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Relative Differential K-theory

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Abstract. Let $\rho : Y \to X$ be a smooth map between two smooth compact manifolds. We define the relative differential K-theory group $\check{K}^*(\rho)$ and show that it fits into a six-term exact sequence. We define $\check{K}^*(\rho, \mathbb{R}/\mathbb{Z})$, the K-theory of ρ with $\mathbb{R}/\mathbb{Z}$ coefficients. It turns out that $\check{K}^*(\rho, \mathbb{R}/\mathbb{Z})$ is isomorphic to the group of homomorphisms from the relative K-homology of ρ [8] to $\mathbb{R}/\mathbb{Z}$ up to a degree-shift by one.

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1. Introduction

Differential K-theory is a generalized differential cohomology theory introduced by Freed and Hopkins [5] as a refinement of topological K-theory for a concrete description of RR-fields in string theory. This theory encode geometric as well as topological information. Roughly speaking, differential K-theory combines topological K-theory with differential forms [3–7, 9]. Benameur and Maghfoul [2] pointed out the relevance to differential K-characters of a description of differential flat K-theory. The group of differential K-characters on a smooth compact manifold X is defined as the K-theoretic version of the group of Cheeger-Simons differential characters on X using the (M, E, f)-picture of Baum-Douglas for K-homology. Recall that a geometric K-cycle of Baum-Douglas over X is a triple (M, E, f) such that: M is a smooth compact $Spin^c$ manifold without boundary, E is a Hermitian vector bundle over M with a fixed Hermitian connection ∇^E, and $f : M \to X$ is a smooth map. Let $C_*(X)$ be the semigroup for the disjoint union of equivalence classes of K-cycles over X generated by direct sum and vector bundle modification [1]. A differential K-character on X is a semigroup homomorphism $h : C_*(X) \to \mathbb{R}/\mathbb{Z}$ such that its restriction to the boundaries is given by the following formula:

$$h(\partial W, \varepsilon|_{\partial W}, g|_{\partial W}) := \int_W g^*(w)Ch(\varepsilon)Td(W) \; mod \; \mathbb{Z},$$

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where w is a closed differential form on X with integer K-periods [2], $Ch(\varepsilon)$ is the Chern form of the connection ∇^{ε} on ε, and $Td(W)$ is the Todd form of the tangent bundle of W.

The purpose of this paper is to construct $\check{K}^*(\rho)$, the relative differential K-theory of a smooth map $\rho : Y \to X$ between two smooth compact manifolds. To motivate our construction, the group $\check{K}^*(\rho)$ must recovers the usual non-relative group of differential K-characters on X. We define the K-theory of $\rho : Y \to X$ with $\mathbb{R}/\mathbb{Z}$ coefficients and show that it is isomorphic to the group of homomorphisms from the relative K-homology of ρ [8] to $\mathbb{R}/\mathbb{Z}$ up to a degree-shift by one.

The paper is organized as follows:
In Section 2, we recall the definition of the group of differential K-characters and study some of its properties. In Section 3, we define the relative differential K-theory of a smooth map $\rho : Y \to X$ between two smooth compact manifolds and show that it fits into a six-term exact sequence. Finally, section 4 is concerned with the definition of the K-theory of a smooth map $\rho : Y \to X$ with $\mathbb{R}/\mathbb{Z}$ coefficients and the construction of an isomorphism between this group and the group of homomorphisms from the relative K-homology of ρ [8] to $\mathbb{R}/\mathbb{Z}$.

2. Differential K-characters

In this section, we give the construction of the group of differential K-characters following [2]. As mentioned in the introduction, in this construction we use the (M, E, f)-picture of Baum-Douglas for K-homology [1].

Definition 1. *Let X be a smooth compact manifold. A K-chain over X is a triple (W, ε, g) such that*

- *W is a smooth compact $Spin^c$ manifold;*
- *ε is a Hermitian vector bundle over W with a fixed Hermitian connection ∇^{ε}; and*
- *$g : W \to X$ is a smooth map.*

There are no connectedness requirements made upon W, and hence the bundle ε can have different fibre dimensions on the different connected components of W. It follows that disjoint union

$$(W, \varepsilon, g) \sqcup (W', \varepsilon', g') := (W \sqcup W', \varepsilon \sqcup \varepsilon', g \sqcup g')$$

is a well-defined operation on the set of K-chains over X.

