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On Statistical and Ideal Convergence of Sequences of Bounded Linear Operators

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Abstract. Let (A_n) be a sequence of bounded linear operators from a separable Banach space X into a Banach space Y. Suppose that Φ is a countable fundamental set of X and the ideal $\mathscr{I}$ of subsets of $\mathbb{N}$ has property (AP). The sequence (A_n) is said to be $b^*\mathscr{I}$-convergent if it is pointwise $\mathscr{I}$-convergent and there exists an index set K such that $\mathbb{N}\setminus K \in \mathscr{I}$ and $(A_k x)_{k\in K}$ is bounded for any $x \in X$. We prove that the sequence (A_n) is $b^*\mathscr{I}$-convergent if and only if $(\|A_n\|)$ is $\mathscr{I}$-bounded and $(A_n\phi)$ is $\mathscr{I}$-convergent for any $\phi \in \Phi$. Applications of this Banach–Steinhaus type theorem are related to some sequence-to-sequence matrix transformations and to the weak $\mathscr{I}$-convergence in Banach spaces.

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1. Introduction and Preliminaries

Let $\mathbb{N} = \{1, 2, \dots\}$ and let X, Y be two normed spaces over the field $\mathbb{K}$ of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. A subset Φ of X is called fundamental if the linear span of Φ is dense in X. By $B(X,Y)$ we denote the space of all bounded linear operators from X into Y. As usual, the dual of X is defined by $X' = B(X,\mathbb{K})$. By $\omega(X)$ we denote the set of all X-valued sequences. We write $\sup_n$, $\lim_n$ and $\sum_n$ instead of $\sup_{n\in\mathbb{N}}$, $\lim_{n\to\infty}$ and $\sum_{n=1}^{\infty}$, respectively. By an *index set* we mean any infinite set $\{k_i\} \subset \mathbb{N}$ with $k_i < k_{i+1}$ for each $i \in \mathbb{N}$.

Let $A_n \in B(X,Y)$ $(n \in \mathbb{N})$. The following theorems of functional analysis are well known (see, for example, [11] or [17]).

Theorem 1 (Principle of uniform boundedness). *Let X be a Banach space. If $\sup_n \|A_n x\| < \infty$ for every $x \in X$, then*

$$\sup_n \|A_n\| < \infty. \tag{1}$$

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Theorem 2 (Banach–Steinhaus). *Let X, Y be two Banach spaces and let Φ be a fundamental set of X. The limit $\lim_n A_n x$ exists for any $x \in X$ if and only if* (1) *holds and $\lim_n A_n \phi$ exists for every $\phi \in \Phi$. Moreover, the limit operator A_0, $A_0 x = \lim_n A_n x$ is bounded and linear, i.e., $A_0 \in B(X,Y)$, and $\|A_0\| \le \sup_n \|A_n\|$. If $A \in B(X,Y)$, then $\lim_n A_n x = Ax$ for any $x \in X$ if and only if* (1) *holds and $\lim_n A_n \phi = A\phi$ ($\phi \in \Phi$).*

The first idea of statistical convergence appeared, under the name of almost convergence, in the first edition (Warsaw, 1935) of the monograph [25] of Zygmund. Since 1951 when Fast [7] (see also [23] and [22]) introduced statistical convergence of number sequences in terms of asymptotic density of subsets of $\mathbb{N}$, several applications and generalizations of this notion have been investigated (for references see [4] and [6]). For instance, Maddox [20] and Kolk [13] considered the statistical convergence of sequences taking values in a locally convex space or a normed space, respectively. An another extension of statistical convergence is related to generalized densities.

Let $T = (t_{nk})$ be a non-negative regular matrix of scalars (i.e., $t_{nk} \ge 0$ $(n,k \in \mathbb{N})$ and $\lim_n \sum_k t_{nk} u_k = \lim_k u_k$ for any convergent scalar sequence (u_k)). A set $K \subset \mathbb{N}$ is said to have *T-density* $\delta_T(K)$ if the limit

$$\delta_T(K) = \lim_n \sum_{k \in K} t_{nk}$$

exists (cf. [9]).

A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is called *T-statistically convergent* to a point $l \in X$, briefly $st_T\text{-}\lim x_k = l$, if

$$\delta_T(\{k : \|x_k - l\| \ge \varepsilon\}) = 0$$

for every $\varepsilon > 0$ (see [3, Definition 7] and [14, p. 44]).

If T is the identity matrix I, then T-statistical convergence is just the ordinary convergence in X and if T is the Cesàro matrix C_1, then T-statistical convergence is statistical convergence as defined by Fast [7].

