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Unitary Addition Cayley Signed Graphs

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Abstract. A *signed graph* (or *sigraph* in short) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$ and $\sigma : E \rightarrow \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$. For a positive integer n, the *unitary addition Cayley graph* G_n is the graph whose vertex set is Z_n, the ring of integers modulo n and if U_n denotes set of all units of the ring, then two vertices a and b are adjacent if and only if $a + b \in U_n$. For a positive integer n, the *unitary addition Cayley sigraph* $\Sigma_n = (\Sigma_n^u, \sigma)$ is defined as the sigraph, where Σ_n^u is the unitary addition Cayley graph and for an edge ab of Σ_n,

$$\sigma(ab) = \begin{cases} + & \text{if } a \in U_n \text{ or } b \in U_n, \\ - & \text{otherwise.} \end{cases}$$

In this paper, we have obtained a characterization of balanced and clusterable unitary addition Cayley sigraphs. Further, we have established a characterization of canonically consistent unitary addition Cayley sigraphs Σ_n, where n has at most two distinct odd prime factors.

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1. Introduction

For standard terminology and notation in graph theory, we refer the reader to Harary [30] and West [45] and to Zaslavsky [46, 47] for sigraphs. Throughout the text, we consider finite, undirected graphs with no loops or multiple edges.

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1.1. Sigraphs and Some Basic Notions and Notations

A *signed graph* (or, *sigraph* in short; see [29]) is an ordered pair $S = (S^u, \sigma)$, where S^u is a graph $G = (V, E)$, called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ is a function from the edge set E of S^u into the set $\{+, -\}$, called the *signature* of S. Let

$$E^+(S) = \{e \in E(G) : \sigma(e) = +\}$$

and

$$E^-(S) = \{e \in E(G) : \sigma(e) = -\}.$$

The elements of $E^+(S)$ and $E^-(S)$ are called *positive* and *negative* edges of S, respectively. A sigraph is *all-positive* (*all-negative*) if all its edges are positive (negative); further, it is said to be *homogeneous* if it is either all-positive or all-negative and *heterogeneous* otherwise.

The *positive* (*negative*) *degree* of a vertex $v \in V(S)$ denoted by $d^+(v)(d^-(v))$ is the number of positive (negative) edges incident on the vertex v and $d(v) = d^+(v) + d^-(v)$. The *negation* $\eta(S)$ of a sigraph S is a sigraph obtained from S by negating the sign of every edge of S, in the sense that to find $\eta(S)$ we change the sign of every edge to its opposite in S.

A *positive* (*negative*) *section* of a subsigraph S' of a sigraph S is a maximal edge-induced connected subsigraph in S consisting of only the positive (negative) edges of S; in particular, a positive (negative) section in a heterogeneous cycle of S is essentially a maximal all-positive (all-negative) path in the cycle.

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a bijective function $f : V_1 \to V_2$ such that for all $v_1, v_2 \in V$: $v_1v_2 \in E_1 \Leftrightarrow f(v_1)f(v_2) \in E_2$. Two sigraphs S_1 and S_2 are isomorphic if there is an isomorphism between their underlying graphs that preserves edge signs.

A cycle in a sigraph S is said to be *positive* if it contains an even number of negative edges. A given sigraph S is said to be *balanced* if every cycle in S is positive (*see* [29]); balanced sigraphs were first defined and characterized by Harary [29]. A spectral characterization of balanced sigraphs was given by Acharya [1]. Harary and Kabell [31, 32] developed a simple algorithm to detect balanced sigraphs and also enumerated them.

1.2. The Notion of Balance in a Sigraph

Harary [29] derived the following structural criterion called *partition criterion* for balance in sigraphs.

Theorem 1 ([29]). *A sigraph S is balanced if and only if its vertex set $V(S)$ can be partitioned into two subsets V_1 and V_2, one of them possibly empty, such that every positive edge joins two vertices in the same subset and every negative edge joins two vertices from different subsets.*

The following important lemma on balanced sigraphs is given by Zaslavsky:

Lemma 1 ([48]). *A sigraph in which every chordless cycle is positive, is balanced.*

1.3. The Notion of Clustering in a Sigraph

A signed graph is said to be *clusterable* if its vertex set can be partitioned into pairwise disjoint subsets, called *clusters*, such that every negative edge joins vertices in different clusters and every positive edge joins vertices in the same cluster; we shall call such a partition a *Davis partition*, after its originator [19], or a *clustering* [16]. Clearly, every graph, treated as an all-positive sigraph, is clusterable with its entire vertex set forming a single cluster. Next, every heterogeneous sigraph is balanced if and only if it is clusterable with exactly two clusters [29]; this particular Davis partition is known as *Harary bipartition* [46]. Davis [19] characterized clusterable signed graphs as precisely those in which no cycle has exactly one negative edge (also, see [16]).

Theorem 2 ([19]). *A siraph S is clusterable if and only if S contains no cycle with exactly one negative edge.*

1.4. The Notions of Consistency and Sign-Compatibility in a Sigraph

A *marked sigraph* is an ordered pair $S_\mu = (S, \mu)$ where $S = (S^u, \sigma)$ is a sigraph and

$$\mu : V(S^u) \to \{+, -\}$$

is a function from the vertex set $V(S^u)$ of S^u into the set $\{+, -\}$, called a *marking* of S. A cycle Z in S_μ is said to be *consistent* if it contains an even number of negative vertices. A given sigraph S is said to be *consistent* if every cycle in it is consistent [2]; for digraphs, the notion was due to Beineke and Harary [11, 12]. In particular, σ induces a unique marking μ_σ defined by

$$\mu_\sigma(v) = \prod_{e_j \in E_v} \sigma(e_j), v \in V(S),$$

is called the *canonical marking* (or, *$\mathscr{C}$-marking* in short) of S, where E_v is the set of edges e_j incident at v in S [40].

Now, if every vertex of a given sigraph S is canonically marked, then a cycle Z in S is said to be *canonically consistent* (*$\mathscr{C}$-consistent*) if it contains an even number of negative vertices and the given sigraph S is said be *$\mathscr{C}$-consistent* if every cycle in it is $\mathscr{C}$-consistent. Thus, the original notion of consistent graphs due to Beineke and Harary [11, 12] reduces to that of trivial $\mathscr{C}$-consistency, when all the vertices receive '+'. Although consistent digraphs were neatly characterized in [11, 12], the problem of characterizing consistent marked graphs was declared open by Beineke and Harary [11]; subsequently, it was solved successfully by many authors (see [46] for a comprehensive appraisal). However, characterization of $\mathscr{C}$-consistent sigraphs is still an open problem.

A sigraph S is *sign-compatible* [40] if there exists a marking μ of its vertices such that the end vertices of every negative edge receive '-1' marks in μ and no positive edge in S has both of its ends assigned '-1' marks in μ. *sign-incompatible* otherwise. The notion of sign-compatibility arises naturally in the characterization of line sigraphs [5].

1.5. Some Notions of Derived Sigraphs

There are many notions of sigraphs derived from a given sigraph, generically addressed here as 'derived sigraphs'. Some of them considered in our investigations include the following ones.

For a sigraph S, Behzad and Chartrand [10] defined its *line sigraph*, $L(S)$ as the sigraph in which the edges of S are represented as vertices, two of these vertices are defined adjacent whenever the corresponding edges in S have a vertex in common, any such edge ef is defined to be negative whenever both e and f are negative edges in S.

