EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 2, 2014, 140-155
ISSN 1307-5543 – www.ejpam.com

Affine Subspaces of the Lie Algebra $\mathfrak{se}(1,1)$

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Abstract. We classify the full-rank affine subspaces (resp. parametrized affine subspaces) of the semi-Euclidean Lie algebra $\mathfrak{se}(1,1)$. The equivalence relations under consideration are motivated by the study of invariant control affine systems. Exhaustive lists of equivalence representatives are obtained, along with classifying conditions.

2010 Mathematics Subject Classifications: 22E60, 93B27

Key Words and Phrases: Lie algebra, affine subspace, equivalence

1. Introduction

A left-invariant control affine system, evolving on a (real, finite-dimensional) Lie group, consists of a family of left-invariant vector fields and a class of "admissible controls". The family of vector fields is affinely parametrized by the control values. A (typical) control is a piecewise continuous curve $u(\cdot)$ in some control set $\mathbb{R}^\ell$. Such a control system on a (matrix) Lie group G is written, in classical notation, as (cf. [11, 16])

$$\dot{g} = g(A + u_1 B_1 + u_2 B_2 + \cdots + u_\ell B_\ell), \quad g \in \mathsf{G},\ u \in \mathbb{R}^\ell. \tag{1}$$

Here $A, B_1, \ldots, B_\ell$ are elements of the Lie algebra $\mathfrak{g}$. These systems provide a fertile geometric setting for various problems in mathematical physics, mechanics, elasticity, and differential geometry [3, 8, 10].

There are two natural equivalence relations for left-invariant control affine systems, namely state space equivalence and detached feedback equivalence (cf. [9, 15]). These equivalence relations are significant in that they establish a one-to-one correspondence between the trajectories of equivalent systems. Two systems are state space equivalent if one can smoothly transform one system into the other, while keeping the controls fixed. For detached feedback equivalence (a weaker equivalence relation), invariant feedback transformations of the

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controls are also permitted. It turns out that these two equivalence relations can be entirely characterized at the level of Lie algebras [4]. More precisely, two systems are state space equivalent (resp. detached feedback equivalent) if and only if the associated parametrized affine subspaces (resp. affine subspaces) are related by a Lie algebra isomorphism. (For a system (1), the associated parametrized affine subspace is given by $\Pi : u \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$, whereas the associated affine subspace is given by $\Gamma = A + \langle B_1, \ldots, B_\ell \rangle$.) Several classes of systems have recently been classified under these equivalence relations [1, 2, 5–7].

In this paper we classify, under the aforementioned equivalence relations, the parametrized affine subspaces (resp. affine subspaces) of the semi-Euclidean Lie algebra $\mathfrak{se}(1,1)$. We classify first the affine subspaces of $\mathfrak{se}(1,1)$. Using these results, we then classify the parametrized affine subspaces. Both classifications are organized by distinguishing between the homogeneity and dimension of the affine subspaces involved. Exhaustive lists of class representatives are obtained, along with associated classifying conditions. A tabulation of the main results is appended.

2. Affine Subspaces and Equivalence

An ℓ-dimensional affine subspace of a Lie algebra $\mathfrak{g}$ is written as

$$\Gamma = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle \tag{2}$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{g}$ and $B_1, \ldots, B_\ell$ are linearly independent. If $A \in \Gamma^0$, we say that Γ is *homogeneous*; otherwise, it is *inhomogeneous*. Γ is referred to as an $(\ell, 0)$-affine subspace if it is homogeneous, and as an $(\ell, 1)$-affine subspace, otherwise. We say that two affine subspaces Γ and Γ' are $\mathfrak{L}$-*equivalent* if there exists a Lie algebra automorphism ψ such that $\psi \cdot \Gamma = \Gamma'$. Note that $\Gamma = A + \Gamma^0$ and $\Gamma' = A' + \Gamma'^0$ are $\mathfrak{L}$-equivalent if and only if there exists an automorphism ψ such that $\psi \cdot \Gamma^0 = \Gamma'^0$ and $\psi \cdot A \in \Gamma'$.

A related concept is that of a parametrized affine subspace, i.e., an (injective) affine $\mathfrak{g}$-valued map. More precisely, an ℓ-dimensional *parametrized affine subspace* is a map

$$\Pi : \mathbb{R}^\ell \to \mathfrak{g}, \quad (u_1, \ldots, u_\ell) \mapsto A + u_1 B_1 + \cdots + u_\ell B_\ell$$

where $B_1, \ldots, B_\ell$ are linearly independent. Whenever convenient, we shall specify Π by simply writing $\Pi : A + u_1 B_1 + \cdots + u_\ell B_\ell$. We say that two parametrized affine subspaces Π and Π' are $\mathfrak{P}$-*equivalent* if there exists an automorphism $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \circ \Pi = \Pi'$. Clearly $\Pi : A + u_1 B_1 + \cdots + u_\ell B_\ell$ is $\mathfrak{P}$-equivalent to $\Pi' : A' + u_1 B_1' + \cdots + u_\ell B_\ell'$ if and only if there exists an automorphism ψ such that $\psi \cdot A = A'$ and $\psi \cdot B_i = B_i'$.

An affine subspace is said to have *full rank* if it generates the entire Lie algebra. (For control systems on Lie groups, the full-rank condition is necessary for controllability). Similarly, a parametrized affine subspace has full rank if its image has full rank. The full-rank property is invariant under both $\mathfrak{L}$-equivalence and $\mathfrak{P}$-equivalence. Throughout, we assume that all affine subspaces (resp. parametrized affine subspaces) under consideration have full rank.

3. Classification

The (real) three-dimensional semi-Euclidean Lie algebra

$$\mathfrak{se}(1,1) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ x_1 & 0 & x_3 \\ x_2 & x_3 & 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}$$

has standard basis

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The commutator relations are given by

$$[E_2, E_3] = -E_1, \quad [E_3, E_1] = E_2, \quad [E_1, E_2] = 0.$$

Remark. $\mathfrak{se}(1,1)$ *is the Lie algebra of the semi-Euclidean group. This matrix Lie group is the group of motions of the Minkowski plane* $\mathbb{R}^{1,1}$*. The signature* $(-1,1)$ *for the Lorentz metric corresponds to the standard basis* (E_1, E_2, E_3)*, whereas the signature* $(1,-1)$ *corresponds to the (Bianchi-Behr) basis* $(E_1, E_2, -E_3)$ *[12–14].*

With respect to the standard basis (E_1, E_2, E_3), the group of automorphisms $\mathsf{Aut}(\mathfrak{se}(1,1))$ takes the form

$$\left\{ \begin{bmatrix} x & y & v \\ \varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} : v, w, x, y \in \mathbb{R}, \varsigma \in \{-1, 1\}, \; x^2 \neq y^2 \right\}.$$

The subsets $\langle E_1, E_2 \rangle$ and $\langle E_1 + E_2 \rangle \cup \langle E_1 - E_2 \rangle$ are invariant.