A further extension of statistical convergence was given in [16] by means of ideals. Recall that a subfamily $\mathscr{I}$ of the family $2^{\mathbb{N}}$ of all subsets of $\mathbb{N}$ is called an *ideal* if for each $K, L \in \mathscr{I}$ we have $K \bigcup L \in \mathscr{I}$ and for each $K \in \mathscr{I}$ and each $L \subset K$ we have $L \in \mathscr{I}$. An ideal $\mathscr{I}$ is called *non-trivial* if $\mathscr{I} \ne \emptyset$ and $\mathbb{N} \notin \mathscr{I}$. A non-trivial ideal $\mathscr{I}$ is called *admissible* if $\mathscr{I}$ contains all finite subsets of $\mathbb{N}$. Any non-trivial ideal $\mathscr{I}$ defines a *filter*

$$\mathscr{F}(\mathscr{I}) = \{K \subset \mathbb{N} : \mathbb{N} \setminus K \in \mathscr{I}\}.$$

For example,

$$\mathscr{I}_T = \{K \subset \mathbb{N} : \delta_T(K) = 0\}$$

is an admissible ideal and the $\mathscr{I}_T$-convergence coincides with the T-statistical convergence.

An admissible ideal $\mathscr{I} \subset 2^{\mathbb{N}}$ is said to have *property* (AP) if for every countable family of mutually disjoint sets $K_1, K_2, \dots$ from $\mathscr{I}$ there exist sets $L_1, L_2, \dots$ from $2^{\mathbb{N}}$ such that the symmetric differences $K_i \Delta L_i$ $(i \in \mathbb{N})$ are finite and $L = \bigcup_i L_i \in \mathscr{I}$.

Remark 1 ([1], Proposition 1). *The property* (AP) *is equivalent to the property* (P)*: for every countable family of sets* $K_1, K_2, \dots$ *from* $\mathscr{I}$ *there exist a set* $K \in \mathscr{I}$ *such that the differences* $K_i \setminus K$ $(i \in \mathbb{N})$ *are finite.*

A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is said to be *$\mathscr{I}$-convergent* to $l \in X$, briefly $\mathscr{I}$-$\lim_k x_k = l$, if for each $\varepsilon > 0$ the set $\{k \in \mathbb{N} : \|x_k - l\| \geq \varepsilon\}$ belongs to $\mathscr{I}$ [16, Definition 3.1]. With the $\mathscr{I}$-convergence are closely related the following two notions. A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is said to be *$\mathscr{I}^*$-convergent* to $l \in X$, briefly $\mathscr{I}^*$-$\lim x_k = l$, if there exists an index set $K = (k_i)$ such that $K \in \mathscr{F}(\mathscr{I})$ and $\lim_i x_{k_i} = l$ in X [16, Definition 3.2]). A sequence $\mathfrak{x} = (x_k) \in \omega(X)$ is said to be *$\mathscr{I}$-bounded*, briefly $x_k = O_{\mathscr{I}}(1)$, if there exists an index set $K = (k_i)$ such that $K \in \mathscr{F}(\mathscr{I})$ and the sequence (k_i) is bounded in X (cf. [10]). In the special case $\mathscr{I} = \mathscr{I}_T$ we write $O_{st_T}(1)$ instead of $O_{\mathscr{I}}(1)$.

We remark that the $\mathscr{I}^*$-convergence of number sequences was introduced already by Freedman [8] as $\mathscr{I}$-near convergence.

It is easy to see that $\mathscr{I}^*$-convergence implies $\mathscr{I}$-convergence and every $\mathscr{I}^*$-convergent sequence is $\mathscr{I}$-bounded.

The following characterization of $\mathscr{I}$-convergence is important for us.

Proposition 1 ([16, Theorem 3.2]). *If the ideal* $\mathscr{I}$ *has property* (AP)*, then* $\mathscr{I}$-$\lim x_k = l$ *in a Banach space* X *if and only if* $\mathscr{I}^*$-$\lim x_k = l$.

By $c_{\mathscr{I}}(X)$ we denote the set of all $\mathscr{I}$-convergent X-valued sequences. Let $\ell_\infty(X)$, $c(X)$ and $c_0(X)$ be the sets of all bounded, convergent and convergent to zero X-valued sequences, respectively. For $1 \leq p < \infty$ let $\ell_p(X)$ be the set of sequences $(x_k) \in \omega(X)$ such that $\sum_k \|x_k\|^p < \infty$.

Using Proposition 1 and Theorem 2, we proved in [15] the following Banach–Steinhaus type theorem for $\mathscr{I}$-convergence.