For a sigraph S, Gill [24] defined its $\times$*-line sigraph* $L_\times(S)$ as follows: the $L_\times(S)$ is a sigraph defined on the line graph $L(S^u)$ of the graph S^u by assigning to each edge ef of $L(S^u)$, the product of signs of the adjacent edges e and f of S.

For a sigraph S, Acharya and Sinha [6] defined its *common-edge sigraph* $C_E(S)$ as the sigraph whose vertex set is the set of pairs of adjacent edges in S and two vertices of $C_E(S)$ are adjacent if the corresponding pairs of adjacent edges of S have exactly one edge in common, with the sign same as that of their common edge.

The *semi-total line graph* $T_1(G)$ [38] of a graph G is the graph whose vertex set is

$$V(G) \cup E(G)$$

where $V(G)$ and $E(G)$ are vertex set and edge set of G, respectively and in $T_1(G)$ two vertices are adjacent if and only if (i) they are adjacent edges in G, or (ii) one is a vertex and the other is an edge in G incident to it. Sinha *et al.* [44] extended this notion of semi-total line graphs to the theory of sigraphs as follows:

Let $S = (V, E, \sigma)$ be any sigraph. Its *semi-total line sigraph* $T_1(S)$ has $T_1(S^u)$ as its underlying graph and for any edge uv of $T_1(S^u)$,

$$\sigma_{T_1}(uv) = \begin{cases} \sigma(u)\sigma(v) & \text{if } u, v \in E, \\ \sigma(v) & \text{if } u \in V \text{ and } v \in E. \end{cases}$$

1.6. Unitary Cayley Graph and its Sigraph Varieties

Let Γ be a group and B be a subset of Γ such that B does not contain the identity of Γ. Assume $B^{-1} = \{b^{-1} : b \in B\} = B$. The *Cayley graph* $X' = Cay(\Gamma, B)$ is an undirected graph having vertex set $V(X') = \Gamma$ and edge set $E(X') = \{ab : ab^{-1} \in B\}$, where $a, b \in \Gamma$. The Cayley graph X' is a regular graph of degree $|B|$. Its connected components are the right cosets of the subgroup generated by B. Therefore, if B generates Γ, then X' is a connected graph. The books on algebraic graph theory by Biggs [14] and by Godsil and Royle [25] provide many information regarding Cayley graphs.

For a positive integer n, the *unitary Cayley graph* X_n is the graph whose vertex set is Z_n, the ring of integers modulo n and if U_n denotes set of all its units then two vertices a and b are adjacent if and only if $(a - b) \in U_n$. The unitary Cayley graph X_n is then the same as $X_n = Cay(Z_n, U_n)$. The structure and various properties of unitary Cayley graphs have been studied in literature (see [7, 9, 13, 15, 20–23, 34, 37, 39]).

Let Γ be an abelian group and B be a subset of Γ. The *addition Cayley graph*

$$G' = Cay^+(\Gamma, B)$$

is the graph having the vertex set $V(G') = \Gamma$ and the edge set $E(G') = \{ab : a + b \in B\}$, where $a, b \in \Gamma$. Several properties of addition Cayley graphs have been discussed in literature (see [8, 17, 18, 26–28, 35, 36]).

For a positive integer n, the *unitary addition Cayley graph* G_n is the graph whose vertex set is Z_n, the integers modulo n and if U_n denotes set of all units of the ring Z_n, then two vertices a and b are adjacent if and only if $a + b \in U_n$. The unitary addition Cayley graph G_n may also be defined as, $G_n = Cay^+(Z_n, U_n)$. Some properties of unitary addition Cayley graphs have been studied in literature (see [43]).

Theorem 3 ([43]). *The unitary addition Cayley graph G_n is isomorphic to the unitary Cayley graph X_n if and only if n is even.*

Some examples of unitary addition Cayley graphs are displayed in Figure 1.

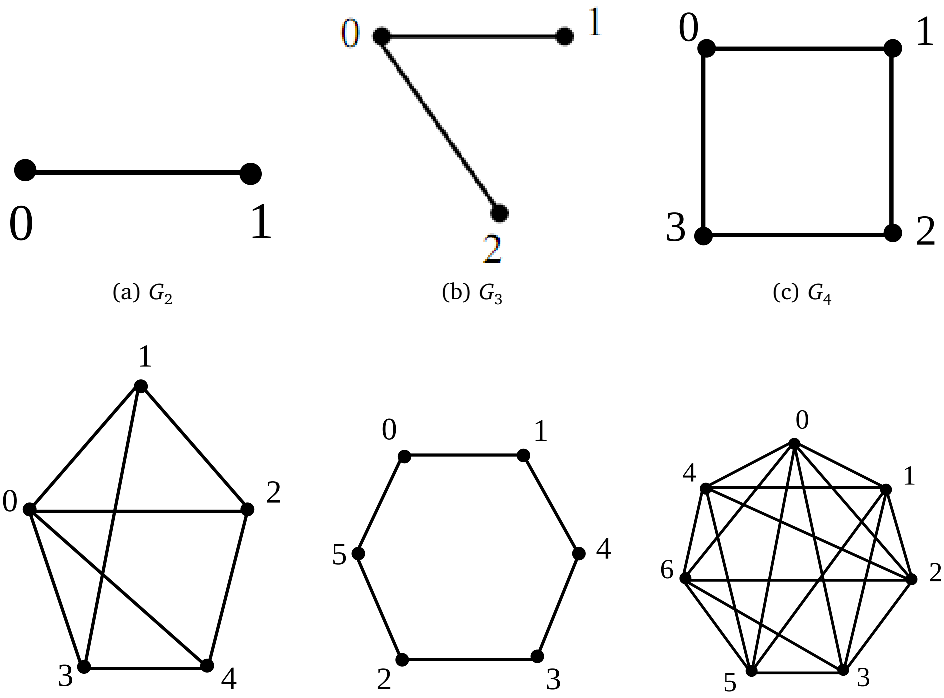
Our aim in this paper is to introduce an extension of the notion of unitary addition Cayley graphs in a natural way to the theory of sigraphs and study their fundamental properties.

2. Unitary Addition Cayley Sigraphs

We introduce the definition of a unitary addition Cayley sigraph as follows:

Definition 1. *For a positive integer n, the* unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$ *is defined as the sigraph, where Σ_n^u is the unitary addition Cayley graph and for an edge ab of Σ_n,*

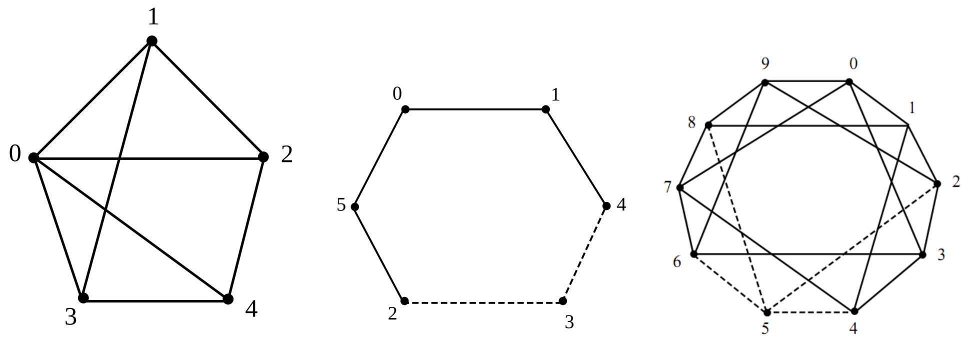
$$\sigma(ab) = \begin{cases} + & \text{if } a \in U_n \text{ or } b \in U_n, \\ - & \text{otherwise.} \end{cases}$$



(a) G_2 (b) G_3 (c) G_4

(d) G_5 (e) G_6 (f) G_7

Figure 1: Some examples of unitary addition Cayley graphs



(a) Σ_5 (b) Σ_6 (c) Σ_{10}

Figure 2: Some examples of unitary addition Cayley sigraphs

Three examples of unitary addition Cayley sigraphs for $n = 5, 6, 10$ are displayed as (a), (b) and (c) respectively in Figure 2. Throughout the text, we consider $n \geq 2$.