We now classify, under $\mathfrak{L}$-equivalence (resp. $\mathfrak{P}$-equivalence), all full-rank affine subspaces (resp. parametrized affine subspaces) of $\mathfrak{se}(1,1)$. We outline the approach followed in classifying these objects. First, we distinguish between the dimension and the homogeneity of the affine subspaces; this yields four types of affine subspaces. The invariant subsets allow us to distinguish between various (families of) equivalence classes. In each case, we simplify an arbitrary affine subspace (resp. parametrized affine subspace) by successively applying automorphisms. Finally, we verify that all the candidates for class representatives are distinct and not equivalent. Families of representatives are typically parametrized by constants $\alpha > 0$, $\boldsymbol{\beta} = (\beta_i)$ and $\boldsymbol{\gamma} = (\gamma_i)$, where $\beta_i \neq 0$, $\gamma_i \in \mathbb{R}$.

Remark. *On* $\mathfrak{se}(1,1)$ *(in fact, on any three-dimensional Lie algebra), the full-rank condition for an affine subspace* (2) *can be characterized as follows. No* $(1,0)$*-affine subspace has full rank. A* $(1,1)$*-affine subspace has full rank if and only if* A*,* B_1 *and* $[A, B_1]$ *are linearly independent, whereas a* $(2,0)$*-affine subspace has full rank if and only if* B_1*,* B_2 *and* $[B_1, B_2]$ *are linearly independent. Also, it is clear that any* $(2,1)$*-affine subspace or* $(3,0)$*-affine subspace has full rank.*

3.1. Affine Subspaces

We begin by classifying the affine subspaces of $\mathfrak{se}(1,1)$. Such a classification has been obtained elsewhere [7]. However, for the sake of completeness, we include full proofs here. We denote by E_3^* the corresponding element of the dual basis.

Theorem 1. *Any* $(1,1)$*-affine subspace* $\Gamma = A + \Gamma^0$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following affine subspaces*

$$\begin{cases} \Gamma_1^{(1,1)} = E_1 + \langle E_3 \rangle & E_3^*(\Gamma^0) \neq \{0\} \\ \Gamma_{2,\alpha}^{(1,1)} = \alpha E_3 + \langle E_1 \rangle & E_3^*(\Gamma^0) = \{0\}. \end{cases}$$

Here $\alpha > 0$*, with different values of the parameter yielding distinct (non-equivalent) class representatives.*

Proof. Suppose that $E_3^*(\Gamma^0) \neq \{0\}$. Then $\Gamma = a_1 E_1 + a_2 E_2 + \langle b_1 E_1 + b_2 E_2 + E_3 \rangle$ and

$$\begin{bmatrix} \frac{a_1}{a_1^2 - a_2^2} & -\frac{a_2}{a_1^2 - a_2^2} & 0 \\ -\frac{a_2}{a_1^2 - a_2^2} & \frac{a_1}{a_1^2 - a_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \Gamma = E_1 + \langle E_3 \rangle = \Gamma_1^{(1,1)};$$

as Γ has full rank, we have $a_1^2 \neq a_2^2$. Thus Γ is $\mathfrak{L}$-equivalent to $\Gamma_1^{(1,1)}$.

Suppose $E_3^*(\Gamma^0) = \{0\}$. Then $\Gamma = a_1 E_1 + a_2 E_2 + a_3 E_3 + \langle b_1 E_1 + b_2 E_2 \rangle$ and

$$\begin{bmatrix} \frac{b_1}{b_1^2 - b_2^2} & -\frac{b_2}{b_1^2 - b_2^2} & 0 \\ -\frac{\operatorname{sgn}(a_3) b_2}{b_1^2 - b_2^2} & \frac{\operatorname{sgn}(a_3) b_1}{b_1^2 - b_2^2} & 0 \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \\ 0 & 0 & 1 \end{bmatrix} \cdot \Gamma = |a_3| E_3 + \langle E_1 \rangle = \Gamma_{2,\alpha}^{(1,1)}$$

where $\alpha = |a_3| > 0$. Due to the full-rank assumption, we have $b_1^2 \neq b_2^2$. Hence Γ is $\mathfrak{L}$-equivalent to $\Gamma_{2,\alpha}^{(1,1)}$.

As $\langle E_1, E_2 \rangle$ is an invariant subspace, $\Gamma_1^{(1,1)}$ cannot be $\mathfrak{L}$-equivalent to $\Gamma_{2,\alpha}^{(1,1)}$. It is easy to show that $\Gamma_{2,\alpha}^{(1,1)}$ and $\Gamma_{2,\alpha'}^{(1,1)}$ are $\mathfrak{L}$-equivalent only if $\alpha = \alpha'$. □

If $\Gamma = \langle A, B \rangle$ is a $(2,0)$-affine subspace, then $A + \langle B \rangle$ is a $(1,1)$-affine subspace and hence is $\mathfrak{L}$-equivalent to either $\Gamma_1^{(1,1)}$ or $\Gamma_{2,\alpha}^{(1,1)}$. Thus Γ is $\mathfrak{L}$-equivalent to $\left\langle \Gamma_1^{(1,1)} \right\rangle$ or $\left\langle \Gamma_{2,\alpha}^{(1,1)} \right\rangle$. Accordingly, we get the following classification of $(2,0)$-affine subspaces.

Corollary 1. *Any* $(2,0)$*-affine subspace is* $\mathfrak{L}$*-equivalent to* $\Gamma^{(2,0)} = \langle E_1, E_3 \rangle$.

Theorem 2. *Any* $(2,1)$*-affine subspace* $\Gamma = A + \Gamma^0$ *is* $\mathfrak{L}$*-equivalent to exactly one of the following affine subspaces*

$$\begin{cases} \Gamma_1^{(2,1)} = E_2 + \langle E_1, E_3 \rangle & E_3^*(\Gamma^0) \neq \{0\},\ E_1 + E_2 \notin \Gamma^0 \text{ and } E_1 - E_2 \notin \Gamma^0 \\ \Gamma_2^{(2,1)} = E_1 + \langle E_1 + E_2, E_3 \rangle & E_3^*(\Gamma^0) \neq \{0\},\ E_1 + E_2 \in \Gamma^0 \text{ or } E_1 - E_2 \in \Gamma^0 \\ \Gamma_{3,\alpha}^{(2,1)} = \alpha E_3 + \langle E_1, E_2 \rangle & E_3^*(\Gamma^0) = \{0\}. \end{cases}$$

Here $\alpha > 0$*, with different values of the parameter yielding distinct (non-equivalent) class representatives.*