Theorem 3 ([15, Theorem 3]). *Let* X *and* Y *be two Banach spaces, where* X *has a countable fundamental set* Φ. *If the ideal* $\mathscr{I}$ *has property* (AP)*, then the sequence* (A_n) *is* $b\mathscr{I}$*-convergent (i.e.,* $(A_n x) \in c_{\mathscr{I}}(Y) \cap \ell_\infty(Y)$ *for any* $x \in X$*) if and only if* (1) *holds and* $(A_n \phi)$ *is* $\mathscr{I}$*-convergent for every* $\phi \in \Phi$. *Thereby, the limit operator* A*,* $Ax = \mathscr{I}$-$\lim A_n x$*, is bounded and linear, and* $\|A\| \leq \sup_n \|A_n\|$.

In this paper we introduce the notion of $b^*\mathscr{I}$-convergence of sequences of bounded linear operators (A_n) and give an analogue of Theorem 3 by finding necessary and sufficient conditions for $b^*\mathscr{I}$-convergence of such sequences (A_n). As applications of this result we characterize infinite summability matrices $\mathfrak{A} = (A_{nk})$ of type $\mathfrak{A} : \lambda(X) \xrightarrow{b^*\mathscr{I}} c(Y)$ with $A_{nk} \in B(X, Y)$ $(n, k \in \mathbb{N})$ and $\lambda \in \{c,\ c_0,\ \ell_1\}$, also consider the weak $b^*\mathscr{I}$-convergence in Banach spaces.

2. Main Theorems

In the following let X, Y be two Banach spaces, $A_n \in B(X, Y)$ $(n \in \mathbb{N})$ and let $\mathscr{I} \subset 2^{\mathbb{N}}$ be a non-trivial admissible ideal.

Recall that a sequence $(x_n) \in \omega(X)$ is said to be *weakly $\mathscr{I}$-convergent* (*weakly T-statistically convergent*) to a point $l \in X$ if $\mathscr{I}$-$\lim x'(x_n) = x'(l)$ (st_T-$\lim x'(x_n) = x'(l)$) for any $x' \in X'$ [2, 21]. We know that every weakly convergent sequence in a Banach space X is bounded. But a weakly $\mathscr{I}$-convergent sequence is not necessary $\mathscr{I}$-bounded (cf. [5, Theorem 1]). Example 2 from [5] shows that the Banach sequence space ℓ_2 contains a weakly statistically null sequence (z_k) with no bounded subsequences. Thus some results of Bhardwaj and Bala [2, Theorem 3.1 and Lemma 3.2] are incorrect. At it, defining $F_n x' = x'(z_n)$ $(x' \in \ell_2', n \in \mathbb{N})$, we get the sequence (F_n) of bounded linear functionals $F_n : \ell_2' \to \mathbb{R}$. Since $\|F_n\| = \|z_n\|$ by the classical Hahn–Banach theorem, the sequence of functionals (F_n) converges statistically to zero for any $x' \in \ell_2'$, but the sequence of norms $(\|F_n\|)$ contains no bounded subsequences. This example justifies the following definition.

Definition 1. *A sequence (A_n) of operators $A_n \in B(X,Y)$ $(n \in \mathbb{N})$ is said to be $b^*\mathscr{I}$-convergent (to $A \in B(X,Y)$) if $\mathscr{I}$-$\lim_n A_n x$ exists ($\mathscr{I}$-$\lim A_n x = Ax$) for any $x \in X$ and there is a set $K \in \mathscr{F}(\mathscr{I})$ such that $(A_k x)_{k \in K}$ is bounded for every $x \in X$. In the special case $\mathscr{I} = \mathscr{I}_T$ we get the notion of b^*T-statistical convergence. The $b^*\mathscr{I}$-limit and the b^*T-statistical limit of (A_n) are denoted, respectively, by $b^*\mathscr{I}$-$\lim_n A_n$ and b^*st_T-$\lim_n A_n$.*

In view of Theorem 1 we can say that a sequence (A_n) is $b^*\mathscr{I}$-convergent if and only if $\mathscr{I}$-$\lim A_n x$ exists for any $x \in X$ and

$$\sup_{k \in K} \|A_k\|) < \infty \text{ for some } K \in \mathscr{F}(\mathscr{I}). \tag{2}$$

Theorem 3 shows that $b\mathscr{I}$-convergence implies $b^*\mathscr{I}$-convergence by the suppositions that X is separable and $\mathscr{I}$ satisfies the condition (AP).

To prove our main theorem we need the following lemma.