Theorem 4 ([43]). *Let m be any vertex of the unitary addition Cayley graph G_n. Then,*

$$d(m)=\begin{cases}\phi(n)-1 & \text{if } n \text{ is odd and } (m,n)=1.\\ \phi(n) & \text{otherwise}\end{cases}.$$

where $\phi(n)$ denotes the Euler totient function that gives the number of primes not exceeding n.

Lemma 2. *For an integer n, if $i\in U_n$ then $(n-i)\in U_n$ and if $i\notin U_n$ then $(n-i)\notin U_n$.*

Proof. Suppose n is any integer. Then, $i\in U_n \Rightarrow (n,i)=1$, where (n,i) is $\gcd(n,i)$. We want to show that $(n-i)\in U_n$. Suppose, on the contrary, $(n-i)\notin U_n$. Then, $(n,n-i)=k \Rightarrow k\mid n$ and $k\mid(n-i)$, whence $n=\alpha k$ and $(n-i)=\beta k$. But, $n=\alpha k$ gives $\alpha k-i=\beta k \Rightarrow (\alpha-\beta)k=i \Rightarrow k\mid i$. Thus, $k\mid i$ and $k\mid n$ imply $(n,i)\neq 1$, a contradiction to our hypothesis. Hence, if $i\in U_n$ then $(n-i)\in U_n$.

Next, suppose $i\notin U_n$. Then, $(n-i)\in U_n \Rightarrow (n,(n-i))=1$. As $i\notin U_n \Rightarrow 1\neq(n,i)=l\Rightarrow l\mid n$ and $l\mid i \Rightarrow n=\alpha l$ and $i=\beta l$. This shows that $n-i=\alpha l-\beta l \Rightarrow (\alpha-\beta)l \Rightarrow l\mid(n-i)$. Thus, $l\mid n$ and $l\mid(n-i)$ imply $(n,(n-i))\neq 1$, a contradiction to the hypothesis. Hence, by contraposition, if $i\notin U_n$ then $(n-i)\notin U_n$. Thus, the result follows.

Theorem 5. *The unitary addition Cayley sigraph $\Sigma_n=(\Sigma_n^u,\sigma_1)$ is isomorphic to the unitary Cayley sigraph $S_n=(S_n^u,\sigma_2)$ if and only if n is even.*

Proof. ***Necessity***: Suppose $\Sigma_n\cong S_n$. Then, $\Sigma_n^u\cong S_n^u$, whence the proof follows by Theorem 3.

Sufficiency: Suppose n is even. Then, by Theorem 3, we get $\Sigma_n^u\cong S_n^u$. Now, consider a function $f:V(G_n)\to V(X_n)$ such that

$$f(m)=\begin{cases}m & \text{if } m \text{ is even}\\ n-m & \text{if } m \text{ is odd.}\end{cases}$$

Let $U_n=\{a_1,a_2,\dots,a_{\phi(n)}\}$. Since the vertex m is adjacent to the vertices of type a_r-m, consider a set

$$A=\{a_1-m,\ a_2-m,\dots,a_{\phi(n)}-m\}.$$

Suppose two vertices i and j are adjacent in G_n, then j is of the form a_r-i, where $a_r\in U_n$.

Case I: If i is even, then $j=a_r-i$ is odd. This implies that $f(i)=i$ and $f(j)=n-j=n-(a_r-i)$. Now, by Theorem 3, we can see that $f(i)$ and $f(j)$ are adjacent in X_n. It is clear that $i\notin U_n$. Now, either $j\in U_n$ or $j\notin U_n$. If $j\in U_n$, then by Lemma 2, $n-j\in U_n$. Then, by the definition of unitary Cayley sigraph $\sigma_2(ij)=+$. Now, $f(i)\notin U_n$ as $f(i)$ is even and $f(j)\in U_n$ as $f(j)=n-j\in U_n$. So, by the definition of unitary addition Cayley sigraph

$$\sigma_1(f(i)f(j))=+.$$

If $j \notin U_n$, then by Lemma 2 $n - j \notin U_n$. Then, by the definition of unitary Cayley sigraph $\sigma_2(ij) = -$ and as in the above argument

$$\sigma_1(f(i)f(j)) = -.$$

Thus, we have

$$\sigma_1(f(i)f(j)) = \sigma_2(ij).$$

Case II: If i is odd, then $j = a_r - i$ is even. Thus, $j \notin U_n$. Again, either $i \in U_n$ or $i \notin U_n$. If $i \in U_n$, then $n - i \in U_n$ and $\sigma_2(ij) = +$ and as in the above argument, $\sigma_1(f(i)f(j)) = +$. If $i \notin U_n$, then $n - i \notin U_n$. Hence, by the same argument, $\sigma_2(ij) = -$ and $\sigma_1(f(i)f(j)) = -$. Now, one can easily verify that f is one-to-one and onto function that preserves adjacency as well as sign of the edges. Hence, $\Sigma_n \cong S_n$.

3. Balance in Σ_n

In this section, we establish a characterization of balanced unitary addition Cayley sigraphs. We recall a known result first.

Theorem 6 ([43]). *The unitary addition Cayley graph G_n, $n \geq 2$, is bipartite if and only if either $n = 3$ or n is even.*

Lemma 3. *For the unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$, if $n = p^a$, where p is a prime number, then Σ_n is an all-positive sigraph.*

Proof. For the unitary addition Cayley sigraph Σ_n, if $n = p^a$, then U_n consists of all the numbers less than n, which are not multiples of p. Suppose αp and βp are two numbers less than n and multiples of p. Then, by the definition of the unitary addition Cayley sigraph, we have a negative edge only when αp is adjacent to βp. Now, we have three possibilities, viz., $\alpha p + \beta p < n$, $\alpha p + \beta p = n$ or $\alpha p + \beta p > n$. When, $\alpha p + \beta p < n$, we see that $\alpha p + \beta p \notin U_n$ as it is a number less than n and a multiple of p. Secondly, when $\alpha p + \beta p = n$, we get $\alpha p + \beta p = 0 \notin U_n$. When $\alpha p + \beta p > n$, there exists an integer k such that $\alpha p + \beta p = n + k = k$, which is again a number less than n and a multiple of p. This implies $\alpha p + \beta p \notin U_n$. Thus, in each case αp is not adjacent with βp since their addition $\alpha p + \beta p \notin U_n$. Thus, Σ_n is an all-positive sigraph.

