k = \mathbb{R}$. Recall that $\mathfrak{osp}(1,2n)$ consists of matrices of the form

$$M = \begin{bmatrix} 0 & A_1 & A_2 \\ A_2^t & B & C \\ -A_1^t & D & -B^t \end{bmatrix}$$

where A_1 and A_2 are $(1 \times n)$-matrices, B is a $(n \times n)$-matrix, C and D are symmetric $(n \times n)$-matrices. Let $e_{i,j}$ be matrices whose entries are 1 for $i = j$ and 0 else. Then for $n \ge 2$, the following forms a basis of $\mathfrak{osp}(1,2n)$:

$$\begin{aligned} \mathfrak{B} =\{ & e_{i,i} - e_{i+n,i+n}, \quad e_{i,i+n}, \quad e_{i+n,i}, \quad e_{i,j} - e_{j+n,i+n}, \quad e_{i,j+n} + e_{j,i+n}, \\ & e_{n+i,j} + e_{n+j,i}, \quad e_{1,j} - e_{n+j,1}, \quad e_{1,j+n} + e_{j,1};\ 2 \le i,j \le n,\ i < j\}. \end{aligned}$$

Assume that $\mathbb{R}^n$ is given the coordinates $\big(x_1, x_2, \dots, x_n\big)$, and let $\frac{\partial}{\partial x^i,}$ be the unit vector fields parallel to the x_i axes respectively. It is easy to show in the case $\mathfrak{osp}(1,2)$ that the Lie superalgebra generated by the family $\mathfrak{B}$ below of vector fields (endowed with the superbracket of vector fields) is isomorphic to the orthosymplectic Lie superalgebra $\mathfrak{osp}(1,2)$:

$$\mathfrak{B} = \{E_{23}, e_{23}, e_{32}, o_{23}, o_{32}\}$$

where

$$\begin{aligned} E_{23} &:= x_2 \frac{\partial}{\partial x^2} - x_3 \frac{\partial}{\partial x^3}, \\ e_{23} &:= x_2 \frac{\partial}{\partial x^3}, \\ e_{32} &:= x_3 \frac{\partial}{\partial x^2}, \end{aligned}$$

$$o_{23} := x_1\frac{\partial}{\partial x^2} - x_3\frac{\partial}{\partial x^1},$$
$$o_{32} := x_1\frac{\partial}{\partial x^3} + x_2\frac{\partial}{\partial x^1}.$$

The remaining of this section details the proof that there are isomorphisms of super vector space

$$H_r(\mathfrak{osp}(1,2);\ \mathbb{R}) \cong \begin{cases} \mathbb{R}, & \text{for } r=0\\ 0, & \text{for } r=1,2\\ \langle E_{23}\wedge e_{23}\wedge e_{32}\rangle = \langle E_{23}\wedge o_{23}\wedge o_{32}\rangle = & \\ \langle e_{23}\wedge o_{23}\wedge o_{23} - e_{32}\wedge o_{32}\wedge o_{32}\rangle, & \text{for } r=3\\ 0, & \text{for } r=4. \end{cases}$$

3.1. Zero and First Homology Groups

Notice that in the Tanaka complex, we have the boundary maps $d_0:\mathbb{R}\to 0$ an $d_1:\mathfrak{osp}(1,2)\to\mathbb{R}$ with $d_1(b)=0$ for all $b\in\mathfrak{osp}(1,2)$. So $\ker d_0=\mathbb{R}$, $\mathrm{Im} d_1 = 0$. So

$$H_0(\mathfrak{osp}(1,2)\ ;\ \mathbb{R}) = \frac{\ker d_0}{Imd_1} = \frac{\mathbb{R}}{0} = \mathbb{R}.$$