Isomorphism. *Two K-chains (W, ε, g) and (W', ε', g') over X are isomorphic if there exists a diffeomorphism $h : W \to W'$ such that*

- *h preserves the $Spin^c$ structures;*
- *$h^*\varepsilon' \cong \varepsilon$; and*

- *the diagram*

$$\begin{array}{ccc} W & \xrightarrow{h} & W' \\ {\scriptstyle g}\downarrow & \swarrow {\scriptstyle g'} & \\ X & & \end{array}$$

commutes.

A K-cycle is a K-chain (M,E,f) without boundary; that is $\partial M = \emptyset$. The boundary $\partial(W,\varepsilon,g)$ of a K-chain (W,ε,g) is the K-cycle $(\partial W, \varepsilon|_{\partial W}, g|_{\partial W})$.

We are going to construct an Abelian group from the set of isomorphism classes of K-cycles over X so as to obtain the geometric K-homology group of X. In order to define the relation in this group we need to introduce several kinds of relations involving K-cycles.

Vector bundle modification. *Let (W,ε,g) be a K-chain over X and let H be a $Spin^c$ Euclidean vector bundle over W with even-dimensional fibers. Let 1_W denote the trivial real line bundle over W. We denote by $\widehat{W} := S(H \oplus 1_W)$, the unit sphere bundle of $H \oplus 1_W$. Let $\pi : \widehat{W} \to W$ be the bundle projection. The $Spin^c$ structures on TW and H induce a $Spin^c$ structure on $T\widehat{W}$. Let $S = S_- \oplus S_+$ be the $\mathbb{Z}_2$-graded bundle of Clifford modules over W associated with the $Spin^c$ structure on H. We denote by H_0 and H_1 the pullbacks of S_- and S_+, respectively, to H by the bundle projection $H \to W$. Then H acts on H_0 and H_1 by Clifford multiplication map: $H_0 \xrightarrow{\sigma} H_1$.*

The manifold $\widehat{W}$ can be thought of as formed of two copies, $B_0(H)$ and $B_1(H)$, of the unit ball bundle of H (carrying opposite $Spin^c$ structures) glued together by the identity map of $S(H)$:

$$\widehat{W} = B_0(H) \cup_{S(H)} B_1(H).$$

The vector bundle $\widehat{H}$ over $\widehat{W}$ is obtained by putting H_0 over $B_0(H)$ and H_1 over $B_1(H)$ and then clutching these two vector bundles along $S(H)$ by the isomorphism σ. The process of obtaining the K-chain $(\widehat{W}, \widehat{H} \otimes \pi^*\varepsilon, g \circ \pi)$ from (W,ε,g) is called vector bundle modification.
Note that

$$\partial(\widehat{W}, \widehat{H} \otimes \pi^*\varepsilon, g \circ \pi) = (\widehat{\partial W}, \widehat{H|_{\partial W}} \otimes \pi^*(\varepsilon|_{\partial W}), g|_{\partial W} \circ \pi|_{\widehat{\partial W}}).$$

Definition 2. *We define the set $C_*(X)$ as the quotient of the set of isomorphism classes of K-cycles over X by the equivalence relation $\sim$ generated by the relations of*

- *direct sum: if $E = E_1 \oplus E_2$, then $(M,E_1,f) \sqcup (M,E_2,f) \sim (M, E_1 \oplus E_2, f)$; and*
- *vector bundle modification.*

An operation on $C_*(X)$ is given by disjoint union,

$$(M,E,f) \sqcup (M',E',f') := (M \sqcup M', E \sqcup E', f \sqcup f').$$

This operation turns $C_*(X)$ into an Abelian semigroup. Since the relation $\sim$ preserves the parity of the dimension of M in K-cycles (M,E,f), one can define the subsemigroup $C_0(X)$ (resp.

$C_1(X)$) consisting of classes of K-cycles (M,E,f) for which all connected components of M are of even (resp. odd) dimension. Then $C_*(X) = C_0(X) \oplus C_1(X)$ has a natural $\mathbb{Z}_2$-grading.