We shall now establish the following characterization of balanced unitary addition Cayley sigraphs.

Theorem 7. *The unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$ is balanced if and only if either n is even or it does not have more than one distinct prime factors.*

Proof. ***Necessity***: Suppose the unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$ is balanced. Assume that the conclusion is false. Suppose n is odd and it has at least two distinct prime factors. So, let $n = p_1^{a_1} p_2^{a_2} \ldots p_m^{a_m}$, where all $p_1, p_2, \ldots, p_m$ are distinct primes, $p_1 \neq 2$ and

$p_1 < p_2 < \cdots < p_m$.

In the unitary addition Cayley graph Σ_n^u, p_1 is adjacent with 1 as $p_1 + 1$ is not a multiple of any p_i's for $i = 1, 2, \ldots, m$ *i.e.,* $p_1 + 1 \in U_n$. Now, we claim that p_1 and p_2 are also adjacent in Σ_n^u. If possible, suppose p_1 and p_2 are not adjacent in Σ_n^u. This shows, $p_1 + p_2 \notin U_n$. Then, $p_1 + p_2$ is a multiple of some p_i's for $i = 1, 2, \ldots, m$. Suppose $p_1 + p_2$ is a multiple of p_1. Then,

$$\begin{aligned} p_1 + p_2 &= \alpha p_1 \\ p_2 &= \alpha p_1 - p_1 \\ &= (\alpha - 1)p_1 \end{aligned}$$

for some positive integer α, which is not possible. Similarly, we can see that $p_1 + p_2$ is not a multiple of p_2. Now, the possibilities are $i = 3, 4, \ldots, m$. Suppose $p_1 + p_2 = \alpha p_i$ for $i = 3, 4, \ldots, m$. Since $p_1 + p_2$ is even, α is even and is at least 2, for any positive integer α. But as $p_1 < p_2 < p_i$, $p_1 + p_2$ is always less than any multiple of p_i for $i = 3, 4, \ldots, m$. Thus, $p_1 + p_2$ is not a multiple of any p_i's for $i = 1, 2, \ldots, m$. So $p_1 + p_2 \in U_n$. This shows that p_1 and p_2 are adjacent in Σ_n^u. Now, if p_2 is adjacent with 1 in Σ_n^u, then we have a cycle

$$Z = (p_1, p_2, 1, p_1)$$

in Σ_n. Clearly, p_1 and p_2 do not belong to U_n and $1 \in U_n$. Then, by the definition of Σ_n, Z has exactly one negative edge p_1p_2. Thus, Z is a negative cycle in Σ_n. This implies that Σ_n is not balanced. Now, suppose p_2 is not adjacent with 1 in Σ_n^u, *i.e.*, $p_2 + 1 \not{i}nU_n$. Then, $p_2 + 1$ is multiple of one of the p_i's for $i = 1, 2, \ldots, m$. Clearly, i cannot exceed 1, as $p_2 < p_3 \cdots < p_m$. So, the only possibility is $i = 1$, whence $p_2 + 1$ is a multiple of p_1. Then,

$$p_2 + 1 = \alpha p_1 \tag{1}$$

for some positive integer α.

By Lemma 2, it is clear that $n - p_2 \notin U_n$. Now, we claim $n - p_2$ is adjacent with 1 *i.e.*, $n - p_2 + 1 = n - (p_2 - 1) \in U_n$. If $p_2 - 1 \in U_n$, then by Lemma 2, $n - p_2 + 1 = n - (p_2 + 1) \in U_n$. Suppose $p_2 - 1 \notin U_n$. Then, $p_2 - 1$ is a multiple of one of the p_i's for $i = 1, 2, \ldots, m$ and by the same argument as above $i = 1$, whence $p_2 - 1$ is a multiple of p_1. Then, $p_2 - 1 = \beta p_1$. But, from equation 1, $p_2 = \alpha p_1 - 1$. This implies,

$$\begin{aligned} p_2 - 1 &= \beta p_1 \\ \alpha p_1 - 1 - 1 &= \beta p_1 \\ \alpha p_1 - 2 &= \beta p_1 \\ \alpha p_1 - \beta p_1 &= 2 \\ (\alpha - \beta)p_1 &= 2. \end{aligned}$$

This is not possible as p_1 is at least 3. Thus, $p_2 - 1$ is not a multiple of any of the p_i's, whence $p_2 - 1 \in U_n$. Hence, $n - p_2 + 1 = n - (p_2 - 1) \in U_n$, whence $n - p_2$ is adjacent with 1 in Σ_n^u.

Now, $n - p_2 + p_1 = n - (p_2 - p_1)$. Since $p_1 < p_2 < \cdots p_m$, $p_2 - p_1$ is not a multiple of any of the p_i's for $i = 2, 3, \ldots m$. Also, $p_2 - p_1$ is not a multiple of p_1. This shows that $p_2 - p_1 \in U_n$ and by Lemma 2, $n - (p_2 - p_1) \in U_n$. This shows that $n - p_2$ is adjacent with p_1 in Σ_n. Thus, we have a cycle

$$Z' = (p_1, n - p_2, 1, p_1)$$

in Σ_n. Clearly, p_1 and $n - p_2$ do not belong to U_n and $1 \in U_n$. Then, by the definition of Σ_n, Z' has exactly one negative edge $p_1(n - p_2)$. Thus, Z' is a negative cycle in Σ_n. This implies that Σ_n is not balanced, a contradiction to the hypothesis. So, by contraposition, the conditions are satisfied.

Sufficiency: Suppose n is even. Then, U_n does not contain any multiple of 2. Then, by Theorem 6, Σ_n is bipartite, whence all its cycles are even. Therefore, every cycle in Σ_n contains alternately either even-odd or odd-even labeled vertices. Without loss of generality, let

$$Z'' = (e_1, o_1, e_2, o_2, \ldots, e_m, o_m, e_1)$$

be a cycle of even length in Σ_n. Clearly, $e_i \notin U_n \forall i = 1, 2, \ldots, m$.

Case(i): Suppose $o_j \in U_n \forall j = 1, 2, \ldots, m$. Then, all the edges in Z'' are positive.

Case(ii): Suppose $o_j \notin U_n$ for some $j = 1, 2, \ldots, m$. Then, Z'' contains two negative edges $e_j o_j$ and $o_j e_{j+1}$ with respect to each $o_j \notin U_n$. Thus, Z'' contains an even number of negative edges. Since Z'' is an arbitrary cycle in Σ_n, using Lemma 1, we conclude Σ_n is balanced.

Next, suppose n is odd and it does not have more than one distinct prime factors. That means, $n = p^a$. Now, using Lemma 3, Σ_n is an all-positive sigraph which is trivially balanced. Hence the theorem.

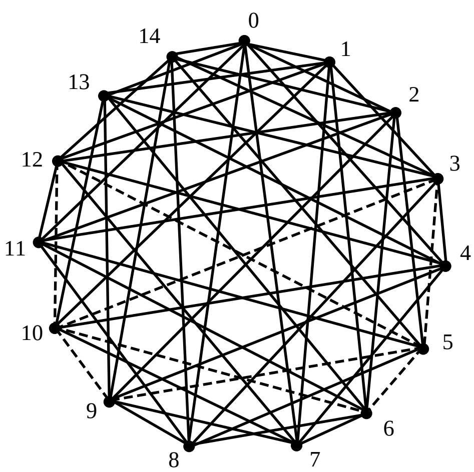


Figure 3: Smallest unbalanced unitary addition Cayley sigraph

The smallest heterogeneous unbalanced unitary addition sigraph is Σ_{15}, which is shown in Figure 3.

