EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 1, 2014, 77-85
ISSN 1307-5543 – www.ejpam.com

The Relation $\mathscr{B}$ and Minimal bi–ideals in Γ–semigroups

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Abstract. In this paper we introduce the relation $\mathscr{B}$ "to generate the same principal bi-ideal" in Γ–semigroups. One of the main results that are proved here is the analogue of the Green's Theorem for Γ–semigroups, which we call the Green's Theorem for the relation $\mathscr{B}$ in Γ–semigroups. Applying our Green's Theorem for relation $\mathscr{B}$ in Γ–semigroups, we prove that any bi-ideal of a Γ–semigroup without zero is minimal if and only if it is a Γ–subgroup. Further, we prove that, if a Γ–semigroup M without zero has a cancellable element contained in a minimal bi-ideal B of M, then M is a Γ–group. Finally, we prove that, if for elements a, c of a Γ–semigroup without zero we have $a\mathscr{D}c$ and the principal bi-ideal $(a)_b$ and principal quasi-ideal $(a)_q$ are minimal, then $(a)_b = (a)_q$ and the principal bi-ideal $(c)_b$ and the principal quasi-ideal $(c)_q$ are minimal too, and $(c)_b = (c)_q$.

Key Words and Phrases: Γ–semigroup, Green's theorem, quasi–ideal, bi–ideal, Γ–group.

1. Introduction

The notion of Γ–semigroup is introduced by Sen in [8]. Let M and Γ be non–empty sets. Any map from $M \times \Gamma \times M$ to M will be called a Γ–multiplication in M and is denoted by $(\cdot)_\Gamma$. The result of this Γ–multiplication for $a, b \in M$ and $\gamma \in \Gamma$ is denoted by $a\gamma b$. According to Sen and Saha [9], a Γ–semigroup is an ordered pair $(M, (\cdot)_\Gamma)$, where M and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ–multiplication in M for which the following proposition:

$$\forall (a, b, c, \alpha, \beta) \in M^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c)$$

is true.

In the literature there are many examples of Γ–semigroups, but the following example, which is inspired from Hestenes's rings [3] is the most well known one.

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Example 1. *The* Γ*–semigroup* M *of all* $m \times n$ *matrices with entries from a field* F*, where* Γ *is the set of all* $n \times m$ *matrices, with entries from* F*. The result of* Γ*–multiplication in* M *for two* $m \times n$ *matrices* A, B *and an* $n \times m$ *matrix* C *is the usual product* ACB*.*

Note that every plain semigroup S can be considered as a Γ–semigroup by taking as Γ a singelton $\{1\}$, where 1 is the identity element of S, when S has a such element, or it is a symbol not representing an element of S, and the Γ–multiplication in S is defined by $a1b = ab$, where ab is the usual product in plain semigroup S.

Similarly to the definition of relations $\mathscr{R}_{plain}$, $\mathscr{L}_{plain}$, $\mathscr{H}_{plain}$ and $\mathscr{D}_{plain}$ in plain semigroup, Saha in [7] has introduced the analogue relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{D}$ in a Γ–semigroup M, which are called the Green's relations in the Γ–semigroup M. In this paper we define the relation $\mathscr{B}$ in a Γ–semigroup M such that $a\mathscr{B}c$ if and only if $(a)_b = (c)_b$, where $(a)_b$ and $(c)_b$ are the principal bi–ideals generated by elements a, c of M respectively. The definition of relation $\mathscr{B}$ in Γ-semigroups mimics the definition of relation $\mathscr{B}$ in plain semigroup introduced in [4]. We show that in Γ–semigroups the relation $\mathscr{B}$ is different from Green's relation $\mathscr{H}$. One of our main results claims that the analogue of Green's Theorem for Green's relation $\mathscr{H}_{plain} = \mathscr{R}_{plain} \cap \mathscr{L}_{plain}$ holds true for the relation $\mathscr{B}$ in Γ -semigroups. This theorem we call Green's Theorem for the relation $\mathscr{B}$ in Γ–semigroups. From this theorem, as a particular case, we get a Green's Theorem for the relation $\mathscr{B}$ in plain semigroups. Then we use our theorem for the relation $\mathscr{B}$ in Γ–semigroups to prove that any bi-ideal of a Γ–semigroup without zero is minimal if and only if it is a Γ–subgroup. As a corollary of the above result we get the analogue of the result for minimal quasi-ideal in Γ–semigroup [6], which states that "if a Γ–semigroup M without zero has a cancellable element contained in a minimal bi-ideal B of M, then M is a Γ–group". From this result we get the analogue of result for plain semigroups which state that "if a semigroup S without zero has a cancellable element contained in a minimal bi–ideal, then S is group".