Now using the identity (2) and the basis of $\mathfrak{osp}(1,2)$ provided above, we obtain the following superbrackets:

$$\begin{array}{lll} [E_{23},e_{23}]=2e_{23} & [E_{23},o_{32}]=o_{32} & [e_{32},o_{32}]=-o_{23}\\ [E_{23},e_{32}]=-2e_{32} & [e_{23},o_{23}]=-o_{32} & [o_{23},o_{32}]=E_{23}\\ [e_{23},e_{32}]=E_{23} & [e_{23},o_{32}]=0 & [o_{23},o_{23}]=-2e_{32}\\ [E_{23},o_{23}]=-o_{23} & [e_{32},o_{23}]=0 & [o_{32},o_{32}]=2e_{23} \end{array}$$

Remark 1. *The set $\{E_{23},e_{23},e_{32}\}$ constitutes the even part of $\mathfrak{osp}(1,2)$ and generates the Lie algebra $\mathfrak{sl}(2)$ i.e., $\mathfrak{osp}_{\bar{0}}(1,2)\cong\mathfrak{sl}(2)$. The set $\{o_{23},o_{32}\}$ constitutes the odd part of $\mathfrak{osp}(1,2)$ and is isomorphic to a 2-dimensional standard representation of $\mathfrak{sl}(2)$.*

To calculate the first homology group, notice that from the boundary map d_1 above, $\ker d_1 = \mathfrak{osp}(1,2)$. Now by definition of Tanaka's complex, the boundary map d_2 is given by:

$$\begin{array}{ll} d(E_{23}\wedge e_{23}) = -[E_{23},e_{23}] = -2e_{23} & d(e_{23}\wedge o_{32}) = -[e_{23},o_{32}] = 0\\ d(E_{23}\wedge e_{32}) = -[E_{23},e_{32}] = 2e_{32} & d(e_{32}\wedge o_{23}) = -[e_{32},o_{23}] = 0\\ d(e_{23}\wedge e_{32}) = -[e_{23},e_{32}] = -E_{23} & d(e_{32}\wedge o_{32}) = -[e_{32},o_{32}] = o_{23}\\ d(E_{23}\wedge o_{23}) = -[E_{23},o_{23}] = o_{23} & d(o_{23}\wedge o_{32}) = -[o_{23},o_{32}] = -E_{23}\\ d(E_{23}\wedge o_{32}) = -[E_{23},o_{32}] = -o_{32} & d(o_{23}\wedge o_{23}) = -[o_{23},o_{23}] = 2e_{32}\\ d(e_{23}\wedge o_{23}) = -[e_{23},o_{23}] = o_{32} & d(o_{32}\wedge o_{32}) = -[e_{32},o_{32}] = -2e_{23}. \end{array}$$

From these formulas, it is clear that $\mathrm{Im}d_2 = \mathfrak{osp}(1,2)$. So

$$H_1(\mathfrak{osp}(1,2)\ ;\ \mathbb{R}) = \frac{kerd_1}{\mathrm{Im}d_2} = \frac{\mathfrak{osp}(1,2)}{\mathfrak{osp}(1,2)} = 0$$

3.2. Second Homology Group

From the boundary map d_2 above, we have

$$\begin{aligned}\ker d_2 =< & E_{23}\wedge o_{32}+e_{23}\wedge o_{23},\ e_{23}\wedge o_{32},\ e_{32}\wedge o_{23}\\ & E_{23}\wedge e_{23}-o_{32}\wedge o_{32},\ e_{23}\wedge e_{32}-o_{23}\wedge o_{32}\\ & E_{23}\wedge e_{32}-o_{23}\wedge o_{23},\ E_{23}\wedge o_{23}-e_{32}\wedge o_{32}>.\end{aligned}$$