4. Clusterability of Σ_n

In this section, we discuss clusterability of unitary addition Cayley sigraphs and obtain the following somewhat surprising result.

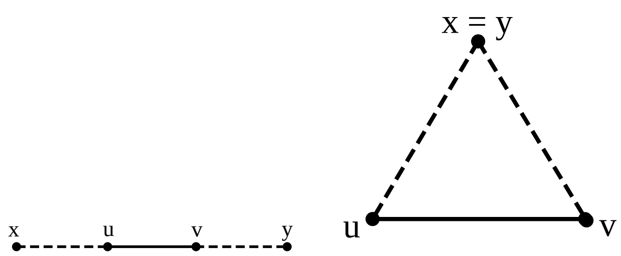
Theorem 8. *A unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$ is clusterable if and only if it is balanced.*

Proof. ***Sufficiency***: Suppose the unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$ is balanced. Then, by the definition of clusterability, Σ_n is clusterable with two clusters.

Necessity: Suppose unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$ is clusterable. If possible, suppose Σ_n is not balanced. Then, by Theorem 7, n is odd with at least two distinct prime factors. So, let $n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$, where all of $p_1, p_2, \dots, p_m$ are distinct primes, $p_1 \neq 2$ and $p_1 < p_2 < \cdots < p_m$. Now, as in the proof of Theorem 7, we have at least one of the cycles $Z = (p_1, p_2, 1, p_1)$ and $Z' = (p_1, n - p_2, 1, p_1)$ in Σ_n. Clearly, p_1 and p_2 do not belong to U_n and $1 \in U_n$. Then, by the definition of Σ_n, Z has exactly one negative edge $p_1 p_2$. Also, p_1 and $n - p_2$ do not belong to U_n and $1 \in U_n$. Then, again by the definition of Σ_n, Z' has exactly one negative edge $p_1(n - p_2)$. Thus, in each case we have a cycle with exactly one negative edge. This shows that Σ_n is not clusterable, a contradiction to the hypothesis. Thus, Σ_n is balanced. Hence, the theorem.

5. Sign-compatibility of Σ_n

Theorem 9 ([41]). *A sigraph S is sign-compatible if and only if S does not contain a subsigraph isomorphic to either of the two sigraphs, S_1 formed by taking the path $P_4 = (x, u, v, y)$ with both the edges xu and vy negative and the edge uv positive and S_2 formed by taking S_1 and identifying the vertices x and y (Figure 4).*



(a) S_1 (b) S_2

Figure 4: Two forbidden subsigraphs for a sign-compatible sigraph [40]

Theorem 10. *Every unitary addition Cayley sigraph Σ_n is sign-compatible.*

Proof. Suppose that the unitary addition Cayley sigraph Σ_n is not sign-compatible. Then, by Theorem 9, there is at least one positive edge, say $v_i v_j$ in Σ_n such that there are negative edges on both the vertices, v_i and v_j in. Since $v_i v_j$ is a positive edge in Σ_n, by the definition of Σ_n, at least one of $v_i, v_j \in U_n$. As at least one of $v_i, v_j \in U_n$, again by the definition of Σ_n, there is no negative edge on at least one vertex, a contradiction to the hypothesis. Hence, Σ_n is sign-compatible.

It has been shown elsewhere that all line sigraphs are sign-compatible [5]. Hence, in view of Theorem 10 the question arises whether any unitary addition Cayley sigraph is a line sigraph. The answer of this question is given in Theorem 12.

Theorem 11. *Unitary addition Cayley graph G_n is a line graph if and only if $n \in \{2, 3, 4, 6\}$.*

Proof. ***Necessity***: Suppose unitary addition Cayley graph G_n is a line graph. If possible, suppose $n \notin \{2, 3, 4, 6\}$.

Case I: Suppose n is a prime number. Clearly, in this case $n \geq 5$. Since n is prime, U_n contains all numbers from 1 to $(n-1)$. Now, 0 is adjacent with all the vertices of G_n. Also, for any other vertex i in G_n, i is not adjacent only with $(n-i)$ as $i + (n-i) = n = 0 \neq U_n$. Thus, for any two vertices i and j in G_n such that $i \neq j \neq 0$, we have an induced subgraph in G_n, which is shown in Figure 5.

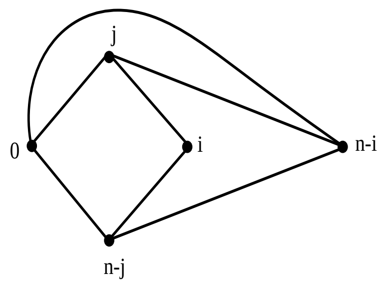


Figure 5: Showing an induced subgraph of G_n, which is forbidden for G_n to be a line graph.

This is one of the Beineke's nine forbidden subgraphs for line graph [30]. This shows that G_n is not a line graph, a contradiction to the hypothesis.

Case II: Suppose n is not a prime number. Clearly, 1 is (always) adjacent with 0 in G_n. Also, 1 is adjacent with p_1, as $p_1 + 1 \in U_n$, where p_1 is the smallest multiple of n. Suppose a is some number such that $ap_1 = n$. Now,

$$\begin{aligned} 1 + (a-1)p_1 &= 1 + ap_1 - p_1 \\ &= 1 + n - p_1 \\ &= n - (p_1 - 1). \end{aligned}$$

Since $p_1 - 1 \in U_n$, by Lemma 2, $n - (p_1 - 1) \in U_n$. Thus, 1 and $(a-1)p_1$ are adjacent in G_n. Also, 0 is not adjacent with p_1 and $(a-1)p_1$ as their addition is a multiple of p_1. Similarly, p_1 and $(a-1)p_1$ are not adjacent in G_n as their addition is a multiple of p_1. Thus, we have an induced subgraph in G_n, which is shown in Figure 6. Again, we have a forbidden subgraph $K_{1,3}$ for a line graph showing that G_n is not a line graph, a contradiction to the hypothesis. Hence, the condition is satisfied.

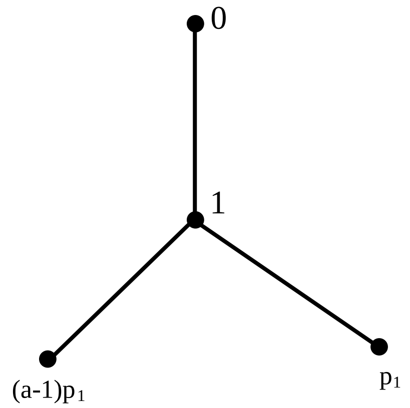


Figure 6: Showing $K_{1,3}$ as an induced subgraph of G_n, which is forbidden for G_n to be a line graph.

Sufficiency: Suppose $n = 2, 3, 4$ or 6. The corresponding graphs are shown in Figure 7, which are line graphs of P_3, P_4, C_4 and C_6, respectively. Hence, the result.