Lemma 1. *Suppose that the ideal $\mathscr{I}$ has property* (AP) *and let $z_{kj} \in X$ $(k, j \in \mathbb{N})$. If $\mathscr{I}$-$\lim_k z_{kj} = z_j$ for any $j \in \mathbb{N}$, then there exists an index set $N = (n_i)$ such that $N \in \mathscr{F}(\mathscr{I})$ and $\lim_i z_{n_i,j} = z_j$ for any $j \in \mathbb{N}$.*

Proof. Assume that $\mathscr{I}$-$\lim_k z_{kj} = z_j$ $(j \in \mathbb{N})$. Since $\mathscr{I}$ has property (AP), by Proposition 1 there exist index sets $K_j = \{k_i(j)\}$ $(j \in \mathbb{N})$ such that

$$\lim_i z_{k_i(j),j} = z_j \quad (j \in \mathbb{N}) \tag{3}$$

and $K_j' = \mathbb{N} \setminus K_j \in \mathscr{I}$ for any $j \in \mathbb{N}$. Because of Remark 1 we can find the set $N' \in \mathscr{I}$ such that the differences $K_j' \setminus N'$ $(j \in \mathbb{N})$ are finite. Now, for $N = \mathbb{N} \setminus N'$ we have that $N \in \mathscr{F}(\mathscr{I})$ and the differences $N \setminus K_j$ are finite. Consequently, denoting $N = (n_i)$, from (3) it follows that $\lim_i z_{n_i,j} = z_j$ for any $j \in \mathbb{N}$. □

Theorem 4. *Let X and Y be two Banach spaces, where X has a countable fundamental set Φ. If the ideal $\mathscr{I}$ has property* (AP). *A sequence (A_n) of operators $A_n \in B(X,Y)$ is $b^*\mathscr{I}$-convergent if and only if $(\|A_n\|)$ is $\mathscr{I}$-bounded, i.e.,* (2) *holds, and $(A_n\phi)$ is $\mathscr{I}$-convergent for every $\phi \in \Phi$. Thereby, the limit operator A_0, $A_0 x = \mathscr{I}$-$\lim A_n x$, is bounded and linear, and $\|A_0\| \le \sup_{k \in K} \|A_k\|$. If $A \in B(X,Y)$, then $b^*\mathscr{I}$-$\lim_n A_n = A$ if and only if $(\|A_n\|)$ is $\mathscr{I}$-bounded and $\mathscr{I}$-$\lim_n A_n \phi = A\phi$ $(\phi \in \Phi)$.*

Proof. If (A_n) is $b^*\mathscr{I}$-convergent ($b^*\mathscr{I}$-$\lim_n A_n = A$), then (2) is satisfied and $\mathscr{I}$-$\lim A_n\phi$ exists ($\mathscr{I}$-$\lim A_n\phi = A\phi$) for every $\phi \in \Phi$.

Conversely, assume that (2) holds and $\mathscr{I}$-$\lim A_n\phi_j$ exists (or $\mathscr{I}$-$\lim A_n\phi_j = A\phi_j$) for every $j \in \mathbb{N}$, where $\Phi = \{\phi_j\}$. Applying Lemma 1 to $z_{nj} = A_n\phi_j$ (and $z_j = A\phi_j$), we fix an index set $N = (n_i) \in \mathscr{F}(\mathscr{I})$ such that $\lim_i A_{n_i}\phi_j$ exists ($\lim_i A_{n_i}\phi_j = A\phi_j$) for any $j \in \mathbb{N}$. Since the set $M = N \cap K$ also belongs to $\mathscr{F}(\mathscr{I})$, denoting $M = (m_i)$, we have that $\lim_i A_{m_i}\phi_j$ exists ($\lim_i A_{m_i}\phi_j = A\phi_j$) for any $j \in \mathbb{N}$ and $\sup_i \|A_{m_i}\| < \infty$. So, by Theorem 2, the limit $A_0 x = \lim_i A_{m_i} x$ exists ($\lim_i A_{m_i} x = Ax$) for any $x \in X$, $A_0 \in B(X, Y)$ and $\|A_0\| \le \sup_i \|A_{m_i}\|$. The proof is completed if we remark that $\lim_i A_{m_i} x = \mathscr{I}$-$\lim A_n x$ by Proposition 1. □

It is known that the ideal $\mathscr{I}_T = \{K \subset \mathbb{N} : \delta_T(K) = 0\}$ defined by a non-negative regular matrix T has the property (AP) (see [9, Proposition 3.2]). Since $\mathscr{I}_T$-convergence coincides with T-statistical convergence, from Theorem 4 we immediately get the following Banach–Steinhaus type theorem for b^*T-statistical convergence.