Now by definition of Tanaka's complex, the boundary map d_3 is given by:

$$d(E_{23}\wedge e_{23}\wedge e_{32})=0$$
$$d(E_{23}\wedge e_{23}\wedge o_{23})=-e_{23}\wedge o_{23}-E_{23}\wedge o_{32} \tag{3}$$
$$d(E_{23}\wedge e_{23}\wedge o_{32})=-3e_{23}\wedge o_{32} \tag{4}$$
$$d(E_{23}\wedge e_{32}\wedge o_{23})=3e_{32}\wedge o_{23} \tag{5}$$
$$d(E_{23}\wedge e_{32}\wedge o_{32})=e_{32}\wedge o_{32}-E_{23}\wedge o_{23} \tag{6}$$
$$d(e_{23}\wedge e_{32}\wedge o_{23})=-E_{23}\wedge o_{23}+e_{32}\wedge o_{32}$$
$$d(e_{23}\wedge e_{32}\wedge o_{32})=-E_{23}\wedge o_{32}-e_{23}\wedge o_{23}$$
$$d(E_{23}\wedge o_{23}\wedge o_{32})=0$$
$$d(e_{23}\wedge o_{23}\wedge o_{32})=o_{32}\wedge o_{32}-E_{23}\wedge e_{23}$$
$$d(e_{32}\wedge o_{23}\wedge o_{32})=-E_{23}\wedge e_{32}+o_{23}\wedge o_{23} \tag{7}$$
$$d(E_{23}\wedge o_{23}\wedge o_{23})=2o_{23}\wedge o_{23}+2e_{32}\wedge E_{23}$$
$$d(E_{23}\wedge o_{32}\wedge o_{32})=-2o_{32}\wedge o_{32}+2E_{23}\wedge e_{23} \tag{8}$$
$$d(e_{23}\wedge o_{23}\wedge o_{23})=2o_{23}\wedge o_{32}-2e_{23}\wedge e_{32} \tag{9}$$
$$d(e_{23}\wedge o_{32}\wedge o_{32})=0$$
$$d(e_{32}\wedge o_{23}\wedge o_{23})=0$$
$$d(e_{32}\wedge o_{32}\wedge o_{32})=2o_{23}\wedge o_{32}-2e_{23}\wedge e_{32}$$
$$d(o_{23}\wedge o_{23}\wedge o_{23})=6e_{32}\wedge o_{23}$$
$$d(o_{23}\wedge o_{23}\wedge o_{32})=2e_{32}\wedge o_{32}-2E_{23}\wedge o_{23}$$
$$d(o_{23}\wedge o_{32}\wedge o_{32})=-2e_{23}\wedge o_{23}-2E_{23}\wedge o_{32}$$
$$d(o_{32}\wedge o_{32}\wedge o_{32})=-6e_{23}\wedge o_{32}.$$

From the formulas (3)–(9) above, it is clear that $\mathrm{Im} d_3=\ker d_2$. Therefore

$$H_2(\mathfrak{osp}(1,2)\ ;\ \mathbb{R})=0.$$

3.3. Third Homology Group

From the boundary map d_3 above, we have

$$\ker d_3 =<E_{23}\wedge e_{23}\wedge e_{32},\ E_{23}\wedge o_{23}\wedge o_{32},\ e_{23}\wedge o_{32}\wedge o_{32},\ e_{32}\wedge o_{23}\wedge o_{23},$$

$$2E_{23}\wedge e_{23}\wedge o_{32}-o_{32}\wedge o_{32}\wedge o_{32},\ 2E_{23}\wedge e_{32}\wedge o_{23}-o_{23}\wedge o_{23}\wedge o_{23},$$
$$2E_{23}\wedge e_{23}\wedge o_{23}-o_{23}\wedge o_{32}\wedge o_{32},\ 2E_{23}\wedge e_{32}\wedge o_{32}-o_{23}\wedge o_{23}\wedge o_{32},$$
$$E_{23}\wedge e_{23}\wedge o_{23}-e_{23}\wedge e_{32}\wedge o_{32},\ E_{23}\wedge e_{32}\wedge o_{32}-e_{23}\wedge e_{32}\wedge o_{23},$$
$$E_{23}\wedge o_{23}\wedge o_{23}-2e_{32}\wedge o_{23}\wedge o_{32},\ E_{23}\wedge o_{32}\wedge o_{32}+2e_{23}\wedge o_{23}\wedge o_{32},$$
$$e_{23}\wedge o_{23}\wedge o_{23}-e_{32}\wedge o_{32}\wedge o_{32}>.$$