At last, we show that if for the elements a, c of Γ–semigroup M without zero, we have $a\mathscr{D}c$ and principal bi-ideal $(a)_b$ and principal quasi–ideal $(a)_q$ are minimal, then $(a)_b = (a)_q$ and the principal bi-ideal $(c)_b$ and the principal quasi-ideal $(c)_q$ are minimal too, and $(c)_b = (c)_q$.

At the end of this paper we raise an open problem.

2. Preliminaries

We give some notions and present some auxiliary results that will be used throughout the paper.

Let M be a Γ–semigroup and A, B be subsets of M. We define the set

$$A\Gamma B = \{a\gamma b \in M | a \in A, b \in B \text{ and } \gamma \in \Gamma\}.$$

For simplicity we write $a\Gamma B$ instead of $\{a\}\Gamma B$, $A\Gamma b$ in place of $A\Gamma\{b\}$, and $a\Gamma b$ instead of $\{a\}\Gamma\{b\}$.

Analogously with the definitions in plain semigroups there are given the following definitions in Γ–semigroups.

Definition 1. *Let M be a Γ–semigroup. A non–empty subset M_1 of M is said to be a Γ–subsemigroup of M if $M_1 \Gamma M_1 \subseteq M_1$.*

Definition 2. *A right [left] ideal of a Γ–semigroup M is a non–empty subset R [L] of M such that $R\Gamma M \subseteq R$, [$M\Gamma L \subseteq L$].*

Definition 3. *A quasi–ideal of a Γ–semigroup M is a non–empty subset Q of M such that $Q\Gamma M \cap M\Gamma Q \subseteq Q$.*

Definition 4. *A bi–ideal of a Γ–semigroup M is a Γ–subsemigroup B of M such that $B\Gamma M\Gamma B \subseteq B$.*

Similarly to the plain semigroups, it is easy to prove the following two propositions:

Proposition 1. *Every quasi–ideal of a Γ–semigroup M is a bi–ideal of M.*

Proposition 2. *The intersection of any set of bi-ideals of a Γ–semigroup M is an empty set or is a bi-ideal of M.*

Theorem 1 ([1])**.** *Let A be a nonempty subset of a Γ–semigroup M. Then*

$$(A)_b = A \cup A\Gamma A \cup A\Gamma M\Gamma A,$$

where $(A)_b$ is the smallest bi–ideal of M containing A, i.e. the intersection of bi–ideals of M containing A.

Let M be a Γ–semigroup and $\gamma \in \Gamma$ is a fixed element. As in [9], we define the multiplication $\circ$ in M by $a \circ b = a\gamma b$. It is obvious that $\circ$ is associative, hence we obtain a semigroup $(M, \circ)$ which is shortly denoted by M_γ.

A zero of a Γ–semigroup M is an element 0 of M such that for all $a \in M$ and $\gamma \in \Gamma$ we have $a\gamma 0 = 0\gamma a = 0$.