Bordism. *Two K-cycles (M,E,f) and (M',E',f') over X are bordant if there exists a K-chain (W,ε,g) such that the two K-cycles $\partial(W,\varepsilon,g)$ and $(M,E,f) \sqcup (-M',E',f')$ are isomorphic, where $-M'$ denotes M' with the $Spin^c$ structure on its tangent bundle TM' reversed [1].*

The bordism relation induces a well-defined equivalence relation $\sim_b$ on $C_*(X)$. This relation is compatible with the semigroup structure, and then the quotient set $C_*(X)/\sim_b$ turns out to be an Abelian semigroup. The Abelian semigroup $C_*(X)/\sim_b$ is in fact an Abelian group. The additive inverse of the class of a K-cycle is obtained by reversing the $Spin^c$ structure:

$$-[M,E,f] = [-M,E,f].$$

The neutral element is represented by the empty manifold, or any K-cycle bordant to the empty manifold.

Definition 3. *The quotient group $C_*(X)/\sim_b$ is denoted by $K_*(X)$ and called the geometric K-homology group of X. It has a natural $\mathbb{Z}_2$-grading:*

$$K_*(X) = K_0(X) \oplus K_1(X).$$

The geometric construction of K-homology is functorial. If $\rho : Y \to X$ is a smooth map between two smooth compact manifolds, then the induced homomorphism

$$\rho_* : K_*(Y) \to K_*(X)$$

of $\mathbb{Z}_2$-graded Abelian groups is given on classes of K-cycles $[M,E,f] \in K_*(Y)$ by

$$\rho_*[M,E,f] := [M,E,\rho \circ f].$$

Since vector bundles over M extend to vector bundles over $M \times [0,1]$, it follows by bordism that $K_*(\rho) := \rho_*$ depend only on the smooth homotopy classes of ρ.

Let X be a smooth compact manifold. Let $L_*(X)$ be the quotient of the set of isomorphism classes of K-chains over X by $\sim$. Note that the boundary map on the set of K-chains over X descends to a boundary map

$$\partial : L_*(X) \to C_{*-1}(X) \subset L_{*-1}(X).$$

Let $\Omega^*(X)$ be the graded algebra of real-valued differential forms on X. Let $\varphi : \Omega^*(X) \to Hom(L_*(X),\mathbb{R})$ be the map defined by

$$\varphi_w(W,\varepsilon,g) := \int_W g^*(w)Ch(\varepsilon)Td(W),$$

where $Ch(\varepsilon)$ is the Chern form of the connection ∇^ε on ε and $Td(W)$ is the Todd form of the tangent bundle of W. The set of K-periods of a real-valued differential form $w \in \Omega^*(X)$ is the subset $\varphi_w(C_*(X))$ of $\mathbb{R}$. The Abelian group of closed real-valued differential forms on X with integer K-periods is denoted by $\Omega_0^*(X)$. It has a natural $\mathbb{Z}_2$-grading:

$$\Omega_0^*(X) = \Omega_0^{even}(X) \oplus \Omega_0^{odd}(X).$$

Example 1. *Let X be a smooth compact manifold. Let F be a Hermitian vector bundle over X with a Hermitian connection ∇. An example of a form with integer K-periods is given by the Atiyah-Singer index theorem applied to the positive part of the Dirac operator associated to the $Spin^c$ structure on a $Spin^c$ compact manifold M in K-cycles (M,E,f) with coefficients in $E\otimes f^*F$:*

$$Ind([D^+(E\otimes f^*F)]) = \int_M f^*(Ch(\nabla))Ch(E)Td(M) \in \mathbb{Z}.$$

Definition 4. *(i) Let X be a smooth compact manifold. A differential K-character on X is a homomorphism of semigroups*

$$h : C_*(X) \to \mathbb{R}/\mathbb{Z}$$

such that its restriction to the boundaries is given by the following formula:

$$h(\partial(W,\varepsilon,g)) := \int_W g^*(w)Ch(\varepsilon)Td(W) \ mod\ \mathbb{Z},$$

where w is a closed real-valued differential form on X with integer K-periods.

(ii) The set of differential K-characters on X is denoted by $\hat{K}^*(X)$. It is an Abelian group which has a natural $\mathbb{Z}_2$-grading:

$$\hat{K}^*(X) = \hat{K}^0(X) \oplus \hat{K}^1(X).$$

The differential form w associated to h, indicated above, is unique. It will be denoted by $\delta_0(h)$. Thus we have a homomorphism

$$\delta_0 : \hat{K}^*(X) \to \Omega_0^{*+1}(X).$$

Note that a differential form $v \in \Omega^*(X)$ determines a differential K-character $\widetilde{\varphi_v}$ on X by setting

$$\widetilde{\varphi_v}(M,E,f) := \int_M f^*(v)Ch(E)Td(M) \ mod\ \mathbb{Z}.$$

It is easy to see that $\delta_0(\widetilde{\varphi_v}) = dv$.