Now by definition of Tanaka's complex, the boundary map d_4 is given by :

$$d(E_{23}\wedge e_{23}\wedge e_{32}\wedge o_{23})=e_{23}\wedge e_{32}\wedge o_{23}-E_{23}\wedge e_{32}\wedge o_{32} \tag{10}$$
$$d(E_{23}\wedge e_{23}\wedge e_{32}\wedge o_{32})=-e_{23}\wedge e_{32}\wedge o_{32}+E_{23}\wedge e_{23}\wedge o_{23} \tag{11}$$
$$d(E_{23}\wedge e_{23}\wedge o_{23}\wedge o_{32})=-E_{23}\wedge o_{32}\wedge o_{32}-2e_{23}\wedge o_{23}\wedge o_{32} \tag{12}$$
$$d(E_{23}\wedge e_{32}\wedge o_{23}\wedge o_{32})=2e_{32}\wedge o_{23}\wedge o_{32}-E_{23}\wedge o_{23}\wedge o_{23} \tag{13}$$
$$d(e_{23}\wedge e_{32}\wedge o_{23}\wedge o_{32})=-E_{23}\wedge o_{23}\wedge o_{32}+e_{32}\wedge o_{32}\wedge o_{32} \tag{14}$$
$$-e_{23}\wedge o_{23}\wedge o_{23}-E_{23}\wedge e_{23}\wedge e_{32} \tag{15}$$
$$d(E_{23}\wedge e_{23}\wedge o_{23}\wedge o_{23})=-2E_{23}\wedge o_{23}\wedge o_{32}+2E_{23}\wedge e_{23}\wedge e_{32} \tag{16}$$
$$d(E_{23}\wedge e_{23}\wedge o_{32}\wedge o_{32})=-4e_{23}\wedge o_{32}\wedge o_{32} \tag{17}$$
$$d(E_{23}\wedge e_{32}\wedge o_{23}\wedge o_{23})=4e_{32}\wedge o_{23}\wedge o_{23} \tag{18}$$
$$d(E_{23}\wedge e_{32}\wedge o_{32}\wedge o_{32})=-2E_{23}\wedge o_{23}\wedge o_{32}+2E_{23}\wedge e_{23}\wedge e_{32}$$
$$d(e_{23}\wedge e_{32}\wedge o_{23}\wedge o_{23})=-E_{23}\wedge o_{23}\wedge o_{23}+2e_{32}\wedge o_{23}\wedge o_{32}$$
$$d(e_{23}\wedge e_{32}\wedge o_{32}\wedge o_{32})=-E_{23}\wedge o_{32}\wedge o_{32}-2e_{23}\wedge o_{23}\wedge o_{32}$$
$$d(E_{23}\wedge o_{23}\wedge o_{23}\wedge o_{23})=3o_{23}\wedge o_{23}\wedge o_{23}-6E_{23}\wedge e_{32}\wedge o_{23} \tag{19}$$
$$d(E_{23}\wedge o_{23}\wedge o_{23}\wedge o_{32})=o_{23}\wedge o_{23}\wedge o_{32}-2E_{23}\wedge e_{32}\wedge o_{32} \tag{20}$$