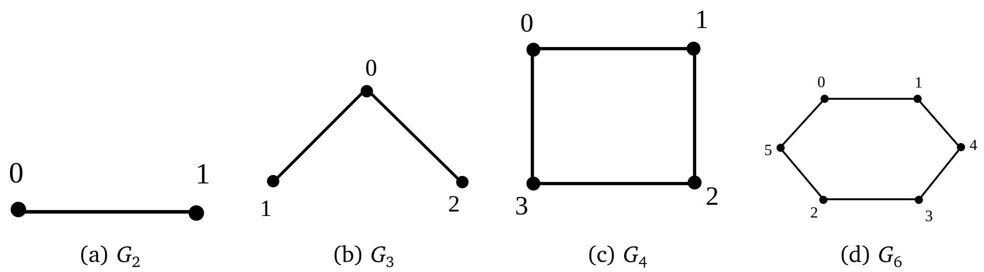


Figure 7: Showing G_2, G_3, G_4 and G_6

Theorem 12. *Unitary addition Cayley sigraph Σ_n is a line sigraph if and only if $n \in \{2, 3, 4, 6\}$.*

Proof. ***Necessity***: Suppose the unitary addition Cayley sigraph Σ_n is a line sigraph. If possible, suppose $n \notin \{2, 3, 4, 6\}$. Then, by Theorem 11, Σ_n^u is not a line graph, a contradiction to the hypothesis. Hence, $n \in \{2, 3, 4, 6\}$.

Sufficiency: Now, suppose $n \in \{2,3,4,6\}$. The corresponding sigraphs $\Sigma_2, \Sigma_3, \Sigma_4$ and Σ_6 and the sigraphs whose line sigraphs are these sigraphs are shown in Figure 8. Hence, Σ_n is a line sigraph for $n \in \{2,3,4,6\}$.

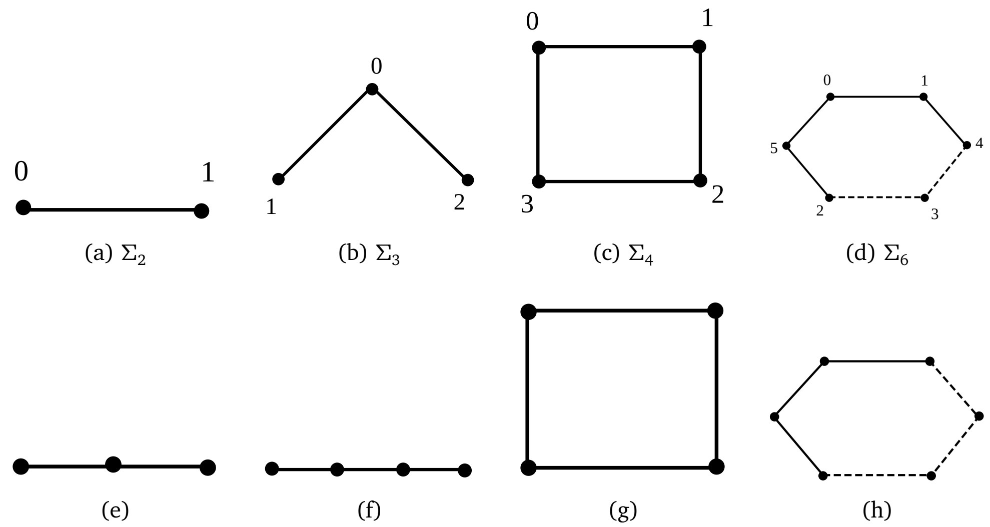


Figure 8: Showing Σ_2, Σ_3, Σ_4 and Σ_6 and the sigraphs whose line sigraphs are these sigraphs

Remark 1. *Unitary addition Cayley sigraph Σ_n is a product line sigraph if and only if $n \in \{2,3,4,6\}$.*

Proof. Suppose the unitary addition Cayley sigraph Σ_n is a product line sigraph. Since, for any given sigraph S, the underlying graphs of the line sigraph $L(S)$ and the product line sigraph $L_\times(S)$ are the same, the condition follows from Theorem 11.

Conversely, suppose $n \in \{2,3,4,6\}$. By Theorem 7 for these values of n, Σ_n is balanced. Since the product line sigraph of any sigraph is always balanced and its underlying structure is the line graph (see [3]), the result follows from Theorem 7 and Theorem 11.

Unitary addition Cayley sigraphs $\Sigma_2, \Sigma_3, \Sigma_4$ and Σ_6 and the sigraphs whose product line sigraphs are these sigraph are shown in Figure 9.

6. $\mathscr{C}$-consistency of Σ_n

Now, we present a characterization of $\mathscr{C}$-consistent unitary addition Cayley sigraphs.

Theorem 13 ([33]). *Let G be a marked graph and T be a spanning tree of G. Then, G is consistent if and only if G satisfies the following two conditions:*

(i) each fundamental cycle relative to T is positive, and

(ii) the two end vertices of any common path between each pair of fundamental cycles relative to T have the same mark.

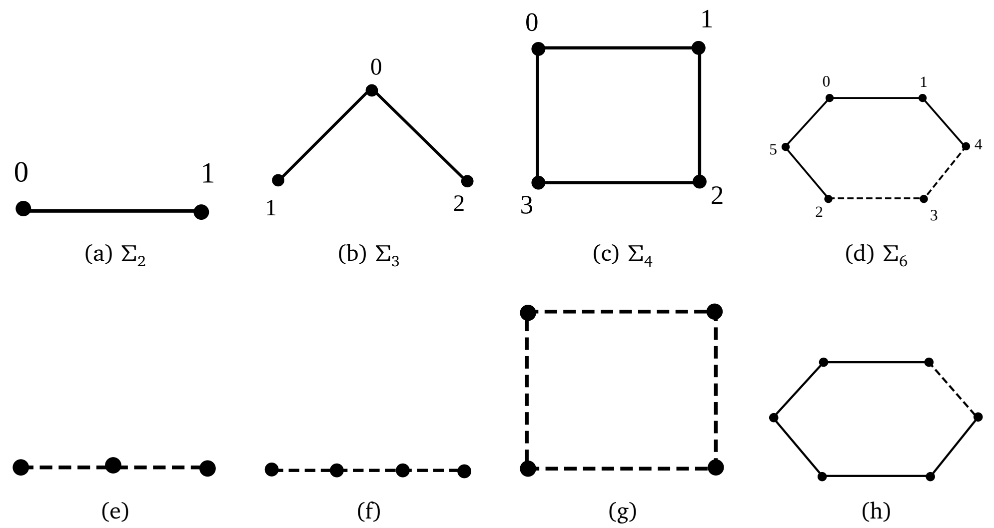


Figure 9: Showing Σ_2, Σ_3, Σ_4 and Σ_6 and the sigraphs whose product line sigraphs are these sigraphs

Theorem 14 ([42]). *The unitary Cayley sigraph $S_n = (S_n^u, \sigma)$, where n has at most two distinct odd prime factors, is $\mathscr{C}$-consistent if and only if n is either odd or n is 2, 6 or a multiple of 4.*

Lemma 4. *In the unitary addition Cayley sigraph Σ_n, if $n = 2p_1^{a_1}$, where p_1 is an odd prime, then the negative degree of the vertex 2 of Σ_n is odd.*

Proof. Suppose $n = 2p_1^{a_1}$ in Σ_n, where p_1 is an odd prime. By the definition of Σ_n, negative edges are incident at the vertex 2 of Σ_n only when 2 is adjacent to multiples of p_1. Since addition of 2 and any even multiple of p_1 is an even number and U_n does not contain an even number, the vertex 2 is not adjacent to any even multiple of p_1. Now, the number of odd multiples of p_1 are $p_1^{a_1-1}$. Now, 2 is adjacent with all the odd multiples of p_1 as their addition with 2 is neither a multiple of 2 nor a multiple of p_1. 2 is negatively adjacent with $p_1^{a_1-1}$. Since p_1 is an odd prime, $d^-(2)$ is odd.