Theorem 5. *Suppose that T is a non-negative regular matrix and X has a countable fundamental set Φ. A sequence (A_n) of operators $A_n \in B(X, Y)$ is b^*T-statistically convergent if and only if* (2) *holds and st_T-$\lim A_n\phi$ exists for any $\phi \in \Phi$. In this case the limit operator A_0, $A_0 x = st_T$-$\lim A_n x$ ($x \in X$), belongs to $B(X, Y)$ and $\|A_0\| \le \sup_{k\in K} \|A_k\|$. If $A \in B(X, Y)$, then b^*st_T-$\lim_n A_n = A$ if and only if $(\|A_n\|)$ is $\mathscr{I}$-bounded and st_T-$\lim_n A_n\phi = A\phi$ ($\phi \in \Phi$).*

3. Some Applications

Let $\lambda(X)$ be a subspace of $\omega(X)$, $\mu(Y)$ a subspaces of $\omega(Y)$ and $\mathfrak{A} = (A_{nk})$ an infinite matrix of operators $A_{nk} \in B(X, Y)$ ($n, k \in \mathbb{N}$). We say that $\mathfrak{A}$ *maps* $\lambda(X)$ *into* $\mu(Y)$, and write $\mathfrak{A} : \lambda(X) \to \mu(Y)$, if for all $\mathfrak{x} = (x_k) \in \lambda(X)$ the series $\mathfrak{A}_n\mathfrak{x} = \sum_k A_{nk}x_k$ ($n \in \mathbb{N}$) converge and the sequence $\mathfrak{A}\mathfrak{x} = (\mathfrak{A}_n\mathfrak{x})$ belongs to $\mu(Y)$.

It is well known that $c(X)$, $c_0(X)$ and $\ell_\infty(X)$ are Banach spaces with the norm $\|\mathfrak{x}\|_\infty = \sup_k \|x_k\|$, and $\ell_p(X)$ is Banach space with the norm $\|\mathfrak{x}\|_p = \left(\sum_k \|x_k\|^p\right)^{1/p}$ if $1 \le p < \infty$.

For $x \in X$ and $n \in \mathbb{N}$ let $\mathfrak{e}(x) = (x, x, \dots)$ be constant sequence and $\mathfrak{e}^k(x) = (e^k_j(x))$ the sequence with $e^k_j(x) = x$ if $j = k$ and $e^k_j(x) = 0$ otherwise. It is not difficult to see that if Φ is a (countable) fundamental set in X, then $\mathscr{E}_0(\Phi) = \{\mathfrak{e}^k(\phi) : k \in \mathbb{N},\ \phi \in \Phi\}$ is a (countable) fundamental set in Banach spaces $c_0(X)$ and $\ell_p(X)$, and $\mathscr{E}_0(\Phi) \bigcup \mathscr{E}_1(\Phi)$ with $\mathscr{E}_1(\Phi) = \{\mathfrak{e}(\phi) : \phi \in \Phi\}$ is a (countable) fundamental set in Banach space $c(X)$.

Using Theorem 2, Zeller [24] (see also [19]) and Kangro [12] characterized the matrices $\mathfrak{A} : c(X) \to c(Y)$, $\mathfrak{A} : c_0(X) \to c(Y)$ and $\mathfrak{A} : \ell_1(X) \to c(Y)$ as follows.

Theorem 6. *Let $\mathfrak{A} = (A_{nk})$ be an infinite matrix with $A_{nk} \in B(X, Y)$. Then:*

(i) $\mathfrak{A} : c(X) \to c(Y)$ *if and only if*

$$G_n = \sup_r \sup_{\|x_k\| \le 1} \left\| \sum_{k=1}^r A_{nk}x_k \right\| < \infty \quad (n \in \mathbb{N}), \tag{4}$$

$$\sup_n G_n < \infty, \tag{5}$$

$$\exists \lim_n A_{nk}x \quad (k \in \mathbb{N},\ x \in X), \tag{6}$$

$$\exists \lim_m \sum_{k=1}^{m} A_{nk}x \quad (n \in \mathbb{N},\ x \in X), \tag{7}$$

$$\exists \lim_n \sum_k A_{nk}x \quad (x \in X); \tag{8}$$

(ii) $\mathfrak{A} : c_0(X) \to c(Y)$ *if and only if* (4)–(6) *hold;*

(iii) $\mathfrak{A} : \ell_1(X) \to c(Y)$ *if and only if* (6) *is satisfied and*

$$H_n = \sup_k \left\| A_{nk} \right\| < \infty \quad (n \in \mathbb{N}), \tag{9}$$

$$\sup_n H_n < \infty,$$

Remark 2. *It is not difficult to see, using Theorem 2, that in Theorem 6 it suffices to require the fulfillment of conditions* (6)–(8) *for all elements* ϕ *from a fundamental set* Φ *of* X*.*

The notion of $b^*\mathscr{I}$-convergence of sequences of bounded linear operators leads us to the definition of new type summability maps.