$$d(E_{23}\wedge o_{23}\wedge o_{32}\wedge o_{32})=-o_{23}\wedge o_{32}\wedge o_{32}+2E_{23}\wedge e_{23}\wedge o_{23} \tag{21}$$
$$d(E_{23}\wedge o_{32}\wedge o_{32}\wedge o_{32})=-3o_{32}\wedge o_{32}\wedge o_{32}+6E_{23}\wedge e_{23}\wedge o_{32} \tag{22}$$
$$d(e_{23}\wedge o_{23}\wedge o_{23}\wedge o_{23})=3o_{23}\wedge o_{23}\wedge o_{32}-6e_{23}\wedge e_{32}\wedge o_{23}$$
$$d(e_{23}\wedge o_{23}\wedge o_{23}\wedge o_{32})=2o_{23}\wedge o_{32}\wedge o_{32}-2e_{23}\wedge e_{32}\wedge o_{32}-2E_{23}\wedge e_{23}\wedge o_{23}$$
$$d(e_{23}\wedge o_{23}\wedge o_{32}\wedge o_{32})=o_{32}\wedge o_{32}\wedge o_{32}-2E_{23}\wedge e_{23}\wedge o_{32}$$
$$d(e_{23}\wedge o_{32}\wedge o_{32}\wedge o_{32})=0$$
$$d(e_{32}\wedge o_{23}\wedge o_{23}\wedge o_{23})=0$$
$$d(e_{32}\wedge o_{23}\wedge o_{23}\wedge o_{32})=o_{23}\wedge o_{23}\wedge o_{23}-2E_{23}\wedge e_{32}\wedge o_{23}$$
$$d(e_{32}\wedge o_{23}\wedge o_{32}\wedge o_{32})=2o_{23}\wedge o_{23}\wedge o_{32}-2E_{23}\wedge e_{32}\wedge o_{32}-2e_{23}\wedge e_{32}\wedge o_{23}$$
$$d(e_{32}\wedge o_{32}\wedge o_{32}\wedge o_{32})=3o_{23}\wedge o_{32}\wedge o_{32}-6e_{23}\wedge e_{32}\wedge o_{32}$$
$$d(o_{23}\wedge o_{23}\wedge o_{23}\wedge o_{23})=12e_{32}\wedge o_{23}\wedge o_{23}$$
$$d(o_{23}\wedge o_{23}\wedge o_{23}\wedge o_{32})=6e_{32}\wedge o_{23}\wedge o_{32}-3E_{23}\wedge o_{23}\wedge o_{23}$$
$$d(o_{23}\wedge o_{23}\wedge o_{32}\wedge o_{32})=2e_{32}\wedge o_{32}\wedge o_{32}-4E_{23}\wedge o_{23}\wedge o_{32}-2e_{23}\wedge o_{23}\wedge o_{23} \tag{23}$$
$$d(o_{23}\wedge o_{32}\wedge o_{32}\wedge o_{32})=-3E_{23}\wedge o_{32}\wedge o_{32}-6e_{23}\wedge o_{23}\wedge o_{32}$$
$$d(o_{32}\wedge o_{32}\wedge o_{32}\wedge o_{32})=-12e_{23}\wedge o_{32}\wedge o_{32}.$$