We can measure the size of $\hat{K}^*$ by inserting it in a certain exact sequence.
We have the short exact sequence

$$0 \to Hom(K_*(X),\mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{K}^*(X) \overset{\delta_0}{\to} \Omega_0^{*+1}(X) \to 0.$$

This, together with the fact that the only K-cycles on pt are $(pt,\mathbb{C}^k,id_{pt})$, implies that

$$\hat{K}^0(pt) \cong \mathbb{R}/\mathbb{Z} \text{ and } \hat{K}^1(pt) \cong \mathbb{Z}.$$

The construction of $\hat{K}^*(X)$ is functorial. If $\rho : Y \to X$ is a smooth map between two smooth compact manifolds, then the induced homomorphism

$$\rho^* : \hat{K}^*(X) \to \hat{K}^*(Y)$$

of $\mathbb{Z}_2$-graded Abelian groups is given on differential K-characters on X by

$$\rho^*(h)(M, E, f) := h(\rho_*(M, E, f)) \text{ for all } (M, E, f) \in C_*(X).$$

It is obvious that $\delta_0(\rho^*(h)) = \rho^*(\delta_0(h))$.

Let X be a smooth compact manifold. Let i be the inclusion $pt \hookrightarrow X$. Set

$$\tilde{K}^*(X) := \ker[\hat{K}^*(X) \xrightarrow{i^*} \hat{K}^*(pt)].$$

Since the short exact sequence

$$0 \to \tilde{K}^*(X) \hookrightarrow \hat{K}^*(X) \xrightarrow{i^*} \hat{K}^*(pt) \to 0$$

is split, we obtain isomorphisms

$$\hat{K}^0(X) \cong \tilde{K}^0(X) \oplus \mathbb{R}/\mathbb{Z} \text{ and } \hat{K}^1(X) \cong \tilde{K}^1(X) \oplus \mathbb{Z}.$$

3. Relative Differential K-theory

In this section, we define the relative differential K-theory of a smooth map between two smooth compact manifolds and show that it fits into a six-term exact sequence.

Let X be a smooth compact manifold. Let $A \subseteq \mathbb{R}$ be a subring of the reals. A K-cochain over X with coefficients in A is a semigroup homomorphism from $L_*(X)$ to A. The set of K-cochains over X with coefficients in A is denoted by $L^*(X,A)$. The set $L^*(X,A)$ is an Abelian group and a coboundary map on $L^*(X,A)$ is defined by transposition:

$$\delta h(W, \varepsilon, g) := h(\partial(W, \varepsilon, g)).$$

We set

$$\check{L}^*(X) = L^*(X, \mathbb{Z}) \times L^{*-1}(X, \mathbb{R}) \times \Omega_0^*(X),$$

and define a coboundary map $\hat{\delta} : \check{L}^*(X) \to \check{L}^{*+1}(X)$ by the formula:

$$\hat{\delta}(c, h, w) := (-\delta c, -\varphi_w + c + \delta h, 0).$$

Let $\rho : Y \to X$ be a smooth map between two smooth compact manifolds. We define the set of relative K-cochains $\check{L}^*(\rho)$ as the direct product $\check{L}^*(X) \times \check{L}^{*-1}(Y)$. A coboundary map $\check{\delta} : \check{L}^*(\rho) \to \check{L}^{*+1}(\rho)$ is given by setting

$$\check{\delta}(S, T) := (\hat{\delta} S, \rho^* S - \hat{\delta} T).$$

Elements of $\ker[\check{L}^*(\rho) \xrightarrow{\check{\delta}} \check{L}^{*+1}(\rho)]$ are called K-cocycles and those of $img[\check{L}^{*-1}(\rho) \xrightarrow{\check{\delta}} \check{L}^*(\rho)]$ are called K-coboundaries. Let $\check{Z}^*(\rho)$ be the set of K-cocycles and $\check{B}^*(\rho)$ the set of K-coboundaries.

Definition 5. *We define the relative differential K-theory group* $\check{K}^*(\rho)$ *as the quotient group* $\check{Z}^*(\rho)/\check{B}^*(\rho)$.