Definition 2. *Let* $\lambda(X)$ *and* $\mu(Y)$ *be two linear subspaces of* $\omega(X)$ *and* $\omega(Y)$*, respectively, and let* $\mathscr{I} \subset 2^{\mathbb{N}}$ *be a non-trivial admissible ideal. We say that a matrix* $\mathfrak{A}$ *maps* $\lambda(X)$ *in the sense of* $b^*\mathscr{I}$*-convergence into* $\mu(Y)$*, and write* $\mathfrak{A} : \lambda(X) \xrightarrow{b^*\mathscr{I}} \mu(Y)$*, if* $\mathscr{I}\text{-}\lim \mathfrak{A}_n \mathfrak{x}$ *exists for any* $\mathfrak{x} \in \lambda(X)$ *and there is an index set* $N = (n_i)$ *from* $\mathscr{F}(\mathscr{I})$ *such that the submatrix* $\mathfrak{A}_{(N)} = (a_{n_i,k})$ *maps* $\lambda(X)$ *into* $\ell_\infty(Y)$*. In the case of* $\mathscr{I} = \mathscr{I}_T$ *we get the matrices of type* $\mathfrak{A} : \lambda(X) \xrightarrow{b^*st_T} \mu(Y)$*.*

Based on Theorems 4 and 5, we describe the matrices $\mathfrak{A} : \lambda(X) \xrightarrow{b^*\mathscr{I}} c(Y)$ and $\mathfrak{A} : \lambda(X) \xrightarrow{b^*st_T} c(Y)$, where $\lambda \in \{c,\ c_0,\ \ell_p\}$.

Proposition 2. *Let* $\mathfrak{A} = (A_{nk})$ *be an infinite matrix with* $A_{nk} \in B(X,Y)$*. Suppose that* X *has a countable fundamental set* Φ *and the ideal* $\mathscr{I}$ *has property* (AP)*. Then:*

(i) $\mathfrak{A} : c(X) \xrightarrow{b^*\mathscr{I}} c(Y)$ *if and only if* (4) *and* (7) *hold,*

$$G_n = O_{\mathscr{I}}(1), \tag{10}$$

$$\exists \mathscr{I}\text{-}\lim_n A_{nk}\phi \quad (k \in \mathbb{N},\ \phi \in \Phi), \tag{11}$$

$$\exists \mathscr{I}\text{-}\lim_n \sum_k A_{nk}\phi \quad (\phi \in \Phi); \tag{12}$$

(ii) $\mathfrak{A} : c_0(X) \xrightarrow{b^*\mathscr{I}} c(Y)$ *if and only if* (4), (10) *and* (11) *hold;*

(iii) $\mathfrak{A} : \ell_1(X) \xrightarrow{b^*\mathscr{I}} c(Y)$ *if and only if* (9) *is satisfied and* (H_n) *is* $\mathscr{I}$*-bounded.*

Proof. The equality $\mathfrak{A}_n^{(r)}\mathfrak{x} = \sum_{k=1}^{r} A_{nk}x_k$ defines a linear operator $\mathfrak{A}_n^{(r)}$ on $c(X)$ and $c_0(X)$ for any $n, r \in \mathbb{N}$. Since

$$\|\mathfrak{A}_n^{(r)}\| = \left\|\sum_{k=1}^{r} A_{nk}x_k\right\|,$$

by Theorem 2 we get that the series $\mathfrak{A}_n\mathfrak{x}$ ($n \in \mathbb{N}$) converge for all $\mathfrak{x} \in c(X)$ and $\mathfrak{A}_n \in B(c(X), Y)$ if and only if (4), (7) are satisfied. Similarly, $\mathfrak{A}_n \in B(c_0(X), Y)$ if and only if (4) holds. Now, applying Theorem 4 to the operators $\mathfrak{A}_n$, we have that $\mathfrak{A} : c(X) \xrightarrow{b^*\mathscr{I}} c(Y)$ (or $\mathfrak{A} : c_0(X) \xrightarrow{b^*\mathscr{I}} c(Y)$) if and only if (10) holds and $(\mathfrak{A}_n\mathfrak{y})$ is $\mathscr{I}$-convergent for any $\mathfrak{y} \in \mathscr{E}_1(\Phi)$ (respectively, $\mathfrak{y} \in \mathscr{E}_0(\Phi)$). But this reduces to (11) and (12) because $\mathfrak{A}_n\mathfrak{e}_k(\phi) = A_{nk}\phi$ and $\mathfrak{A}_n\mathfrak{e}(\phi) = \sum_k A_{nk}\phi$.