From the formulas (10)–(13), (17)–(25) above, it is clear that all the cycles but $E_{23} \wedge e_{23} \wedge e_{32}$, $E_{23} \wedge o_{23} \wedge o_{32}$, $e_{23} \wedge o_{23} \wedge o_{23} - e_{32} \wedge o_{32} \wedge o_{32}$ are boundaries, so are zero in homology. By (15) and (16) or (26), these remaining three cycles differ by a boundary, so they generate the same homology class. Therefore

$$H_3(\mathfrak{osp}(1,2)\ ;\ \mathbb{R}) = \langle E_{23} \wedge e_{23} \wedge e_{32} \rangle = \langle E_{23} \wedge o_{23} \wedge o_{32} \rangle = \langle e_{23} \wedge o_{23} \wedge o_{23} - e_{32} \wedge o_{32} \wedge o_{32} \rangle.$$

3.4. Fourth Homology Group

In this subsection, $e^{\wedge k}$ stands for $\underbrace{e \wedge e \wedge \ldots \wedge e}_{\text{k-times}}$. From the boundary map d_4 above, we have

$$\begin{aligned}\ker d_4 = \langle & e_{23} \wedge o_{32}^{\wedge 3},\ e_{32} \wedge o_{23}^{\wedge 3},\ E_{23} \wedge e_{23} \wedge o_{23}^{\wedge 2} - E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2},\\ & 3E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - o_{23} \wedge o_{32}^{\wedge 3},\ 3E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} - o_{23}^{\wedge 3} \wedge o_{32},\\ & 3E_{23} \wedge e_{23} \wedge o_{32}^{\wedge 2} - o_{32} \wedge o_{32}^{\wedge 3},\ 3E_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} - o_{23}^{\wedge 4},\\ & E_{23} \wedge o_{23}^{\wedge 3} - 3e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32},\ E_{23} \wedge o_{32}^{\wedge 3} + 3e_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2},\\ & E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - e_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2},\ E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} - e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} \rangle.\end{aligned}$$

Now by definition of Tanaka's complex, the boundary map d_5 is given by:

$$d(E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}) = E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{23} - E_{23} \wedge e_{32} \wedge o_{32} \wedge o_{32} \tag{24}$$

$$d(E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2}) = 2e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{23} - 2E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} \tag{25}$$

$$d(E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2}) = 2E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - 2e_{23} \wedge e_{32} \wedge o_{32} \wedge o_{32} \tag{26}$$

$$\begin{aligned}
d(E_{23} \wedge e_{23} \wedge o_{23}^{\wedge 3}) &= e_{23} \wedge o_{23}^{\wedge 3} - 3E_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32} + 6E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{23}\\
d(E_{23} \wedge e_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32}) &= -2E_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} + 2E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32}\\
&\quad - e_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32}\\
d(E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2}) &= E_{23} \wedge o_{32}^{\wedge 3} - 3e_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2}
\end{aligned}$$

$$d(E_{23} \wedge e_{23} \wedge o_{32}^{\wedge 3}) = -5e_{23} \wedge o_{32}^{\wedge 3} \tag{27}$$

$$d(E_{23} \wedge e_{32} \wedge o_{23}^{\wedge 3}) = 5e_{32} \wedge o_{23}^{\wedge 3} \tag{28}$$

$$d(E_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32}) = -E_{23} \wedge o_{23}^{\wedge 3} + 3e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32} \tag{29}$$

$$\begin{aligned}
d(E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2}) &= 2E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{23} - 2E_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32}\\
&\quad + e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2}\\
d(E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 3}) &= 6E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32} - 3E_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} + e_{32} \wedge o_{32}^{\wedge 3}\\
d(e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 3}) &= -E_{23} \wedge o_{23}^{\wedge 3} + 3e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32}\\
d(e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32}) &= -E_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32} - 2E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{23}\\
&\quad + 2e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2} - e_{23} \wedge o_{23}^{\wedge 3}\\
d(e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2}) &= -E_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} - 2E_{23} \wedge e_{23} \wedge e_{32} \wedge o_{32}\\
&\quad - 2e_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32} + e_{32} \wedge o_{32}^{\wedge 3}
\end{aligned}$$