Lemma 5 ([42]). *In the unitary Cayley sigraph S_n, if $n = 2p_1^{a_1}p_2^{a_2}$, where p_1 and p_2 are distinct odd primes, then the negative degree of the vertex 2 of S_n is odd.*

Lemma 6. *In the unitary addition Cayley sigraph Σ_n, if $n = 2p_1^{a_1}p_2^{a_2}$, where p_1 and p_2 are distinct odd primes, then the negative degree of the vertex 2 of Σ_n is odd.*

Proof. Given that $n = 2p_1^{a_1}p_2^{a_2}$, where p_1 and p_2 are distinct odd primes, since n is even, $\Sigma_n \cong S_n$ by Theorem 5. Since 2 is an even number, by the consideration of mapping in Theorem 5, vertex 2 of S_n is mapped to the vertex 2 in Σ_n and by Lemma 5, negative degree of the vertex 2 in Σ_n is odd.

Lemma 7. *In the unitary addition Cayley sigraph Σ_n, if $n = p_1^{a_1} p_2^{a_2}$, where n is odd, then the negative degree of the vertices of Σ_n that are multiples of p_1 or p_2 is even.*

Proof. Given that $n = p_1^{a_1} p_2^{a_2}$, where n is odd, and p_1 and p_2 are distinct odd primes it follows from the definition of Σ_n, that the negative edges are incident at the vertex p_1 when p_1 is adjacent to multiples of p_2 which do not have p_1 as the factor. Every multiple of p_2, which does not contain any multiple of p_1, is adjacent with p_1 as its addition is neither a multiple of p_1 nor a multiple of p_2. Thus,

$$\begin{aligned} d^-(p_1) &= p_1^{a_1} p_2^{a_2-1} - p_1^{a_1-1} p_2^{a_2-1} \\ &= p_1^{a_1-1} p_2^{a_2-1}(p_1 - 1). \end{aligned}$$

Since p_1 and p_2 are odd, $d^-(p_1)$ is even. This formula works for any multiple of p_1 except those which have p_2 as a factor. Similarly,

$$\begin{aligned} d^-(p_2) &= p_1^{a_1-1} p_2^{a_2} - p_1^{a_1-1} p_2^{a_2-1}. \\ &= p_1^{a_1-1} p_2^{a_2-1}(p_2 - 1). \end{aligned}$$

Since p_1 and p_2 are odd, $d^-(p_2)$ is even. This formula works for any multiple of p_2 except those which have p_1 as a factor. And the negative degree of the vertices of Σ_n that are multiples of $p_1 p_2$ is zero. Thus, the negative degree of the vertices of Σ_n that are multiples of p_1 or p_2 is even.

Theorem 15. *The unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$, where n has at most two distinct odd prime factors, is $\mathscr{C}$-consistent if and only if n is either odd, or n is* 2*,* 6 *or a multiple of* 4*.*

Proof. ***Necessity***: Suppose the unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$ is $\mathscr{C}$-consistent. Let, on contrary, $n \equiv 2 \pmod 4$ with $n \neq 2$ and $n \neq 6$. Then, either $n = 2p_1^{a_1}$ or $n = 2p_1^{a_1} p_2^{a_2}$, where p_1 and p_2 are distinct odd primes.

Case(i): Suppose $n \equiv 0 \pmod 3$. Then, either $n = 2 \times 3^{a_1}$ or $n = 2 \times 3^{a_1} \times p_2^{a_2}$. First, suppose $p_2 \neq 5$ and $p_2 \neq 7$. Then, due to Lemma 4 and Lemma 6,

$$\mu_\sigma(2) = -.$$

Since the vertex $5 \in U_n$, by the definition of Σ_n, $d^-(5) = 0$. It follows,

$$\mu_\sigma(5) = +.$$

Now, the vertex 5 is adjacent to the vertex 2 since $5 + 2 = 7 \in U_n$. Since $(n-4) + (n-3) = n + (n-7) = n - 7 \in U_n$ as $7 \in U_n$, $(n-4)$ and $(n-3)$ are adjacent in Σ_n^u and $(n-3) + 2 = n - 1 \in U_n$. This implies $(n-3)$ and 2 are also adjacent in Σ_n^u. Similarly, $(n-4)+5 = n+1 = 1 \in U_n$. This implies 5 and $(n-4)$ are adjacent in Σ_n^u. Consider the two cycles, $Z_1 = (2, 5, 0, 1, 4, 3, 2)$ and $Z_2 = (2, 5, (n-4), (n-3), 2)$ in Σ_n. Clearly, the cycles Z_1 and Z_2 share the chord whose end vertices are 2 and 5. Now, if either Z_1 or Z_2 is $\mathscr{C}$-inconsistent cycle, then we have a contradiction to the hypothesis. Therefore, Z_1 and Z_2 are both $\mathscr{C}$-consistent cycles. However, the end vertices 2 and 5 of their common chord are marked oppositely under the canonical marking and this contradicts Theorem 13.

Now, if $n = 2 \times 3^{a_1} \times p_2^{a_2}$, where either $p_2 = 5$ or $p_2 = 7$, then since the vertex $11 \in U_n$, by the definition of Σ_n, $d^-(11) = 0$. It follows,

$$\mu_\sigma(11) = +.$$

Now, the vertex 11 is adjacent to the vertex 2 since $11 + 2 = 13 \in U_n$. Since $(n-12)+(n-1) = n+(n-13) \in U_n$ as $13 \in U_n$, $(n-12)$ and $(n-1)$ are adjacent in Σ_n^u and $(n-1)+2 = n+1 = 1 \in U_n$. This implies, $(n-1)$ and 2 are also adjacent in Σ_n^u. Similarly, $(n-12)+11 = n-1 \in U_n$, which implies 11 and $(n-12)$ are adjacent in Σ_n^u. Now, consider the two cycles, $Z_3 = (11, 2, 9, 4, 7, 6, 11)$ and $Z_4 = (2, 11, (n-12), (n-1), 2)$ in Σ_n. Clearly, the cycles Z_3 and Z_4 share the chord whose end vertices are 2 and 11. As argued above, Z_3 and Z_4 are both $\mathscr{C}$-consistent cycles. However, the end vertices 2 and 11 of their common chord are marked oppositely under the canonical marking, a contradiction to Theorem 13.

Case(ii): Suppose either $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. That means, 3 does not divide n, which implies that the vertex $3 \in U_n$. Now, consider a cycle $Z = (0, 1, 2, (n-1), 0)$ in Σ_n. Since $1 \in U_n$ and $(n-1) \in U_n$, by the definition of Σ_n, $d^-(1) = d^-(n-1) = 0$. It follows that in the cycle Z,

$$\mu_\sigma(1) = \mu_\sigma(n-1) = +.$$

Since the vertex 0 is adjacent to those vertices which belong to U_n, $d^-(0) = 0$. That means,

$$\mu_\sigma(0) = +.$$

Now, due to Lemma 4 and Lemma 6,

$$\mu_\sigma(2) = -.$$

Thus, the cycle Z is $\mathscr{C}$-inconsistent, whence Σ_n is not $\mathscr{C}$-consistent, a contradiction to the hypothesis. Thus, this part of the proof is complete.