Since $\mathfrak{A}_n \in B(\ell_1(X), Y)$ if and only if (9) holds, the statement (iii) also follows by Theorem 4. □

The matrix map $\mathfrak{A} : \ell_p(X) \xrightarrow{b^*\mathscr{I}} c(Y)$ we consider in the special cases $Y = \mathbb{K}$ and $1 < p < \infty$. Then $B(X, Y) = X'$ and so, $A_{nk} \in X'$ ($n, k \in \mathbb{N}$). In this case $\mathfrak{A}_n \in (\ell_p(X))'$ if and only if $(A_{nk})_{k\in\mathbb{N}} \in \ell_q(X')$, i.e., $\sum_k \|A_{nk}\|^q < \infty$, where $1/p + 1/q = 1$. Therefore, denoting $c = c(\mathbb{K})$ and using the same arguments as in the proof of Proposition 2, we get the following result.

Proposition 3. *Let* $\mathfrak{A} = (A_{nk})$ *be an infinite matrix with* $A_{nk} \in X'$. *Suppose that* X *has a countable fundamental set* Φ*, the ideal* $\mathscr{I}$ *has property* (AP) *and* $1 < p < \infty$, $1/p + 1/q = 1$. *Then* $\mathfrak{A} : \ell_p(X) \xrightarrow{b^*\mathscr{I}} c$ *if and only if* (11) *holds and*

$$\sum_k \|A_{nk}\|^q = O_{\mathscr{I}}(1).$$

If $X = Y = \mathbb{K}$, then the matrix map $\mathfrak{A}$ reduces to the transformation $A : \lambda \to \mu$ defined by an infinite scalar matrix $A = (a_{nk})$. Using the fact that for $Y = \mathbb{K}$ we have (see [12, p. 114])

$$\sup_{\|x_k\|\le 1} \left\|\sum_{k=1}^{r} A_{nk}x_k\right\| = \sum_{k=1}^{r} \|A_{nk}\|,$$

from Propositions 2 and 3 we obtain the following corollary.

Corollary 1. *Let* $A = (a_{nk})$ *be an infinite matrix of scalars,* $1 < p < \infty$ *and* $1/p + 1/q = 1$. *If the ideal* $\mathscr{I}$ *has property* (AP)*, then:*

(i) $A : c \xrightarrow{b^*\mathscr{I}} c$ *if and only if*

$$\sum_k |a_{nk}| = O_{\mathscr{I}}(1), \tag{13}$$

$$\exists \mathscr{I}\text{-}\lim_n a_{nk} \quad (k \in \mathbb{N}), \tag{14}$$

$$\exists \mathscr{I}\text{-}\lim_n \sum_k a_{nk};$$

(ii) $A : c_0 \xrightarrow{b^*\mathscr{I}} c$ *if and only if* (13) *and* (14) *hold;*

(iii) $A : \ell_1 \xrightarrow{b^*\mathscr{I}} c$ *if and only if* (14) *is satisfied,* $h_n = \sup_k |a_{nk}| < \infty$ $(n \in \mathbb{N})$ *and* (h_n) *is* $\mathscr{I}$*-bounded;*

(iv) $A : \ell_p \xrightarrow{b^*\mathscr{I}} c$ *if and only if* (14) *is satisfied and*

$$\sum_k |a_{nk}|^q = O_{\mathscr{I}}(1).$$

Letting $\mathscr{I} = \mathscr{I}_T$ in Propositions 2, 3 and Corollary 1, we get the characterizations of analogical matrix maps in the sense of b^*T-statistical convergence. We also remark that the matrix maps in the sense of $b\mathscr{I}$- and bst_T-convergence were studied in [15].

At the beginning of Section 2 we remarked that a weakly $\mathscr{I}$-convergent sequence is not necessary $\mathscr{I}$-bounded. This fact leads us to a new variant of weak $\mathscr{I}$-convergence.