Proof. Suppose that $E_3^*(\Gamma^0) \neq \{0\}$, $E_1 + E_2 \notin \Gamma^0$ and $E_1 - E_2 \notin \Gamma^0$. Then

$$\Gamma = a_1E_1 + a_2E_2 + \langle b_1E_1 + b_2E_2, c_1E_1 + c_2E_2 + E_3\rangle$$

with $b_1^2 \neq b_2^2$. Hence

$$\Gamma' = \begin{bmatrix} 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \Gamma = a_1E_1 + a_2E_2 + \langle b_1E_1 + b_2E_2, E_3\rangle$$

where $b_1a_2 - a_1b_2 \neq 0$ (as Γ is inhomogeneous). Consequently

$$\begin{aligned}
&\begin{bmatrix} \frac{b_1^2-b_2^2}{b_1a_2-a_1b_2} & 0 & 0 \\ 0 & \frac{b_1^2-b_2^2}{b_1a_2-a_1b_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{b_1}{b_1^2-b_2^2} & -\frac{b_2}{b_1^2-b_2^2} & 0 \\ -\frac{b_2}{b_1^2-b_2^2} & \frac{b_1}{b_1^2-b_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \Gamma' \\
&= \frac{a_1b_2 - a_2b_2}{b_1a_2 - a_1b_2}E_1 + E_2 + \left\langle \frac{b_1^2 - b_2^2}{b_1a_2 - a_1b_2}E_1, E_3 \right\rangle \\
&= E_2 + \langle E_1, E_3\rangle = \Gamma_1^{(2,1)}.
\end{aligned}$$

Thus Γ is $\mathfrak{L}$-equivalent to $\Gamma_1^{(2,1)}$.

On the other hand, suppose that $E_3^*(\Gamma^0) \neq \{0\}$ and $E_1 \pm E_2 \in \Gamma^0$. Then

$$\Gamma = a_1E_1 + a_2E_2 + \langle E_1 \pm E_2, b_1E_1 + b_2E_2 + E_3\rangle$$

and

$$\Gamma' = \begin{bmatrix} 1 & 0 & -b_1 \\ 0 & 1 & -b_2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \Gamma = a_1E_1 + a_2E_2 + \langle E_1 \pm E_2, E_3\rangle$$

where $a_1 \mp a_2 \neq 0$ (as Γ is inhomogeneous). Therefore

$$\begin{aligned}
\begin{bmatrix} \frac{a_1}{a_1^2-a_2^2} & -\frac{a_2}{a_1^2-a_2^2} & 0 \\ \mp\frac{a_2}{a_1^2-a_2^2} & \pm\frac{a_1}{a_1^2-a_2^2} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \cdot \Gamma' =& E_1 + \left\langle \frac{a_1 \mp a_2}{a_1^2 - a_2^2}(E_1 + E_2), \pm E_3 \right\rangle \\
=& E_1 + \langle E_1 + E_2, E_3\rangle = \Gamma_2^{(2,1)}.
\end{aligned}$$

Thus Γ is $\mathfrak{L}$-equivalent to $\Gamma_2^{(2,1)}$.

Lastly, suppose that $E_3^*(\Gamma^0) = \{0\}$. Then $\Gamma = a_3E_3 + \langle E_1, E_2\rangle$ and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \operatorname{sgn}(a_3) & 0 \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix} \cdot \Gamma = |a_3|E_3 + \langle E_1, \operatorname{sgn}(a_3)E_2\rangle = \Gamma_{3,\alpha}^{(2,1)}$$

where $\alpha = |a_3| > 0$. Hence Γ is $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{3,\alpha}$.

As $\langle E_1, E_2\rangle$ and $\langle E_1 + E_2\rangle \cup \langle E_1 - E_2\rangle$ are invariant subsets, no two of $\Gamma^{(2,1)}_1$, $\Gamma^{(2,1)}_2$ and $\Gamma^{(2,1)}_{3,\alpha}$ are $\mathfrak{L}$-equivalent. It is a simple matter to show that $\Gamma^{(2,1)}_{3,\alpha}$ is $\mathfrak{L}$-equivalent to $\Gamma^{(2,1)}_{3,\alpha'}$ only if $\alpha = \alpha'$. $\square$

Remark. *There is only one* $(3,0)$*-affine subspace, namely* $\mathfrak{se}(1,1)$ *itself.*

3.2. Parametrized Affine Subspaces

When convenient, a parametrized affine subspace specified by

$$\Pi : \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i + u_3 \sum_{i=1}^{3} d_i E_i$$

will be represented (in matrix form) as

$$\left[\begin{array}{c|ccc} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right].$$

Since any automorphism ψ is identified with its matrix, the composition $\psi \circ \Pi$ becomes a matrix multiplication.

We begin by classifying the parametrized $(1,1)$-affine subspaces.

Theorem 3. *Let* Π *be a parametrization of a* $(1,1)$*-affine subspace* Γ*.*

(i) If Γ *is* $\mathfrak{L}$*-equivalent to* $\Gamma^{(1,1)}_1$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized affine subspaces*

$$\Pi^{(1,1)}_{1,\alpha,\gamma} : \; E_1 + \gamma_1 E_3 + u(\alpha E_3).$$

(ii) If Γ *is* $\mathfrak{L}$*-equivalent to* $\Gamma^{(1,1)}_{2,\alpha}$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized affine subspaces*

$$\Pi^{(1,1)}_{2,\alpha} : \; \alpha E_3 + u E_1.$$

Here $\alpha > 0$ *and* $\gamma_1 \in \mathbb{R}$*, with different values of these parameters yielding distinct (non-equivalent) class representatives.*

Proof. Let $\Pi : \sum_{i=1}^3 a_i E_i + u \sum_{i=1}^3 b_i E_i$. By Theorem 1, Γ is $\mathfrak{L}$-equivalent to $\Gamma^{(1,1)}_1 = E_1 + \langle E_3\rangle$ or $\Gamma^{(1,1)}_{2,\alpha} = \alpha E_3 + \langle E_1\rangle$.

(i) Suppose that Γ is $\mathfrak{L}$-equivalent to $\Gamma^{(1,1)}_1$. Then $b_3 \neq 0$ and

$$\begin{bmatrix} 1 & 0 & -\frac{b_1}{b_3} \\ 0 & \operatorname{sgn}(b_3) & -\frac{\operatorname{sgn}(b_3) b_2}{b_3} \\ 0 & 0 & \operatorname{sgn}(b_3) \end{bmatrix} \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{array}\right] = \left[\begin{array}{c|c} a_1' & 0 \\ a_2' & 0 \\ a_3' & |b_3| \end{array}\right]$$

(for some $a_1', a_2', a_3' \in \mathbb{R}$). Since Γ is $\mathfrak{L}$-equivalent to $\Gamma_1^{(1,1)}$, we have $a_1' \neq 0$, $a_2' \neq 0$ and $(a_1')^2 \neq (a_2')^2$. Accordingly,

$$\begin{bmatrix} \frac{a_1'}{(a_1')^2-(a_2')^2} & -\frac{a_2'}{(a_1')^2-(a_2')^2} & 0 \\ -\frac{a_2'}{(a_1')^2-(a_2')^2} & \frac{a_1'}{(a_1')^2-(a_2')^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} a_1' & 0 \\ a_2' & 0 \\ a_3' & |b_3| \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha \end{array}\right]$$

where $\alpha = |b_3| > 0$ and $\gamma_1 = a_3' \in \mathbb{R}$. Thus Π is $\mathfrak{P}$-equivalent to $\Pi_{1,\alpha,\gamma}^{(1,1)}$.