The construction of relative differential K-theory is functorial. If

$$\begin{array}{ccc} Y' & \xrightarrow{\rho'} & X' \\ \downarrow g & \circlearrowleft & \downarrow f \\ Y & \xrightarrow{\rho} & X \end{array}$$

is a commutative diagram of smooth maps between smooth compact manifolds, then the homomorphism

$$(f,g)^* : \check{K}^*(\rho) \to \check{K}^*(\rho')$$

of $\mathbb{Z}_2$-graded Abelian groups is given on classes of K-cocycles $[S,T] \in \check{K}^*(\rho')$ by

$$(f,g)^*([S,T]) := [f^*S, g^*T].$$

Exact Sequence

Let $(S,T) \in \check{Z}^*(\rho)$. If we set $S = (c_x, h_x, w_x)$ and $T = (c_y, h_y, w_y)$, then the equality $\check{\delta}(S,T) = 0$ implies that:

$$\begin{cases} \delta c_x = 0 \\ \varphi_{w_x} = \delta h_x + c_x \end{cases} \quad \text{and} \quad \begin{cases} \rho^* c_x = -\delta c_y \\ \rho^* h_x = -\varphi_{w_y} + \delta h_y + c_y \\ \rho^* w_x = 0 \end{cases}$$

It follows that the natural homomorphism $\mathbb{R} \xrightarrow{\sim} \mathbb{R}/\mathbb{Z}$ composed with the restriction of h_x to $C_{*-1}(X)$, denoted by $\overline{h_x}$, is a differential K-character on X. Let $j : \check{Z}^*(\rho) \to \hat{K}^{*-1}(X)$ be the map given by $j(S,T) := \overline{h_x}$. It is obvious that $j(\check{\delta}(S,T)) = 0$. Then we obtain a homomorphism from $\check{K}^*(\rho)$ to $\hat{K}^{*-1}(X)$, also denoted by j.

Now, let $h \in \hat{K}^*(Y)$. Since $\mathbb{R}$ is divisible, there is a real K-cochain h' with $\overline{h'} = h$. Set

$$u_{h'} = \varphi_{\delta_0(h)} - \delta h'.$$

It is obvious that $u_{h'} \in L^{*-1}(Y, \mathbb{Z})$. On the other hand, we have

$$\delta u_{h'} = \varphi_{d\delta_0(h)} - (\delta \circ \delta) h' = 0.$$

Therefore, $[0, (u_{h'}, h', \delta_0(h))] \in \check{K}^*(\rho)$. We claim that $[0, (u_{h'}, h', \delta_0(h))]$ is independent of the choice of h'. In fact if h'' is another lift of h, then $\overline{h''} - \overline{h'} = 0$ so that $h'' = h' + c + \delta\gamma$ for same $c \in L^*(Y, \mathbb{Z})$ and $\gamma \in L^{*-1}(Y, \mathbb{R})$. Thus we finally get

$$(0, (u_{h''}, h'', \delta_0(h))) = (0, (u_{h'}, h', \delta_0(h))) - \check{\delta}(0, (c, \gamma, 0)).$$

We define a homomorphism $\theta : \hat{K}^*(Y) \to \check{K}^*(\rho)$ by setting

$$\theta(h) := [0, (u_{h'}, h', \delta_0(h))].$$

Theorem 1. *The following six-term sequence*

$$\begin{array}{ccccc} \hat{K}^0(X) & \xrightarrow{\rho^*} & \hat{K}^0(Y) & \xrightarrow{\theta} & \check{K}^0(\rho) \\ {\scriptstyle j}\uparrow & & & & \downarrow{\scriptstyle j} \\ \check{K}^1(\rho) & \xleftarrow{\theta} & \hat{K}^1(Y) & \xleftarrow{\rho^*} & \hat{K}^1(X) \end{array}$$

is exact.

Proof. Exactness at $\check{K}^1(\rho)$. It is evident that $j\circ\theta = 0$.
Let $[S,T] \in \check{K}^1(\rho)$ with $S = (c_x, h_x, w_x)$ and $T = (c_y, h_y, w_y)$. Assume that $j[S,T] = 0$. Then we have $w_x = 0$, $c_x = -\delta h_x$, and there exist $g \in L^1(X,\mathbb{R})$ and $u \in L^0(X,\mathbb{Z})$ such that $h_x = \delta g + u$. Since

$$(S,T) = (0,(c_y - \rho^* u, h_y - \rho^* g, w_y)) + \check{\delta}((u,g,0),0)$$

and $[0,(c_y - \rho^* u, h_y - \rho^* g, w_y)]$ lies in the image of θ, we get $[S,T] \in img(\theta)$.