$$
\begin{aligned}
d(e_{23} \wedge e_{32} \wedge o_{32}^{\wedge 3}) &= -E_{23} \wedge o_{32}^{\wedge 3} - 3e_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} && (30)\\
d(E_{23} \wedge o_{23}^{\wedge 4}) &= 6E_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} + 4o_{23}^{\wedge 4} \\
d(E_{23} \wedge o_{23}^{\wedge 3} \wedge o_{32}) &= -6E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} + 2o_{23}^{\wedge 3} \wedge o_{32} && (31)\\
d(E_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32}^{\wedge 2}) &= -2E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2} + 2E_{23} \wedge e_{23} \wedge o_{23}^{\wedge 2} \\
d(E_{23} \wedge o_{23} \wedge o_{32}^{\wedge 3}) &= 6E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - 2o_{23} \wedge o_{32}^{\wedge 3} && (32)\\
d(E_{23} \wedge o_{32}^{\wedge 4}) &= 6E_{23} \wedge e_{23} \wedge o_{32}^{\wedge 2} - 4o_{32}^{\wedge 4} \\
d(e_{23} \wedge o_{23}^{\wedge 4}) &= -6e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} + 4o_{23}^{\wedge 3} \wedge o_{32} \\
d(e_{23} \wedge o_{23}^{\wedge 3} \wedge o_{32}) &= 3o_{23}^{\wedge 2} \wedge o_{32}^{\wedge 2} - 6e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} \\
&\quad + 3E_{23} \wedge e_{23} \wedge o_{23}^{\wedge 2} \\
d(e_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32}^{\wedge 2}) &= -4E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - 2e_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2} \\
&\quad + 2o_{23} \wedge o_{32}^{\wedge 3} && (33)\\
d(e_{23} \wedge o_{23} \wedge o_{32}^{\wedge 3}) &= 3E_{23} \wedge e_{23} \wedge o_{32}^{\wedge 2} + o_{32}^{\wedge 4} && (34)\\
d(e_{23} \wedge o_{32}^{\wedge 4}) &= 0 \\
d(e_{32} \wedge o_{23}^{\wedge 4}) &= 0 \\
d(e_{32} \wedge o_{23}^{\wedge 3} \wedge o_{32}) &= 3E_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} + o_{23}^{\wedge 4} && (35)\\
d(e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32}^{\wedge 2}) &= 4E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} + 2e_{23} \wedge e_{32} \wedge o_{23}^{\wedge 2} \\
&\quad - 2o_{23}^{\wedge 3} \wedge o_{32} && (36)\\
d(e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 3}) &= 3o_{23}^{\wedge 2} \wedge o_{32}^{\wedge 2} + 6e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} + 3E_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2} \\
d(e_{32} \wedge o_{32}^{\wedge 4}) &= 6e_{23} \wedge e_{32} \wedge o_{32}^{\wedge 2} + 4o_{23} \wedge o_{32}^{\wedge 3} \\
d(o_{23}^{\wedge 5}) &= 20e_{32} \wedge o_{23}^{\wedge 3} \\
d(o_{23}^{\wedge 4} \wedge o_{32}) &= 12e_{32} \wedge o_{23}^{\wedge 2} \wedge o_{32} - 4E_{23} \wedge o_{23}^{\wedge 3} \\
d(o_{23}^{\wedge 3} \wedge o_{32}^{\wedge 2}) &= 6e_{32} \wedge o_{23} \wedge o_{32}^{\wedge 2} - 6E_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32} - 2e_{23} \wedge o_{23}^{\wedge 3} \\
d(o_{23}^{\wedge 2} \wedge o_{32}^{\wedge 3}) &= -6e_{23} \wedge o_{23}^{\wedge 2} \wedge o_{32} - 6E_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} + 2e_{32} \wedge o_{32}^{\wedge 3} \\
d(o_{23} \wedge o_{32}^{\wedge 4}) &= -12e_{23} \wedge o_{23} \wedge o_{32}^{\wedge 2} - 4E_{23} \wedge o_{32}^{\wedge 3} \\
d(o_{32}^{\wedge 5}) &= -20e_{23} \wedge o_{32}^{\wedge 3}.
\end{aligned}
$$