(*ii*) Suppose that Γ is $\mathfrak{L}$-equivalent to $\Gamma_{2,\alpha}^{(1,1)}$. Then $b_3 = 0$, $a_3 \neq 0$, $b_1^2 \neq b_2^2$ and

$$\begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & \operatorname{sgn}(a_3) & -\frac{\operatorname{sgn}(a_3)a_2}{a_3} \\ 0 & 0 & \operatorname{sgn}(a_3) \end{bmatrix} \left[\begin{array}{c|c} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & b_1 \\ 0 & \operatorname{sgn}(a_3)b_2 \\ |a_3| & 0 \end{array}\right].$$

Furthermore,

$$\begin{bmatrix} \frac{b_1}{b_1^2-b_2^2} & -\frac{\operatorname{sgn}(a_3)b_2}{b_1^2-b_2^2} & 0 \\ -\frac{\operatorname{sgn}(a_3)b_2}{b_1^2-b_2^2} & \frac{b_1}{b_1^2-b_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} 0 & b_1 \\ 0 & \operatorname{sgn}(a_3)b_2 \\ |a_3| & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{array}\right]$$

where $\alpha = |a_3| > 0$. Thus Π is $\mathfrak{P}$-equivalent to $\Pi_{2,\alpha}^{(1,1)}$.

By Theorem 1, $\Gamma_1^{(1,1)}$ and $\Gamma_{2,\alpha}^{(1,1)}$ are not $\mathfrak{L}$-equivalent. Hence $\Pi_{1,\alpha,\gamma}^{(1,1)}$ is not $\mathfrak{P}$-equivalent to $\Pi_{2,\alpha}^{(1,1)}$. Suppose there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that $\psi \circ \Pi_{1,\alpha,\gamma}^{(1,1)} = \Pi_{1,\alpha',\gamma'}^{(1,1)}$. Then

$$\left[\begin{array}{c|c} x + v\gamma_1 & v\alpha \\ \varsigma y + w\gamma_1 & w\alpha \\ \varsigma\gamma_1 & \varsigma\alpha \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \gamma_1' & \varsigma\alpha' \end{array}\right]$$

(for some $v, w \in \mathbb{R}$, $x^2 \neq y^2$ and $\varsigma \in \{-1, 1\}$) which implies that $\alpha = \alpha'$, $\varsigma = 1$ and $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$. If $\psi \circ \Pi_{2,\alpha}^{(1,1)} = \Pi_{2,\alpha'}^{(1,1)}$ for some automorphism ψ, then

$$\left[\begin{array}{c|c} v\alpha & x \\ w\alpha & \varsigma y \\ \varsigma\alpha & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \alpha' & 0 \end{array}\right]$$

and so $\alpha = \alpha'$. □

If $\Pi : A + u_1 B + u_2 C$ is a parametrized $(2,0)$-affine subspace, then $u \mapsto B + uC$ is a parametrized $(1,1)$-affine subspace, and hence is $\mathfrak{P}$-equivalent to $\Pi_{1,\alpha,\gamma}^{(1,1)}$ or $\Pi_{2,\alpha}^{(1,1)}$. We use this fact to arrive at the following classification of the parametrized $(2,0)$-affine subspaces. These representatives parametrize $\Gamma^{(2,0)}$.

Corollary 2. *Let* $\Pi : \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i$ *be a parametrized* $(2,0)$*-affine subspace.* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized affine subspaces*

$$\begin{cases} \Pi^{(2,0)}_{1,\alpha,\gamma} : \gamma_1 E_1 + \gamma_2 E_3 + u_1(E_1 + \gamma_3 E_3) + u_2(\alpha E_3) & c_3 \neq 0 \\ \Pi^{(2,0)}_{2,\alpha,\gamma} : \gamma_1 E_1 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_1 & c_3 = 0. \end{cases}$$

Here $\alpha > 0$ *and* $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, *with different values of these parameters yielding distinct (non-equivalent) class representatives.*

We now proceed to the classification of the parametrized $(2,1)$-affine subspaces.

Lemma 1. *Let* $X = \sum_{i=1}^{3} x_i E_i$, $Y = \sum_{i=1}^{3} y_i E_i$ *and* $Z = z_3 E_3$ *be linearly independent elements of* $\mathfrak{se}(1,1)$ *and let* $\sigma \in \{-1, 1\}$.

(i) If $y_1^2 \neq y_2^2$, *then there exists* $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ *such that*

$$\psi \cdot X = \begin{bmatrix} x_1' \\ \sigma x_2' \\ \sigma x_3 \end{bmatrix}, \quad \psi \cdot Y = \begin{bmatrix} 1 \\ 0 \\ \sigma y_3 \end{bmatrix}, \quad \psi \cdot Z = \begin{bmatrix} 0 \\ 0 \\ \sigma z_3 \end{bmatrix}$$

for some $x_1', x_2' \in \mathbb{R}$.

(ii) If $y_1^2 = y_2^2$ *and* $x_1^2 \neq x_2^2$, *then there exists* $\varrho \in \{-1, 1\}$ *and* $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ *such that*

$$\psi \cdot X = \begin{bmatrix} x_1' \\ 0 \\ \sigma x_3 \end{bmatrix}, \quad \psi \cdot Y = \begin{bmatrix} 1 \\ \varrho\sigma \\ \sigma y_3 \end{bmatrix}, \quad \psi \cdot Z = \begin{bmatrix} 0 \\ 0 \\ \sigma z_3 \end{bmatrix}$$

for some $x_1' \in \mathbb{R}$.

(iii) If $y_1^2 = y_2^2$ *and* $x_1^2 = x_2^2$, *then there exists* $\varrho \in \{-1, 1\}$ *and* $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ *such that*

$$\psi \cdot X = \begin{bmatrix} 1 \\ -\sigma \\ \sigma\varrho x_3 \end{bmatrix}, \quad \psi \cdot Y = \begin{bmatrix} 1 \\ \sigma \\ \sigma\varrho y_3 \end{bmatrix}, \quad \psi \cdot Z = \begin{bmatrix} 0 \\ 0 \\ \sigma\varrho z_3 \end{bmatrix}.$$

Proof. (i) Suppose that $y_1^2 \neq y_2^2$. Then

$$\begin{bmatrix} \frac{y_1}{y_1^2 - y_2^2} & -\frac{y_2}{y_1^2 - y_2^2} & 0 \\ -\frac{\sigma y_2}{y_1^2 - y_2^2} & \frac{\sigma y_1}{y_1^2 - y_2^2} & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} x_1' & 1 & 0 \\ \sigma x_2' & 0 & 0 \\ \sigma x_3 & \sigma y_3 & \sigma z_3 \end{bmatrix}.$$