Exactness at $\hat{K}^1(X)$. For any $[S,T] \in \check{K}^0(\rho)$ with $(S,T) = ((c_x,h_x,w_x),(c_y,h_y,w_y))$, the equality $\check{\delta}(S,T) = 0$, together with the fact that $w_y \in \Omega_0^{odd}(Y)$, implies that

$$\rho^* \circ j[S,T](\sigma) = -\overline{\varphi_{w_y}}(\sigma) + \overline{h_y}(\partial\sigma) = 0 \text{ for all } \sigma \in C_1(Y).$$

Now, let $h \in \ker[\hat{K}^0(X) \xrightarrow{\rho^*} \hat{K}^0(Y)]$. First, we have $\rho^*(\delta_0(h)) = 0$. Furthermore, we can find $f \in L^1(Y,\mathbb{R})$ and $c \in L^0(Y,\mathbb{Z})$ such that

$$\rho^* h' = \delta f + c \text{ and } \rho^* u_{h'} = -\delta c.$$

It is easy to check that $R := ((u_{h'}, h', \delta_0(h)),(c,f,0))$ defines an element in $\check{K}^1(\rho)$ with $j([R]) = h$.

Exactness at $\hat{K}^0(Y)$. For every $h \in \hat{K}^0(X)$,

$$\theta\circ\rho^*(h) = [0,(u_{\rho^* h'}, \rho^* h', \rho^*\delta_0(h))] = [\check{\delta}((u_{h'}, h', \delta_0(h)),0)] = 0.$$

If $f \in \hat{K}^0(Y)$ such that $\theta(f) = 0$, then there exists $((c_x,h_x,w_x),(c_y,h_y,w_y)) \in \check{L}^1(\rho)$ with coboundary $(0,(u_{f'}, f', \delta_0(f)))$. Therefore, we have the equations

$$\begin{cases} \delta c_x = 0 \\ \varphi_{w_x} = \delta h_x + c_x \end{cases} \quad \text{and} \quad \begin{cases} \rho^* c_x + \delta c_y = u_{f'} \\ \rho^* h_x + \varphi_{w_y} - \delta h_y - c_y = f' \\ \rho^* w_x = \delta_0(f) \end{cases}$$

which imply that $\overline{h_x}$ is a differential K-character on X with $\delta_0(\overline{h_x}) = w_x$ and $\rho^*(\overline{h_x}) = f$. □

Remark 1. *Let X be a smooth compact manifold. Let i be the inclusion $pt \hookrightarrow X$. The above exact sequence, together with the fact that $i^* : \hat{K}^{*-1}(X) \to \hat{K}^{*-1}(pt)$ is surjective, implies that $j : \check{K}^*(i) \to \hat{K}^{*-1}(X)$ is injective with $img(j) = \ker(i^*)$. Thus we get an isomorphism*
$\check{K}^*(i) \cong \tilde{K}^{*-1}(X)$.

4. $\mathbb{R}/\mathbb{Z}$ Relative K-theory

This section is concerned with the definition of the K-theory of a smooth map $\rho : Y \to X$ with $\mathbb{R}/\mathbb{Z}$ coefficients and the construction of an isomorphism between this group and the group of homomorphisms from the relative K-homology of ρ [8] to $\mathbb{R}/\mathbb{Z}$.

Let $\rho : Y \to X$ be a smooth map between two smooth compact manifolds. We write $\check{L}^*(\rho,\mathbb{R}/\mathbb{Z})$ for the set of relative K-cochains of the form $((c_x,h_x,0),(c_y,h_y,0))$. The set $\check{L}^*(\rho,\mathbb{R}/\mathbb{Z})$ is in fact an Abelian subgroup of $\check{L}^*(\rho)$. Note that the image of the restriction of $\check{\delta} : \check{L}^*(\rho) \to \check{L}^{*+1}(\rho)$ to $\check{L}^*(\rho,\mathbb{R}/\mathbb{Z})$ is included in $\check{L}^{*+1}(\rho,\mathbb{R}/\mathbb{Z})$.
The Kernel of $\check{\delta} : \check{L}^*(\rho,\mathbb{R}/\mathbb{Z}) \to \check{L}^{*+1}(\rho,\mathbb{R}/\mathbb{Z})$ is denoted by $\check{Z}^*(\rho,\mathbb{R}/\mathbb{Z})$ and the image of $\check{\delta} : \check{L}^{*-1}(\rho,\mathbb{R}/\mathbb{Z}) \to \check{L}^*(\rho,\mathbb{R}/\mathbb{Z})$ is denoted by $\check{B}^*(\rho,\mathbb{R}/\mathbb{Z})$.