From the formulas (24)–(32), (34), (35) above, it is clear that all the cycles but

$$
\begin{gathered}
E_{23} \wedge e_{23} \wedge o_{23} \wedge o_{32} - e_{23} \wedge e_{32} \wedge o_{32} \wedge o_{32}, \\
E_{23} \wedge e_{32} \wedge o_{23} \wedge o_{32} - e_{23} \wedge e_{32} \wedge o_{23} \wedge o_{23}
\end{gathered}
$$

are boundaries, so are zero in homology. For these remaining two cycles, combining (31) and (36) shows that the first is a boundary. Similarly, combining (32) and (33) shows that the second is also boundary. Therefore

$$H_4(\mathfrak{osp}(1,2)\ ;\ \mathbb{R}) = 0.$$

In the following section, we provide the low dimensional cohomology groups with trivial coefficients of $\mathfrak{osp}(1,2)$.

4. Lie Superalgebra Cohomology of $\mathfrak{osp}(1,2)$

Theorem 1. *There are isomorphisms of super vector spaces*

$$H^r(\mathfrak{osp}(1,2);\ \mathbb{R}) \cong \begin{cases} \mathbb{R} & \text{for } r=0, \\ 0 & \text{for } r=1,2, \\ \left\langle E_{23}^* \wedge e_{23}^* \wedge e_{32}^* \right\rangle = \left\langle E_{23}^* \wedge o_{23}^* \wedge o_{32}^* \right\rangle = & \\ \left\langle e_{23}^* \wedge o_{23}^* \wedge o_{23}^* - e_{32}^* \wedge o_{32}^* \wedge o_{32}^* \right\rangle, & \text{for } r=3 \\ 0, & \text{for } r=4. \end{cases}$$

Proof. We use the super vector space isomorphism (see [8, lemma 1.7])

$$H^*(\mathfrak{osp}(1,2);\ \mathbb{R}) \cong Hom(H_*(\mathfrak{osp}(1,2);\ \mathbb{R}),\ \mathbb{R})$$

and the dual basis

$$\begin{array}{lll} E_{23}^* := x_2 dx^2 - x_3 dx^3 & e_{32}^* := x_3 dx^2 & o_{32}^* := x_1 dx^3 + x_2 dx^1 \\ e_{23}^* := x_2 dx^3 & o_{23}^* := x_1 dx^2 - x_3 dx^1 & \end{array}$$

where dx^i is the dual of $\frac{\partial}{\partial x^i}$ with respect to the basis of $\mathfrak{osp}(1,2)$ given in section 2. □

References

[1] D B Fuks. *Cohomology of infinite dimensional Lie algebras,* Consultants Bureau, New York, London, 1986.

[2] D B Fuks and D A Leites. *Cohomology of Lie Superalgebras,* Comptes rendus de l'Academie bulgare des Sciences, 37, 12:1595-1596, 1984.

[3] K Iohara and Y Koga. *Central Extensions of Lie Superalgebras,* Commentarii Mathematici Helvetici, 76, 1:110-154, 2001.

[4] K Iohara and Y Koga. *Second homology of Lie Superalgebras,* Mathematische Nachrichten, 278, 9: 1041-1053, 2005.

[5] V G Kac. *Lie Superalgebras, Advances in Mathematics*, 26, (1977), 8-96.

[6] Y Y Kochetkov. *Homology of Nilpotent Subalgebras of the Lie Superalgebra K(1,1),* Mathematical Notes, 73 2:218-227, 2003.

[7] J Tanaka. *On Homology and Cohomology of Lie Superalgebras with Coefficients in Their Finite-Dimensional Representations*, Proceedings of the Japan Academy, 71, Ser. A:51-53, 1995.

[8] J Tanaka. *Homology and Cohomology of Lie Superalgebras* $\mathfrak{sl}(2,1)$ *with Coefficients in The Space of Finite-Dimensional Irreducible Representations*, Journal of Mathematics of Kyoto University, 35, 4:733-756, 1995.

[9] W Xie and Y Zhang. *Second Cohomology of the Modular Lie Superalgebra of Cartan type K*, Algebra Colloquium, 16, 2:309-324, 2009.