Definition 3. *A sequence* $\mathfrak{x} = (x_n) \in \omega(X)$ *is said to be weakly* $b^*\mathscr{I}$*-convergent to* $l \in X$*, briefly* $wb^*\mathscr{I}\text{-}\lim_n x_n = l$*, if* $\mathfrak{x}$ *is weakly* $\mathscr{I}$*-convergent to* l *and there is a set* $K \in \mathscr{F}(\mathscr{I})$ *such that the sequence* $(x'(x_k))_{k\in K}$ *is bounded for every* $x' \in X'$*. For* $\mathscr{I} = \mathscr{I}_T$ *we get the notion of weak* b^*T*-statistical convergence, in this case we write* $wb^*st_T\text{-}\lim_n x_n = l$.

Using bounded linear functionals $F_z : X' \to \mathbb{R}$, $F_z x' = x'(z)$ $(x' \in X', z \in X)$, we can say that $wb^*\mathscr{I}\text{-}\lim_n x_n = l$ $(wb^*st_T\text{-}\lim_n x_n = l)$ if and only if the sequence (F_{x_n}) is $b^*\mathscr{I}$-convergent (b^*T-statistically convergent) to F_l. Thus, since $\|F_z\| = \|z\|$, by Theorems 4 and 5 we get the following characterizations of these new types of weak convergence.

Proposition 4. *Let* $\mathfrak{x} = (x_n) \in \omega(X)$ *and* $l \in X$*. Assume that* X' *has a countable fundamental set* Φ'.

(i) If $\mathscr{I}$ *is an ideal with the property* (AP)*, then* $wb^*\mathscr{I}\text{-}\lim_n x_n = l$ *if and only if*

$$\|x_n\| = O_{\mathscr{I}}(1), \tag{15}$$

$$\mathscr{I}\text{-}\lim_n \phi'(x_n) = \phi'(l) \quad (\phi' \in \Phi'). \tag{16}$$

(ii) If T *is a regular matrix, then* $wb^*st_T\text{-}\lim_n x_n = l$ *if and only if* (15) *and* (16) *are satisfied with* st_T *instead of* $\mathscr{I}$.

Finally we apply Proposition 4 to Banach sequence spaces $c_0(X)$ and $\ell_p(X)$ with $1 < p < \infty$. It is known that the dual spaces $c_0(X)'$ and $\ell_p(X)'$ are isometrically isomorphic, respectively, to $\ell_1(X')$ and $\ell_q(X')$, where $1/p + 1/q = 1$ (see, for example, [18]). If Φ' is a fundamental set of X', then $\mathscr{E}_0(\Phi')$ is the fundamental set of $\ell_1(X')$ and $\ell_q(X')$. Thus from Proposition 4 we get the following two corollaries.

Corollary 2. *Let* $\mathfrak{x}_n = (x_{ni})$ $(n \in \mathbb{N})$ *and* $\mathfrak{x}_0 = (x_i)$ *be the elements of* $c_0(X)$*. Assume that the dual* X' *has a countable fundamental set* Φ'.

(i) *If $\mathscr{I}$ is an ideal with the property* (AP)*, then* $wb^*\mathscr{I}\text{-}\lim_n \mathfrak{x}_n = \mathfrak{x}_0$ *if and only if*

$$\|\mathfrak{x}_n\|_\infty = O_{\mathscr{I}}(1), \tag{17}$$

$$\mathscr{I}\text{-}\lim_i \phi'(x_{ni}) = \phi'(x_i) \quad (\phi' \in \Phi',\ n \in \mathbb{N}). \tag{18}$$

(ii) *If T is a non-negative regular matrix, then* $wb^*st_T\text{-}\lim_n \mathfrak{x}_n = \mathfrak{x}_0$ *if and only* (17) *and* (18) *hold with st_T instead of $\mathscr{I}$.*

Corollary 3. *Let $\mathfrak{x}_n = (x_{ni})$ $(n \in \mathbb{N})$ and $\mathfrak{x}_0 = (x_i)$ be the elements of $\ell_p(X)$ $(1 < p < \infty)$. Assume that X' has a countable fundamental set Φ'.*

(i) *If $\mathscr{I}$ is an ideal with the property* (AP)*, then* $wb^*\mathscr{I}\text{-}\lim_n \mathfrak{x}_n = \mathfrak{x}_0$ *if and only if* (18) *is true and*

$$\|\mathfrak{x}_n\|_p = O_{\mathscr{I}}(1). \tag{19}$$

(ii) *If T is a non-negative regular matrix, then* $wb^*st_T\text{-}\lim_n \mathfrak{x}_n = \mathfrak{x}_0$ *if and only* (18) *and* (19) *hold with st_T instead of $\mathscr{I}$.*

Proposition 4(ii) and Corollary 3(ii), for $T = C_1$, may be considered as some corrected versions, respectively, of Theorem 3.1 and Lemma 3.2 from [2].

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