(ii) Suppose that $y_1^2 = y_2^2$ and $x_1^2 \neq x_2^2$. Let $y_0 = y_1 \neq 0$. Then $y_2 = \pm y_0$ and

$$\begin{bmatrix} \frac{x_1}{x_1^2 - x_2^2} & -\frac{x_2}{x_1^2 - x_2^2} & 0 \\ \mp\frac{x_2}{x_1^2 - x_2^2} & \pm\frac{x_1}{x_1^2 - x_2^2} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} x_1 & y_0 & 0 \\ x_2 & \pm y_0 & 0 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & y_0' & 0 \\ 0 & y_0' & 0 \\ \pm x_3 & \pm y_3 & \pm z_3 \end{bmatrix}.$$

Furthermore,

$$\begin{bmatrix} \frac{1}{y_0'} & 0 & 0 \\ 0 & \pm\frac{\sigma}{y_0'} & 0 \\ 0 & 0 & \pm\sigma \end{bmatrix} \begin{bmatrix} 1 & y_0' & 0 \\ 0 & y_0' & 0 \\ \pm x_3 & \pm y_3 & \pm z_3 \end{bmatrix} = \begin{bmatrix} x_1'' & 1 & 0 \\ 0 & \varrho\sigma & 0 \\ \sigma x_3 & \sigma y_3 & \sigma z_3 \end{bmatrix}$$

where $\varrho = \pm 1$. (The composition of these two automorphisms yields ψ.)

(*iii*) Suppose that $y_1^2 = y_2^2$ and $x_1^2 = x_2^2$. Let $y_0 = y_1 \neq 0$ and $x_0 = x_1 \neq 0$. Then (since X and Y are linearly independent) $y_2 = \pm y_0$ and $x_2 = \mp x_0$. We have

$$\begin{bmatrix} \frac{1}{y_0} & 0 & 0 \\ 0 & \pm\frac{1}{y_0} & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} x_0 & y_0 & 0 \\ \mp x_0 & \pm y_0 & 0 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} x_0' & 1 & 0 \\ -x_0' & 1 & 0 \\ \pm x_3 & \pm y_3 & \pm z_3 \end{bmatrix}.$$

Moreover,

$$\begin{bmatrix} \frac{x_0'+1}{2x_0'} & \frac{x_0'-1}{2x_0'} & 0 \\ \frac{\sigma(x_0'-1)}{2x_0'} & \frac{\sigma(x_0'+1)}{2x_0'} & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} x_0' & 1 & 0 \\ -x_0' & 1 & 0 \\ \pm x_3 & \pm y_3 & \pm z_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -\sigma & \sigma & 0 \\ \sigma\varrho x_3 & \sigma\varrho y_3 & \sigma\varrho z_3 \end{bmatrix}$$

where $\varrho = \pm 1$. □

Theorem 4. *Let* $\Pi : \sum_{i=1}^3 a_i E_i + u_1 \sum_{i=1}^3 b_i E_i + u_2 \sum_{i=1}^3 c_i E_i$ *be a parametrization of a* $(2,1)$*-affine subspace* Γ.

(i) If Γ *is* $\mathfrak{L}$*-equivalent to* $\Gamma_1^{(2,1)}$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized affine subspaces*

$$\begin{cases} \Pi^{(2,1)}_{1,\alpha,\beta,\gamma} : \gamma_1 E_1 + \beta_1 E_2 + \gamma_2 E_3 + u_1(E_1 + \gamma_3 E_3) + u_2(\alpha E_3) & c_3 \neq 0 \\ \Pi^{(2,1)}_{2,\alpha,\beta,\gamma} : \gamma_1 E_1 + \beta_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_1 & c_3 = 0. \end{cases}$$

(ii) If Γ *is* $\mathfrak{L}$*-equivalent to* $\Gamma_2^{(2,1)}$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized affine subspaces*

$$\begin{cases} \Pi^{(2,1)}_{3,\beta,\gamma} : \beta_1 E_1 + \gamma_1 E_3 + u_1(E_1 + E_2 + \gamma_2 E_3) + u_2(\beta_2 E_3) & c_3 \neq 0,\ \left(\frac{a_1c_3 - a_3c_1}{c_3}\right)^2 \neq \left(\frac{a_2c_3 - a_3c_2}{c_3}\right)^2 \\ \Pi^{(2,1)}_{4,\beta,\gamma} : E_1 - E_2 + \gamma_1 E_3 + u_1(E_1 + E_2 + \gamma_2 E_3) + u_2(\beta_1 E_3) & c_3 \neq 0,\ \left(\frac{a_1c_3 - a_3c_1}{c_3}\right)^2 = \left(\frac{a_2c_3 - a_3c_2}{c_3}\right)^2 \\ \Pi^{(2,1)}_{5,\beta,\gamma} : \beta_1 E_1 + \gamma_1 E_3 + u_1(\beta_2 E_3) + u_2(E_1 + E_2) & c_3 = 0,\ \left(\frac{a_1b_3 - a_3b_1}{b_3}\right)^2 \neq \left(\frac{a_2b_3 - a_3b_2}{b_3}\right)^2 \\ \Pi^{(2,1)}_{6,\beta,\gamma} : E_1 - E_2 + \gamma_1 E_3 + u_1(\beta_1 E_3) + u_2(E_1 + E_2) & c_3 = 0,\ \left(\frac{a_1b_3 - a_3b_1}{b_3}\right)^2 = \left(\frac{a_2b_3 - a_3b_2}{b_3}\right)^2. \end{cases}$$

(iii) If Γ *is* $\mathfrak{L}$*-equivalent to* $\Gamma_{3,\alpha}^{(2,1)}$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized affine subspaces*

$$\begin{cases} \Pi^{(2,1)}_{7,\alpha,\beta,\gamma} : \beta_1 E_3 + u_1(\gamma_1 E_1 + \alpha E_2) + u_2 E_1 & c_1^2 \neq c_2^2 \\ \Pi^{(2,1)}_{8,\beta} : \beta_1 E_3 + u_1(\beta_2 E_1) + u_2(E_1 + E_2) & c_1^2 = c_2^2,\ b_1^2 \neq b_2^2 \\ \Pi^{(2,1)}_{9,\beta} : \beta_1 E_3 + u_1(E_1 - E_2) + u_2(E_1 + E_2) & c_1^2 = c_2^2,\ b_1^2 = b_2^2. \end{cases}$$

Here $\alpha > 0$, $\beta_i \neq 0$ and $\gamma_i \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives.

Proof. By Theorem 2, we have that Γ is $\mathfrak{L}$-equivalent to $\Gamma_1^{(2,1)} = E_2 + \langle E_1, E_3 \rangle$, $\Gamma_2^{(2,1)} = E_1 + \langle E_1 + E_2, E_3 \rangle$ or $\Gamma_{3,\alpha}^{(2,1)} = \alpha E_3 + \langle E_1, E_2 \rangle$.