Theorem 2 ([6])**.** *Let M be any Γ–semigroup without zero and $\gamma \in \Gamma$ a fixed element. Then S_γ is a group if and only if S has not proper quasi-ideals.*

From this theorem we give:

Theorem 3 (citesensaha)**.** *Let M be a Γ–semigroup without zero. If M_γ is a group for some $\gamma \in \Gamma$, then it is a group for all $\gamma \in \Gamma$.*

Definition 5 ([9])**.** *A Γ–semigroup M is called a Γ–group if M_γ is a group for some (hence for all) $\gamma \in \Gamma$.*

Let $\overline{M}$ be a Γ–subsemigroup of a Γ–semigroup M. In the set $\overline{M}$ we have a Γ–multiplication induced by the Γ–multiplication of Γ–semigroup M, $(\cdot)_\Gamma$, denoting it with the same symbol. It is clear that the ordered pair $(\overline{M}, (\cdot)_\Gamma)$ is a Γ–semigroup. From Theorem 3, if $\overline{M}_\gamma$ is a group for some $\gamma \in \Gamma$, then it is group for all $\gamma \in \Gamma$ and so it is a Γ–group. In this case, we will call $\overline{M}$ a Γ–subgroup of the Γ–semigroup M.

Saha has defined in [7] the Green's relations $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ in a Γ–semigroup M as follows:

$$\forall(a,b)\in M^2,\ a\mathscr{R}b \Leftrightarrow (a)_r=(b)_r,$$
$$\forall(a,b)\in M^2,\ a\mathscr{L}b \Leftrightarrow (a)_l=(b)_l,$$
$$\forall(a,b)\in M^2,\ a\mathscr{H}b \Leftrightarrow (a)_r=(b)_r \text{ and } (a)_l=(b)_l,$$

where $(a)_r = a\cup a\Gamma M$, $(b)_r = b\cup b\Gamma M$, $(a)_l = a\cup M\Gamma a$, $(b)_l = b\cup M\Gamma b$, are respectively the principal right ideal generated by a, the principal right ideal generated by b, the principal left ideal generated by a, and the principal left ideal generated by b in Γ–semigroup M.

It turns out that $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ are equivalent relations. The respective equivalence classes of $a\in M$ are denoted by R_a, L_a, H_a.

Proposition 3 ([7])**.** *Let M be a Γ–semigroup. Then we have:*

(i) For every three elements a,b,c of M and for every $\gamma\in\Gamma$

$$a\mathscr{R}b \Rightarrow c\gamma a\mathscr{R}c\gamma b \text{ and } a\mathscr{L}b \Rightarrow a\gamma c\mathscr{L}a\gamma b.$$

(ii) For every two elements $a,b\in M$, $a\mathscr{R}b$ if and only if either $a=b$ or there exist $\alpha,\beta\in\Gamma$ and $c,d\in M$ such that $a=b\alpha c$ and $b=\alpha\beta d$.

(iii) For every two elements $a,b\in M$, $a\mathscr{L}b$ if and only if either $a=b$ or there exist $\alpha,\beta\in\Gamma$ and $c,d\in M$ such that $a=c\alpha b$ and $b=d\beta\alpha$.

(iv) $\mathscr{R}$ and $\mathscr{L}$ commute, that is $\mathscr{R}\circ\mathscr{L}=\mathscr{L}\circ\mathscr{R}$. So, one can define a fourth Green's relation in M which is

$$\mathscr{D}=\mathscr{R}\circ\mathscr{L}=\mathscr{L}\circ\mathscr{R}.$$

The equivalence class of $a\in M$ from $\mathscr{D}$ is denoted by D_a.

In [6], it is defined the relation $\mathscr{Q}$ in a Γ–semigroup M as follows

$$\forall(a,b)\in M^2, a\mathscr{Q}b \Leftrightarrow (a)_q=(b)_q,$$

where $(a)_q = a\cup(a\Gamma M\cap M\Gamma a)$, $(b)_q = b\cup(b\Gamma M\cap M\Gamma b)$, are the principal quasi-ideal generated by a and the principal quasi-ideal generated by b in Γ –semigroup M, respectively.

The relation $\mathscr{Q}$ is an equivalence relation and moreover we have

Proposition 4 ([6])**.** *The relations $\mathscr{H}$ and $\mathscr{Q}$ coincide in every Γ–semigroup M.*

For the relation $\mathscr{H}$, consequently for relation $\mathscr{Q}$, has an analogue of Green's Theorem for plain semigroups, which is called Green's Theorem for Γ–semigroups.