Sufficiency: Next, suppose n is odd, 2, 6 or a multiple of 4.

Case(i): Let n be odd, and $n = p_1^{a_1} p_2^{a_2}$, where p_1 and p_2 are distinct odd primes. Using Lemma 7 one can easily see that all the vertices in Σ_n which are multiples of p_1 and p_2 are even and all other vertices belong to U_n. So, their negative degrees are zero. Hence, all the vertices of Σ_n are marked positively under the canonical marking. Hence, Σ_n is $\mathscr{C}$-consistent.

Case(ii): Suppose $n = 2, 6$ in Σ_n. Then, we can easily verify that Σ_2 and Σ_6 are $\mathscr{C}$-consistent.

Case(iii): Suppose n is a multiple of 4. Here n is even and by Theorem 5 $\Sigma_n \cong S_n$ and by Theorem 14, Σ_n is $\mathscr{C}-$consistent.

7. Balance in Certain Derived Sigraphs

In this section, we consider the conditions that a given sigraph must satisfy in order that its certain derived sigraphs are balanced.

Corollary 1. *For the unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$, its negation sigraph $\eta(\Sigma_n)$ is balanced if and only if either $n = 3$ or n is even.*

Proof. First, suppose $\eta(\Sigma_n)$ is balanced. Assume that the conclusion is false. Suppose n is odd and not equal to 3. Then, $2 \in U_n$. Since $0 + 2 = 2 \in U_n$, 0 and 2 are adjacent in Σ_n^u and $2 + (n-1) = n + 1 = 1 \in U_n$. This implies, 2 and $n-1$ are adjacent in Σ_n^u. Thus, we can consider a triangle $T : (0, 2, n-1, 0)$ in Σ_n. Since $2, n-1 \in U_n$, by the definition of Σ_n all the edges of T are positive. That means, all the edges of the triangle T are negative in $\eta(\Sigma_n)$. Thus, $\eta(\Sigma_n)$ is unbalanced, which contradicts the hypothesis.
Conversely, suppose n is even or $n = 3$. Now, due to Theorem 6, S_n^u is bipartite and due to Theorem 7, Σ_n is balanced. Thus, $\eta(\Sigma_n)$ is balanced.

Theorem 16 ([4]). *For a sigraph S, its line sigraph $L(S)$ is balanced if and only if the following conditions hold:*

(i) for any cycle Z in S,

(a) if Z is all-negative, then Z has even length,

(b) if Z is heterogeneous, then Z has an even number of negative sections with even length, and

(ii) for $v \in S$, if $d(v) > 2$, then there is at most one negative edge incident at v in S.

Corollary 2. *For the unitary addition Cayley sigraph Σ_n, its line sigraph $L(\Sigma_n)$ is balanced if and only if $n = p^a$, where p is a prime number.*

Proof. Suppose $L(\Sigma_n)$ is balanced for the unitary addition Cayley sigraph Σ_n. Assume that the conclusion is false. Let n have at least two distinct prime factors. Suppose p_1 and p_2 are two smallest prime factors of n such that $p_1 < p_2$. It is shown in the proof of Theorem 7, p_1 is adjacent with 1 and p_2. Suppose $\alpha p_2 = n$ for any positive integer α. Now,

$$\begin{aligned}(\alpha-1)p_2 + p_1 &= \alpha p_2 - p_2 + p_1 \\ &= n - p_2 + p_1 \\ &= n - (p_2 - p_1)\end{aligned}$$

Since $p_2 - p_1 \in U_n$, by Lemma 2 $n - (p_2 - p_1) \in U_n$, whence $(\alpha - 1)p_2 + p_1 \in U_n$. This shows that $(\alpha - 1)p_2$ is adjacent with p_1. Clearly, the vertex p_2 and $(\alpha - 1)p_2$ are adjacent to the vertex p_1 with negative edges in Σ_n. That means, $d^-(p_1) \geq 2$ and clearly $d(p_1) > 2$ except $n = 6$ in Σ_n. Thus, condition (ii) of Theorem 16 does not hold for Σ_n and when $n = 6$ it is easy to see that condition (i)(b) does not hold, which implies that $L(\Sigma_n)$ is unbalanced, a contradiction to the hypothesis. Hence $n = p^a$, where p is a prime number. Converse part can be proved easily by using Lemma 3.

Theorem 17 ([6]). *For any sigraph S, $C_E(S)$ is balanced if and only if S is a balanced sigraph such that for every vertex $v \in V(S)$ with $d(v) \geq 3$*

(i) if $d(v) > 3$ then $d^-(v) = 0$

(ii) if $d(v) = 3$ then $d^-(v) = 0$ or $d^-(v) = 2$

(iii) for every x-y path $P_4 = (x, v, w, y)$ of length three, vw is a positive edge in S.

Theorem 18. *For the unitary addition Cayley sigraph $\Sigma_n = (\Sigma_n^u, \sigma)$, its $C_E(\Sigma_n)$ is balanced if and only if $n = p^a$ or $n = 6$, where p is a prime number.*

Proof. Suppose $C_E(\Sigma_n)$ is balanced for the unitary addition Cayley sigraph Σ_n. Assume that the conclusion is false. Let $n \neq 6$ and have at least two distinct prime factors. So, let $n = p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}$, where all of $p_1, p_2, \dots, p_m$ are distinct primes and $p_1 < p_2 < \dots < p_m$.

Case I: Suppose n is even. Clearly, $p_1 = 2 \notin U_n$. p_2 can never be adjacent with any number in U_n because U_n contains only odd numbers as n is even and then sum of these two elements will be always even and does not belong to U_n. Since $p_2 \notin U_n$, by Theorem 4, we have $d(p_2) = \phi(n)$. So all the degrees of p_2 are negative and greater than 3. Thus, the condition (i) of Theorem 17 does not hold for Σ_n, which implies that $C_E(\Sigma)$ is unbalanced, a contradiction to the hypothesis.

Case II: Now, suppose n is odd. We have already shown that p_1 is adjacent p_2. Clearly, $d(p_1) > 3$ and $d^-(p_1) \geq 1$. Thus, condition (i) of Theorem 17 does not hold for Σ_n, which implies that $C_E(\Sigma)$ is unbalanced, a contradiction to the hypothesis. Hence $n = p^a$ or $n = 6$, where p is a prime number. Converse part can be proved easily by using Lemma 3.

Theorem 19 ([3]). *The $\times$-line sigraph $L_\times(S)$ of a sigraph S is a balanced sigraph.*

Theorem 20. *For the unitary addition Cayley sigraph Σ_n, its $\times$-line sigraph $L_\times(\Sigma_n)$ is balanced.*

Proof. Result follows from Theorem 19.

Theorem 21 ([44]). *The semi-total line sigraph $T_1(S)$ of a sigraph S is a balanced sigraph.*

Theorem 22. *For the unitary addition Cayley sigraph Σ_n, its semi-total line sigraph $T_1(\Sigma_n)$ is balanced.*

Proof. Result follows from Theorem 21.

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