(i) Assume Γ is $\mathfrak{L}$-equivalent to $\Gamma_1^{(2,1)}$. The affine subspace $\sum_{i=1}^3 b_i E_i + \left\langle \sum_{i=1}^3 c_i E_i \right\rangle$ has full rank (as $\langle E_1, E_3 \rangle$ has full rank) and so the parametrized affine subspace

$$u \mapsto \sum_{i=1}^{3} b_i E_i + u \sum_{i=1}^{3} c_i E_i$$

is $\mathfrak{P}$-equivalent to $\Pi_{1,\alpha,\gamma}^{(1,1)}$ or $\Pi_{2,\alpha}^{(1,1)}$, by Theorem 3. It follows that Π is $\mathfrak{P}$-equivalent to $\Pi_{1,\alpha,\boldsymbol{\beta},\gamma}^{(2,1)}$ when $c_3 \neq 0$ and Π is $\mathfrak{P}$-equivalent to $\Pi_{2,\alpha,\boldsymbol{\beta},\gamma}^{(2,1)}$ when $c_3 = 0$. As Π is inhomogeneous, $\beta_1 \neq 0$.

(ii) Assume Γ is $\mathfrak{L}$-equivalent to $\Gamma_2^{(2,1)}$ (in this case $b_3 \neq 0$ or $c_3 \neq 0$). Suppose that $c_3 \neq 0$. Then

$$\begin{bmatrix} 1 & 0 & -\frac{c_1}{c_3} \\ 0 & 1 & -\frac{c_2}{c_3} \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} a_1' & b_1' & 0 \\ a_2' & b_2' & 0 \\ a_3 & b_3 & c_3 \end{array}\right]$$

where $a_1' = \frac{a_1 c_3 - a_3 c_1}{c_3}$, $a_2' = \frac{a_2 c_3 - a_3 c_2}{c_3}$ and $b_1', b_2' \in \mathbb{R}$. As Γ is $\mathfrak{L}$-equivalent to $\Gamma_2^{(2,1)}$, we have $(b_1')^2 = (b_2')^2$. Suppose $(a_1')^2 \neq (a_2')^2$. By the lemma (with $X = a_1' E_1 + a_2' E_2 + a_3 E_3$, $Y = b_1' E_1 + b_2' E_2 + b_3 E_3$ and $Z = c_3 E_3$) there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1' & b_1' & 0 \\ a_2' & b_2' & 0 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} \beta_1 & 1 & 0 \\ 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & \beta_2 \end{array}\right]$$

for some $\beta_1, \beta_2 \neq 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. Therefore Π is $\mathfrak{P}$-equivalent to $\Pi_{3,\boldsymbol{\beta},\gamma}^{(2,1)}$. On the other hand, suppose that $(a_1')^2 = (a_2')^2$. By the lemma (with X, Y and Z as before) there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1' & b_1' & 0 \\ a_2' & b_2' & 0 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} 1 & 1 & 0 \\ -1 & 1 & 0 \\ \gamma_1 & \gamma_2 & \beta_1 \end{array}\right]$$

for some $\beta_1 \neq 0$ and $\gamma_1, \gamma_2 \in \mathbb{R}$. Hence Π is $\mathfrak{P}$-equivalent to $\Pi_{4,\boldsymbol{\beta},\gamma}^{(2,1)}$.

Suppose that $c_3 = 0$. Then $b_3 \neq 0$ and

$$\begin{bmatrix} 1 & 0 & -\frac{b_1}{b_3} \\ 0 & 1 & -\frac{b_2}{b_3} \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & 0 \end{array}\right] = \left[\begin{array}{c|cc} a_1' & 0 & c_1' \\ a_2' & 0 & c_2' \\ a_3 & b_3 & 0 \end{array}\right]$$

where $a_1' = \frac{a_1 b_3 - b_1 a_3}{b_3}$, $a_2' = \frac{a_2 b_3 - b_2 a_3}{b_3}$ and $c_1', c_2' \in \mathbb{R}$. Since Γ is $\mathfrak{L}$-equivalent to $\Gamma_2^{(2,1)}$, we have $(c_1')^2 = (c_2')^2$. Suppose that $(a_1')^2 \neq (a_2')^2$. By the lemma (with $X = a_1'E_1 + a_2'E_2 + a_3E_3$, $Y = c_1'E_1 + c_2'E_2$ and $Z = b_3E_3$), there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1' & 0 & c_1' \\ a_2' & 0 & c_2' \\ a_3 & b_3 & 0 \end{array}\right] = \left[\begin{array}{c|cc} \beta_1 & 0 & 1 \\ 0 & 0 & 1 \\ \gamma_1 & \beta_2 & 0 \end{array}\right]$$

for some $\beta_1, \beta_2 \neq 0$ and $\gamma_1 \in \mathbb{R}$. Thus Π is $\mathfrak{P}$-equivalent to $\Pi_{5,\boldsymbol{\beta},\boldsymbol{\gamma}}^{(2,1)}$. On the other hand, suppose that $(a_1')^2 = (a_2')^2$. By the lemma (with X, Y and Z as before), there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1' & 0 & c_1' \\ a_2' & 0 & c_2' \\ a_3 & b_3 & 0 \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & 1 \\ -1 & 0 & 1 \\ \gamma_1 & \beta_1 & 0 \end{array}\right]$$

for some $\beta_1 \neq 0$ and $\gamma_1 \in \mathbb{R}$. Hence Π is $\mathfrak{P}$-equivalent to $\Pi_{6,\boldsymbol{\beta},\boldsymbol{\gamma}}^{(2,1)}$.

(*iii*) Assume Γ is $\mathfrak{L}$-equivalent to $\Gamma_{3,\alpha}^{(2,1)}$ (in this case, $b_3 = c_3 = 0$ and $a_3 \neq 0$). We have

$$\begin{bmatrix} 1 & 0 & -\frac{a_1}{a_3} \\ 0 & 1 & -\frac{a_2}{a_3} \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array}\right].$$

Suppose that $c_1^2 \neq c_2^2$. By the lemma (with $X = b_1E_1 + b_2E_2$, $Y = c_1E_1 + c_2E_2$ and $Z = a_3E_3$), there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} 0 & \gamma_1 & 1 \\ 0 & \alpha & 0 \\ \beta_1 & 0 & 0 \end{array}\right]$$

for some $\alpha > 0$, $\beta_1 \neq 0$ and $\gamma_1 \in \mathbb{R}$. Therefore Π is $\mathfrak{P}$-equivalent to $\Pi_{7,\alpha,\boldsymbol{\beta},\boldsymbol{\gamma}}^{(2,1)}$.

Suppose that $c_1^2 = c_2^2$ and $b_1^2 \neq b_2^2$. By the lemma (with $X = b_1E_1 + b_2E_2$, $Y = c_1E_1 + c_2E_2$ and $Z = a_3E_3$), there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} 0 & \beta_2 & 1 \\ 0 & 0 & 1 \\ \beta_1 & 0 & 0 \end{array}\right]$$

for some $\beta_1, \beta_2 \neq 0$. Thus Π is $\mathfrak{P}$-equivalent to $\Pi_{8,\boldsymbol{\beta}}^{(2,1)}$.