Theorem 4 ([6] Green's Theorem for Γ–semigroups)**.** *If the elements $a,b,a\gamma b$ of a Γ–semigroup M all belong to the same $\mathscr{H}$–class H of M, then H is a subgroup of the semigroup M_γ. Moreover, for any two element $h_1,h_2\in H$, the element $h_1\gamma h_2$ belongs to H.*

Theorem 5 ([6]). *A quasi–ideal Q of a Γ–semigroup S without zero is minimal if and only if Q is an $\mathscr{H}$–class.*

Theorem 6 ([6]). *A quasi–ideal Q of a Γ–semigroup S without zero is minimal if and only if Q is a Γ–subgroup of S.*

Theorem 7 ([6]). *Let a,b be two elements of a Γ–semigroup S without zero such that $a\mathscr{D}b$. Then the principal quasi–ideal $(a)_q$ is minimal if and only if the same holds for $(b)_q$.*

3. Main Results

For every element a of a Γ–semigroup M we denote by $(a)_b$ the intersection of all bi-ideals of M that contain a. This bi–ideal, that is, the smallest bi–ideal of M containing a, is called principal bi-ideal of M generated by a. From the Theorem 1 we have

$$(a)_b = a \cup a\Gamma a \cup a\Gamma M\Gamma a.$$

Now, similarly with the definition of the relation $\mathscr{B}$ in plain semigroups [4], we define the relation $\mathscr{B}$ in Γ–semigroup M by

$$\forall (a,c) \in M^2, a\mathscr{B}c \Leftrightarrow (a)_b = (c)_b.$$

So, for every two elements a,c of Γ–semigroups we have

$$a\mathscr{B}c \Leftrightarrow a \cup a\Gamma a \cup a\Gamma M\Gamma a = c \cup c\Gamma c \cup c\Gamma M\Gamma c.$$

Clearly $\mathscr{B}$ is an equivalence relation on M. The equivalence class of M mod $\mathscr{B}$ containing the element $a \in M$ is denoted by B_a.

From Proposition 3, it is clear that $\mathscr{B} \subseteq \mathscr{H}$. The following example shows that the inclusion may be strict.

Example 2. *Consider the set of integers modulo 8,*

$$\mathbb{Z}/8\mathbb{Z} = \{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6},\overline{7}\},$$

and $\Gamma = \{0,1,2\} \subseteq \mathbb{N} \cup \{0\}$.

The result of Γ–multiplication in $M = \mathbb{Z}/8\mathbb{Z}$ for two any elements $\overline{a},\overline{b}$ of M and every element $\gamma \in \Gamma$ is the usual product $\overline{a}\,\overline{\gamma}\,\overline{b}$ of integers modulo 8, $\overline{a},\overline{\gamma},\overline{b}$.

It is clear that $(M = \mathbb{Z}/8\mathbb{Z}, (\cdot)_\Gamma)$ is a Γ–semigroup.

The elements $\overline{2}$ and $\overline{6}$ are $\mathscr{L}$ equivalent since:

$$(\overline{2})_l = \overline{2} \cup \mathbb{Z}/8\mathbb{Z}\Gamma\overline{2} = \{\overline{0},\overline{2},\overline{4},\overline{6}\},$$
$$(\overline{6})_l = \overline{6} \cup \mathbb{Z}/8\mathbb{Z}\Gamma\overline{6} = \{\overline{0},\overline{2},\overline{4},\overline{6}\}.$$

In the Γ–semigroup $(M = \mathbb{Z}/8\mathbb{Z}, (\cdot)_\Gamma)$ the Green's relations $\mathscr{L}$ and $\mathscr{R}$ coincide and so we have $\overline{2}\mathscr{H}\overline{6}$.

The elements $\overline{2}, \overline{6}$ *of* Γ*–semigroup* $M = \mathbb{Z}/8\mathbb{Z}$ *are not* $\mathscr{B}$*–equivalent since:*

$$(\overline{2})_b = \overline{2} \cup \overline{2}\Gamma\overline{2} \cup \overline{2}\Gamma M\Gamma\overline{2} = \{\overline{0}, \overline{2}, \overline{4}\},$$
$$(\overline{6})_b = \overline{6} \cup \overline{6}\Gamma\overline{6} \cup \overline{6}\Gamma M\Gamma\overline{6} = \{\overline{0}, \overline{4}, \overline{6}\},$$

and so $\mathscr{H} \neq \mathscr{B}$.