Definition 6. *We define the relative K-theory of ρ with $\mathbb{R}/\mathbb{Z}$ coefficients, denoted by $\check{K}^*(\rho,\mathbb{R}/\mathbb{Z})$, as the quotient group $\check{Z}^*(\rho,\mathbb{R}/\mathbb{Z})/\check{B}^*(\rho,\mathbb{R}/\mathbb{Z})$.*

It is obvious that $\check{K}^*(\rho,\mathbb{R}/\mathbb{Z})$ is an Abelian subgroup of $\check{K}^*(\rho)$.

Let us recall the six-term exact sequence in section 3:

$$\begin{array}{ccccc} \hat{K}^0(X) & \xrightarrow{\rho^*} & \hat{K}^0(Y) & \xrightarrow{\theta} & \check{K}^0(\rho) \\ j\uparrow & & & & \downarrow j \\ \check{K}^1(\rho) & \xleftarrow{\theta} & \hat{K}^1(Y) & \xleftarrow{\rho^*} & \hat{K}^1(X) \end{array}$$

Note that the image of the restriction of j to $\check{K}^*(\rho,\mathbb{R}/\mathbb{Z})$ is included in $Hom(K_{*-1}(X),\mathbb{R}/\mathbb{Z})$, and the image of the restriction of θ to $Hom(K_*(Y),\mathbb{R}/\mathbb{Z})$ is included in $\check{K}^*(\rho,\mathbb{R}/\mathbb{Z})$.
Let $K^*(X,\mathbb{R}/\mathbb{Z})$, the K-theory of X with $\mathbb{R}/\mathbb{Z}$ coefficients. We have the six-term exact sequence

$$\begin{array}{ccccc} K^0(X,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\rho^*} & K^0(Y,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\theta} & \check{K}^0(\rho,\mathbb{R}/\mathbb{Z}) \\ j\uparrow & & & & \downarrow j \\ \check{K}^1(\rho,\mathbb{R}/\mathbb{Z}) & \xleftarrow{\theta} & K^1(Y,\mathbb{R}/\mathbb{Z}) & \xleftarrow{\rho^*} & K^1(X,\mathbb{R}/\mathbb{Z}) \end{array}$$

obtained from the above exact sequence and after identification of the groups $K^*(X,\mathbb{R}/\mathbb{Z})$ and $Hom(K_*(X),\mathbb{R}/\mathbb{Z})$ following [2].

Now, we show that the group $\check{K}^*(\rho,\mathbb{R}/\mathbb{Z})$ can be identified with the group of homomorphisms from the relative K-homology group $K_*(\rho)$ [8] to $\mathbb{R}/\mathbb{Z}$.

Let us recall the construction of the group $K_*(\rho)$ following [8].
We set

$$L_*(\rho) := L_*(X) \times L_{*-1}(Y)$$

and define a boundary map $\hat{\partial} : L_*(\rho) \to L_{*-1}(\rho)$ by the formula:

$$\hat{\partial}(\alpha,\beta) := (\partial\alpha + \rho_*\beta, -\partial\beta).$$

Let $C_*(\rho)$ denote the kernel of $\hat{\partial}$. There is a well-defined operation on $C_*(\rho)$ given by disjoint union of K-chains,

$$(\alpha,\beta) + (\alpha',\beta') := (\alpha \sqcup \alpha', \beta \sqcup \beta').$$

Bordism. Two elements (α,β) and (α',β') in $C_*(\rho)$ are bordant if there exists $(\sigma,\tau) \in L_{*+1}(\rho)$ such that $(\alpha,\beta) + (-\alpha',-\beta') = \hat{\partial}(\sigma,\tau)$.

Definition 7. *We define the relative K-homology group $K_*(\rho)$ as the group obtained from quotienting $C_*(\rho)$ by the equivalence relation of bordism.*

We denote by $\hat{K}^*(\rho,\mathbb{R}/\mathbb{Z})$ the group of homomorphisms from $K_*(\rho)$ to $\mathbb{R}/\mathbb{Z}$.