Suppose that $c_1^2 = c_2^2$ and $b_1^2 = b_2^2$. By the lemma (with $X = b_1E_1 + b_2E_2$, $Y = c_1E_1 + c_2E_2$ and $Z = a_3E_3$), there exists $\psi \in \mathsf{Aut}(\mathfrak{se}(1,1))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ a_3 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} 0 & 1 & 1 \\ 0 & -1 & 1 \\ \beta_1 & 0 & 0 \end{array}\right]$$

for some $\beta_1 \neq 0$. Hence Π is $\mathfrak{P}$-equivalent to $\Pi^{(2,1)}_{9,\beta}$.

Clearly, parametrized affine subspaces corresponding to different $(2,1)$-affine subspace class representatives ($\Gamma^{(2,1)}_1$, $\Gamma^{(2,1)}_2$ and $\Gamma^{(2,1)}_{3,\alpha}$) cannot be $\mathfrak{P}$-equivalent. No two families in case (*i*), case (*ii*) and case (*iii*) are $\mathfrak{P}$-equivalent, as the subsets $\langle E_1, E_2\rangle$ and $\langle E_1 + E_2\rangle \cup \langle E_1 - E_2\rangle$ are invariant. For each family, it is straightforward to verify that two representatives are $\mathfrak{P}$-equivalent only if their parameters are equal. □

Suppose $\Pi : A + u_1 B + u_2 C + u_3 D$ is a parametrization of a $(3,0)$-affine subspace Γ. We shall denote by $\widehat{\Pi}$ the parametrization $\widehat{\Pi} : B + u_1 C + u_2 D$ of the associated affine subspace $\widehat{\Gamma} = B + \langle C, D\rangle$. As $\widehat{\Pi}$ is a parametrized $(2,1)$-affine subspace, it is $\mathfrak{P}$-equivalent to exactly one of the representatives listed in Theorem 4. Accordingly, we get the following classification of parametrized $(3,0)$-affine subspaces (we again use Theorem 2, the classification of $(2,1)$-affine subspaces, to organize the results).

Corollary 3. *Let* $\Pi : \sum_{i=1}^3 a_i E_i + u_1 \sum_{i=1}^3 b_i E_i + u_2 \sum_{i=1}^3 c_i E_i + u_3 \sum_{i=1}^3 d_i E_i$ *be a parametrization of a* $(3,0)$*-affine subspace* Γ.

(i) *If* $\widehat{\Gamma}$ *is* $\mathfrak{L}$*-equivalent to* $\Gamma^{(2,1)}_1$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized subspaces*

$$\begin{cases} \Pi^{(3,0)}_{1,\alpha,\beta,\gamma} : \sum_{i=1}^3 \gamma_i E_i + u_1(\gamma_4 E_1 + \beta_1 E_2 + \gamma_5 E_3) + u_2(E_1 + \gamma_6 E_2) + u_3(\alpha E_3) & d_3 \neq 0 \\ \Pi^{(3,0)}_{2,\alpha,\beta,\gamma} : \sum_{i=1}^3 \gamma_i E_i + u_1(\gamma_4 E_1 + \beta_1 E_2 + \gamma_5 E_3) + u_2(\alpha E_3) + u_3 E_1 & d_3 = 0. \end{cases}$$

(ii) *If* $\widehat{\Gamma}$ *is* $\mathfrak{L}$*-equivalent to* $\Gamma^{(2,1)}_2$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized subspaces*

$$\begin{cases} \Pi^{(3,0)}_{3,\beta,\gamma} : & \sum_{i=1}^3 \gamma_i E_i + u_1(\beta_1 E_1 + \gamma_4 E_3) + u_2(E_1 + E_2 + \gamma_5 E_3) + u_3(\beta_2 E_3) \\ & \qquad d_3 \neq 0,\ \left(\frac{b_1 d_3 - b_3 d_1}{d_3}\right)^2 \neq \left(\frac{b_2 d_3 - b_3 d_2}{d_3}\right)^2 \\ \Pi^{(3,0)}_{4,\beta,\gamma} : & \sum_{i=1}^3 \gamma_i E_i + u_1(E_1 - E_2 + \gamma_4 E_3) + u_2(E_1 + E_2 + \gamma_5 E_3) + u_3(\beta_1 E_3) \\ & \qquad d_3 \neq 0,\ \left(\frac{b_1 d_3 - b_3 d_1}{d_3}\right)^2 = \left(\frac{b_2 d_3 - b_3 d_2}{d_3}\right)^2 \\ \Pi^{(3,0)}_{5,\beta,\gamma} : & \sum_{i=1}^3 \gamma_i E_i + u_1(\beta_1 E_1 + \gamma_4 E_3) + u_2(\beta_2 E_3) + u_3(E_1 + E_2) \\ & \qquad d_3 = 0,\ \left(\frac{b_1 c_3 - b_3 c_1}{c_3}\right)^2 \neq \left(\frac{b_2 c_3 - b_3 c_2}{c_3}\right)^2 \\ \Pi^{(3,0)}_{6,\beta,\gamma} : & \sum_{i=1}^3 \gamma_i E_i + u_1(E_1 - E_2 + \gamma_4 E_3) + u_2(\beta_1 E_3) + u_3(E_1 + E_2) \\ & \qquad d_3 = 0,\ \left(\frac{b_1 c_3 - b_3 c_1}{c_3}\right)^2 = \left(\frac{b_2 c_3 - b_3 c_2}{c_3}\right)^2. \end{cases}$$

(iii) *If* $\widehat{\Gamma}$ *is* $\mathfrak{L}$*-equivalent to* $\Gamma^{(2,1)}_{3,\alpha}$*, then* Π *is* $\mathfrak{P}$*-equivalent to exactly one of the following parametrized subspaces*

$$\begin{cases} \Pi^{(3,0)}_{7,\alpha,\beta,\gamma} : \sum_{i=1}^3 \gamma_i E_i + u_1(\beta_1 E_3) + u_2(\gamma_4 E_1 + \alpha E_2) + u_3 E_1 & d_1^2 \neq d_2^2 \\ \Pi^{(3,0)}_{8,\beta,\gamma} : \sum_{i=1}^3 \gamma_i E_i + u_1(\beta_1 E_3) + u_2(\beta_2 E_1) + u_3(E_1 + E_2) & d_1^2 = d_2^2,\ c_1^2 \neq c_2^2 \\ \Pi^{(3,0)}_{9,\beta,\gamma} : \sum_{i=1}^3 \gamma_i E_i + u_1(\beta_1 E_3) + u_2(E_1 - E_2) + u_3(E_1 + E_2) & d_1^2 = d_2^2,\ c_1^2 = c_2^2. \end{cases}$$

Here $\alpha > 0$*,* $\beta_i \neq 0$ *and* $\gamma_i \in \mathbb{R}$*, with different values of these parameters yielding distinct (non-equivalent) class representatives.*