For the equivalence relation $\mathscr{B}$ in Γ–semigroup it is true the following theorem, which resembles the Green's Theorem for plain semigroups [10], the Green's Theorem for rings [5], the Green's Theorem for semirings [2], and Green's Theorem for Γ–semigroup [6]. We will call this theorem the Green's Theorem for the relation $\mathscr{B}$ in Γ–semigroups.

Theorem 8. *If the elements* $a, b, a\gamma b$ *of a* Γ*–semigroup* $(M, (\cdot)_\Gamma)$ *all belong to the same* $\mathscr{B}$*-class* B*, then* B *is a* Γ*–subgroup of semigroup* M_γ.

Proof. Since the relation $\mathscr{B}$ is included in the relation $\mathscr{H}$, we have

$$B = B_a \subseteq H_a.$$

Thus the elements $a, b, a\gamma b$ belong to the $\mathscr{H}$–class H_a of Γ –semigroup M. So, by Theorem 3, H_a is a subgroup of semigroup M_γ and therefore there exists the identity e of subgroup H_a and the following equalities are true:

$$a = e\gamma a\gamma e, \quad e = a\gamma a^{-1}\gamma a^{-1}\gamma a,$$

where a^{-1} is the inverse element of a in the subgroup H_a of semigroup M_γ. These equalities show that the principal bi–ideals generated by elements a and e are the same. Thus, the element e belongs to the class $B_a = B$. Now, let x be any element of subgroup H_a. We have the following equalities:

$$x = e\gamma x\gamma e, \quad e = x\gamma x^{-1}\gamma x^{-1}\gamma x,$$

where x^{-1} is the inverse element of the element x of subgroup H_a of semigroup M_γ. These equalities show that $(x)_b = (e)_b$. So, since $e \in B_a$, the element x belongs to B_a. Thus, we have

$$B = B_a = H_a,$$

and consequently $\mathscr{B}$-class B is a subgroup of Γ–semigroup M_γ. □

A element e of a Γ-semigroup M is called *idempotent* if there exists $\gamma \in \Gamma$ such that $e = e\gamma e$.
From the Theorem 8 we get immediately the following:

Corollary 1. *If a* $\mathscr{B}$*–class* B *of a* Γ*–semigroup* M *contains an idempotent* $e = e\gamma e, \gamma \in \Gamma$*, then* B *is a subgroup of semigroup* M_γ.

Since, every plain semigroup S can be considered as a Γ–semigroup, from the Theorem 8 and the Corollary 1, we get the following theorem and corollary to plain semigroups:

Theorem 9. *If the elements* a, b, ab *of a semigroup* S *all belong to the same* $\mathscr{B}$*–class* B*, then* B *is a subgroup of semigroup* S.

Corollary 2. *If an $\mathscr{B}$–class B of a semigroup S contains an idempotent e, then B is a subgroup of semigroup S.*

We can call the Theorem 9 *the Green's Theorem for the relation $\mathscr{B}$* in plain semigroups.

Proposition 5. *Let M be an arbitrary Γ–semigroup. If the idempotent $e = e\gamma e, \gamma \in \Gamma$ together with $a, b \in M$ all belong to the same $\mathscr{B}$-class B, then $e\gamma a = a\gamma e = a$ and $a\gamma b \in B$.*

Proof. Since B contains an idempotent $e = e\gamma e, \gamma \in \Gamma$, then the Corollary 1 implies that B is a subgroup of M_γ. The identity of B is e because $e \circ e = e\gamma e = e$. Since $a \circ e = e \circ a = e$, we have $a\gamma e = a = a\gamma e$. The element a, b belongs to B, therefore $a\gamma b = a \circ b \in B$. □

A bi–ideal B of a Γ–semigroup M without zero is called *minimal* if B does not properly contain any bi–ideal of M.

