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The Distance From a Point to a Compact Convex Set

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Abstract. Let K be a compact convex subset of the plane and $\lambda \in \mathbb{C}\backslash K$, then

$$dist(\lambda, K) = \|(\lambda - N_\mu)^{-1}\|^{-1},$$

where μ is the Lebesgue measure concentrated on K and N_μ be the multiplication operator on $L^2(\mu)$.

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1. Main Result

Let K be a compact convex subset of the plane and μ be the Lebesgue measure concentrated on K, i.e., $\mu = m_2|_K$. Define N_μ on $L^2(\mu)$ by $N_\mu f = zf$ for each f in $L^2(\mu)$. It is easy to check that N_μ is normal. Let $s \in K$ and put $U_n = B(s, \frac{1}{n})$, the disc with center at s and radius $\frac{1}{n}$, so $\mu(U_n) \neq 0$. Since μ is regular then $\mu(U_n) < \infty$. Now define

$$f_n = \frac{1}{\sqrt{\mu(U_n)}} \chi_{U_n},$$

so $\|f_n\|_2 = 1$ and $\|(N_\mu - s)f_n\|_2 \longrightarrow 0$, that is $s \in \sigma(N_\mu)$. Let $s \in K^c$, then there is an open set U with $\mu(U) = 0$ and $s \in U$. Define

$$\psi(z) = \begin{cases} (s-z)^{-1} & \text{if } z \in U^c; \\ 0 & \text{if } z \in U. \end{cases}$$

There is $r > 0$ such that $B(s, r) \subset U$. If $z \in U^c$ then $\frac{1}{|s-z|} < \frac{1}{r}$. Therefore $\|\psi\|_\infty \leq \frac{1}{r}$ a.e. and so $\psi \in L^\infty(\mu)$. Define the operator T on $L^2(\mu)$ by $T(f) = \psi f$, then we have

$$(s - N_\mu)T = T(s - N_\mu) = I$$

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a.e.. Thus s is not in $\sigma(N_\mu)$ and so $\sigma(N_\mu) = K$. Thus for $\lambda \in \mathbb{C}\backslash K$ we have (see [1, Proposition 3.9 p.198]):

$$\|(\lambda - N_\mu)^{-1}\|^{-1} \leq dist(\lambda, K). \quad (1)$$

To prove the inverse inequality, we need to the following concepts which can be found in [2].

For a bounded linear operator T on a Hilbert space $\mathscr{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathscr{H}$ under the quadratic form $x \to < Tx, x >$ associated with the operator. More precisely,

$$W(T) = \{< Tx, x >: x \in \mathscr{H}, \|x\| = 1\}$$

Thus the numerical range of an operator, like the spectrum, is a subset of the complex plane whose geometrical properties should say something about the operator.

One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toeplitz-Hausdorff Theorem. Other important property of $W(T)$ is that its closure contains the spectrum of the operator. $W(T)$ is a connected set and for normal operator N,

$$\overline{W(N)} = co(\sigma(N)), \quad (2)$$

where $\sigma(N)$ is the spectrum of N.

Also we need to the following Theorem which can be found in [3].

Theorem 1. *Let T be a bounded linear operator T on a Hilbert space $\mathscr{H}$ and λ outside $\overline{W(T)}$. Then*

$$dist(\lambda, \overline{W(T)}) \leq \|(\lambda - T)^{-1}\|^{-1}. \quad (3)$$

For the operator N_μ as defined in the first paragraph, we have $\overline{W(N_\mu)} = K$ and the above Theorem implies that:

$$dist(\lambda, K) \leq \|(\lambda - N_\mu)^{-1}\|^{-1}. \quad (4)$$

Now the result follows from (1) and (4).

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[3] F. J. Narcowich. *Analytic properties of the boundary of the numerical range*, Indiana University Mathematics Journal, **29**, 67-77. 1980.