4. Final Remark

In this paper we classified, under $\mathfrak{L}$-equivalence (resp. $\mathfrak{P}$-equivalence), the affine subspaces (resp. parametrized affine subspaces) of the semi-Euclidean Lie algebra $\mathfrak{se}(1,1)$. This can be interpreted as a classification, under detached feedback equivalence (resp. state space equivalence), of left-invariant control affine systems on the semi-Euclidean group $\mathsf{SE}(1,1)$ (i.e., the connected matrix Lie group with Lie algebra $\mathfrak{se}(1,1)$). For instance, by corollary 1, any two-input homogeneous system (1) on $\mathsf{SE}(1,1)$ is detached feedback equivalent to the system $\dot{g} = g(u_1 E_1 + u_2 E_3)$. Likewise, by Corollary 2, any homogeneous two-input system on $\mathsf{SE}(1,1)$ is state space equivalent to exactly one of the systems

$$\dot{g} = g\left(\gamma_1 E_1 + \gamma_2 E_3 + u_1(E_1 + \gamma_3 E_3) + u_2(\alpha E_3)\right)$$
$$\dot{g} = g\left(\gamma_1 E_1 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_1\right).$$

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Appendix

Table 1: Classification of affine subspaces and parametrized affine subspaces

<table>
<tr><th>Type</th><th>Affine subspaces</th><th>Parametrized affine subspaces</th></tr>
<tr><td rowspan="2">$(1,1)$</td><td>$E_1 + \langle E_3 \rangle$</td><td>$\left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \gamma_1 & \alpha \end{array}\right]$</td></tr>
<tr><td>$\alpha E_3 + \langle E_1 \rangle$</td><td>$\left[\begin{array}{c|c} 0 & 1 \\ 0 & 0 \\ \alpha & 0 \end{array}\right]$</td></tr>
<tr><td>$(2,0)$</td><td>$\langle E_1, E_3 \rangle$</td><td>$\left[\begin{array}{c|cc} \gamma_1 & 1 & 0 \\ 0 & 0 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right]$ $\left[\begin{array}{c|cc} \gamma_1 & 0 & 1 \\ 0 & 0 & 0 \\ \gamma_2 & \alpha & 0 \end{array}\right]$</td></tr>
<tr><td rowspan="3">$(2,1)$</td><td>$E_2 + \langle E_1, E_3 \rangle$</td><td>$\left[\begin{array}{c|cc} \gamma_1 & 1 & 0 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right]$ $\left[\begin{array}{c|cc} \gamma_1 & 0 & 1 \\ \beta_1 & 0 & 0 \\ \gamma_2 & \alpha & 0 \end{array}\right]$</td></tr>
<tr><td>$E_1 + \langle E_1 + E_2, E_3 \rangle$</td><td>$\left[\begin{array}{c|cc} \beta_1 & 1 & 0 \\ 0 & 1 & 0 \\ \gamma_1 & \gamma_2 & \beta_2 \end{array}\right]$ $\left[\begin{array}{c|cc} 1 & 1 & 0 \\ -1 & 1 & 0 \\ \gamma_1 & \gamma_2 & \beta_1 \end{array}\right]$
$\left[\begin{array}{c|cc} \beta_1 & 0 & 1 \\ 0 & 0 & 1 \\ \gamma_1 & \beta_2 & 0 \end{array}\right]$ $\left[\begin{array}{c|cc} 1 & 0 & 1 \\ -1 & 0 & 1 \\ \gamma_1 & \beta_1 & 0 \end{array}\right]$</td></tr>
<tr><td>$\alpha E_3 + \langle E_1, E_2 \rangle$</td><td>$\left[\begin{array}{c|cc} 0 & \gamma_1 & 1 \\ 0 & \alpha & 0 \\ \beta_1 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{c|cc} 0 & \beta_2 & 1 \\ 0 & 0 & 1 \\ \beta_1 & 0 & 0 \end{array}\right]$
$\left[\begin{array}{c|cc} 0 & 1 & 1 \\ 0 & -1 & 1 \\ \beta_1 & 0 & 0 \end{array}\right]$</td></tr>
<tr><td colspan="3">$\alpha > 0,\ \beta_i \neq 0,\ \gamma_i \in \mathbb{R}$</td></tr>
</table>

Table 2: Classification of parametrized $(3,0)$-affine subspaces $\Pi : A + u_1 B + u_2 C + u_3 D$

<table>
<tr><th>Classifying conditions</th><th>Parametrized affine subspaces</th></tr>
<tr><td>$E_3^*(\langle C,D\rangle) \neq \{0\}$
$E_1 + E_2, E_1 - E_2 \notin \langle C,D\rangle$</td><td>$\left[\begin{array}{c|ccc} \gamma_1 & \gamma_4 & 1 & 0 \\ \gamma_2 & \beta_1 & 0 & 0 \\ \gamma_3 & \gamma_5 & \gamma_6 & \alpha \end{array}\right]$ $\left[\begin{array}{c|ccc} \gamma_1 & \gamma_4 & 0 & 1 \\ \gamma_2 & \beta_1 & 0 & 0 \\ \gamma_3 & \gamma_5 & \alpha & 0 \end{array}\right]$</td></tr>
<tr><td>$E_3^*(\langle C,D\rangle) \neq \{0\}$
$E_1 \pm E_2 \in \langle C,D\rangle$</td><td>$\left[\begin{array}{c|ccc} \gamma_1 & \beta_1 & 1 & 0 \\ \gamma_2 & 0 & 1 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 & \beta_2 \end{array}\right]$ $\left[\begin{array}{c|ccc} \gamma_1 & 1 & 1 & 0 \\ \gamma_2 & -1 & 1 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 & \beta_1 \end{array}\right]$
$\left[\begin{array}{c|ccc} \gamma_1 & \beta_1 & 0 & 1 \\ \gamma_2 & 0 & 0 & 1 \\ \gamma_3 & \gamma_4 & \beta_2 & 0 \end{array}\right]$ $\left[\begin{array}{c|ccc} \gamma_1 & 1 & 0 & 1 \\ \gamma_2 & -1 & 0 & 1 \\ \gamma_3 & \gamma_4 & \beta_1 & 0 \end{array}\right]$</td></tr>
<tr><td>$E_3^*(\langle C,D\rangle) = \{0\}$</td><td>$\left[\begin{array}{c|ccc} \gamma_1 & 0 & \gamma_4 & 1 \\ \gamma_2 & 0 & \alpha & 0 \\ \gamma_3 & \beta_1 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{c|ccc} \gamma_1 & 0 & \beta_2 & 1 \\ \gamma_2 & 0 & 0 & 1 \\ \gamma_3 & \beta_1 & 0 & 0 \end{array}\right]$
$\left[\begin{array}{c|ccc} \gamma_1 & 0 & 1 & 1 \\ \gamma_2 & 0 & -1 & 1 \\ \gamma_3 & \beta_1 & 0 & 0 \end{array}\right]$</td></tr>
<tr><td colspan="2">$\alpha > 0,\ \beta_i \neq 0,\ \gamma_i \in \mathbb{R}$</td></tr>
</table>