For every K-cocycle (S,T) in $\check{Z}^*(\rho,\mathbb{R}/\mathbb{Z})$ with $(S,T) = ((c_x,h_x,0),(c_y,h_y,0))$, we set

$$\mu(S,T)(\alpha,\beta) := \widetilde{h_x}(\alpha) + \widetilde{h_y}(\beta) \text{ for all } (\alpha,\beta) \in C_{*-1}(\rho).$$

If $(S,T) \in \check{L}^*(\rho,\mathbb{R}/\mathbb{Z})$ with $(S,T) = ((c_x,h_x,0),(c_y,h_y,0))$, then for all $(\sigma,\tau) \in L_*(\rho)$,

$$\begin{aligned}
\mu(S,T)(\hat{\partial}(\sigma,\tau)) =& \mu(S,T)(\partial\sigma + \rho_*\tau, -\partial\tau) \\
=& \widetilde{h_x}(\partial\sigma) + \widetilde{h_x}(\rho_*\tau) - \widetilde{h_y}(\partial\tau) \\
=& \widetilde{\delta h_x}(\sigma) + (\widetilde{\rho^* h_x} - \widetilde{\delta h_y})(\tau) \\
=& \mu(\hat{\delta}(S,T))(\sigma,\tau).
\end{aligned}$$

It follows that μ induces a well-defined homomorphism

$$\check{K}^*(\rho,\mathbb{R}/\mathbb{Z}) \to \hat{K}^{*-1}(\rho,\mathbb{R}/\mathbb{Z}),$$

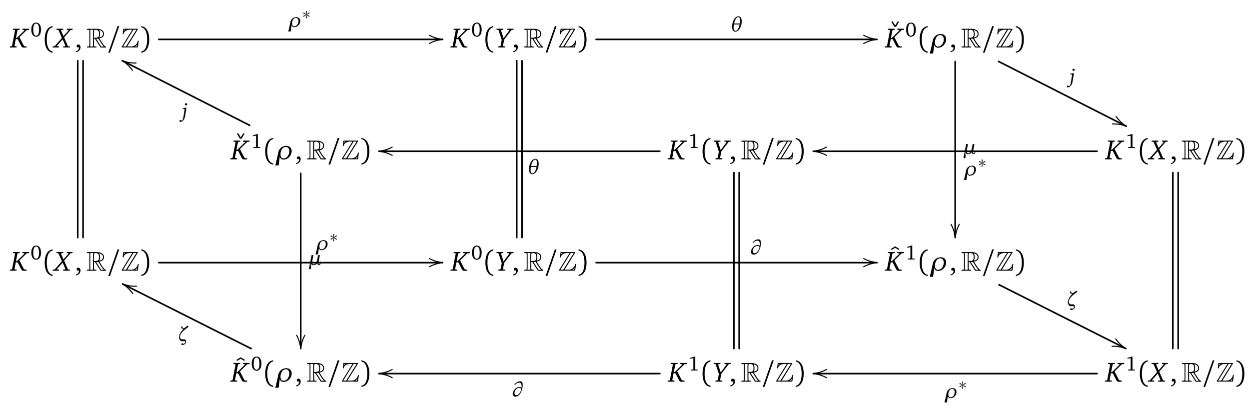
also denoted by μ.

Proposition 1. *The homomorphism $\mu : \check{K}^*(\rho,\mathbb{R}/\mathbb{Z}) \to \hat{K}^{*-1}(\rho,\mathbb{R}/\mathbb{Z})$ turns out to be an isomorphism.*

Proof. Let us recall the six-term exact sequence in [8, p. 8]:

$$\begin{array}{ccccc}
\hat{K}^1(\rho,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\zeta} & K^1(X,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\rho^*} & K^1(Y,\mathbb{R}/\mathbb{Z}) \\
\uparrow \partial & & & & \downarrow \partial \\
K^0(Y,\mathbb{R}/\mathbb{Z}) & \xleftarrow{\rho^*} & K^0(X,\mathbb{R}/\mathbb{Z}) & \xleftarrow{\zeta} & \hat{K}^0(\rho,\mathbb{R}/\mathbb{Z})
\end{array}$$

If we combine this six-term exact sequence with that given by Theorem 1, then we obtain the following commutative diagram



in which the rows are exact sequences. It follows from the five lemma that the homomorphism μ is an isomorphism. $\square$

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