One can prove easily that:

Lemma 1. *A bi–ideal B of a Γ–semigroup M without zero is minimal if and only if B is an $\mathscr{B}$–class.*

Now we will use the Green's Theorem for the relation $\mathscr{B}$ in Γ–semigroup (Theorem 8) to prove a theorem concerning minimal bi–ideals in Γ–semigroup M without zero.

Theorem 10. *A bi–ideal B of a Γ–semigroup M without zero is minimal if and only if B is a Γ–subgroup of M.*

Proof. If B is a minimal bi–ideal of the Γ–semigroup M, then by Lemma 1 all elements of B are $\mathscr{B}$–equivalent. Thus for two elements a, b of B and every $\gamma \in \Gamma$ the elements $a, b, a\gamma b$ all belong to the same $\mathscr{B}$–class of M. Now applying the Green's Theorem for the relation $\mathscr{B}$ in Γ–semigroup (Theorem 8) the $\mathscr{B}$–class B is a subgroup of semigroup M_γ for every $\gamma \in \Gamma$. So, B is a Γ–subsemigroup such that for every $\gamma \in \Gamma$, $B_\gamma = (B, \circ)$ is a group. Thus B is a Γ–subgroup of Γ–semigroup M and it is a $\mathscr{H}$–class.

Conversely, let the bi–ideal B be a Γ–subgroup of M. If B' is a bi–ideal of M contain in B, then

$$B'\Gamma B\Gamma B' \subseteq B'\Gamma M\Gamma B' \subseteq B',$$

that is, B' is a bi–ideal of B, too. Let a be an element of B' and γ an element of Γ. Then, since the semigroup $B_\gamma = (B, \circ)$ is a group, we have

$$B = a \circ B \circ a = a\gamma B\gamma a \subseteq B',$$

whence $B = B'$. This means that B is a minimal bi–ideal of Γ–semigroup M. □

As a particular case we get the following theorem for plain semigroups:

Theorem 11. *A bi–ideal B of a plain semigroup S without zero is minimal if and only if B is a subgroup of S.*

This theorem is proved in [4] by a direct method.

Definition 6. *An element a of a Γ–semigroup M is called cancellable if for two elements $b,c \in M$ and every $\gamma \in \Gamma$ we have*

$$(a\gamma b = a\gamma c \Rightarrow b = c) \wedge (b\gamma a = c\gamma a \Rightarrow b = c).$$

Theorem 12. *If a Γ–semigroup M without zero has a cancellable element contained in a minimal bi–ideal B of M, then M is a Γ–group.*

Proof. By the Theorem 10, the minimal bi–ideal B is a Γ–subgroup of M. Let e be the identity of the group $B_\gamma = (B, \circ)$ for a fixed $\gamma \in \Gamma$ and let a be a cancellable element of M contained in B. Then multiplying both side of the equality $e\gamma a = a$ by any element b of M, we have $b\gamma e\gamma a = b\gamma a$, hence $b\gamma e = b$. Dually we obtain $e\gamma b = b$ for every $b \in M$. Thus e is the identity element of the semigroup $M_\gamma = (M, \circ)$. Since $e \in B$, for any $b \in M$ we have

$$b = e\gamma b\gamma e \in B.$$

So, $M = B$ and consequently M is a Γ–group with zero. □

Since every plain groups is a minimal bi–ideal and has a cancellable element (this is the identity element of the group), therefore from the Theorem 12, we get the following:

Theorem 13. *A semigroup S without zero is a group if and only if it has a cancellable element contained in a minimal bi–ideal B of S.*

Theorem 14. *Let a,c are two elements of a Γ–semigroup without zero such that $a\mathcal{D}c$. If the principal bi–ideal $(a)_b$ and the principal quasi–ideal $(a)_q$ are minimal, then $(a)_b = (a)_q$ and the principal bi–ideal $(c)_b$ and the principal quasi–ideal $(c)_q$ are minimal and $(c)_b = (c)_q$.*

Proof. Assume that bi–ideal $(a)_b$ and quasi–ideal $(a)_q$ are minimal. Firstly we prove that $(a)_b = (a)_q$. It is clear the inclusion $(a)_b \subseteq (a)_q$. Since $(a)_q$ is a minimal quasi–ideal, then Theorem 5 implies that $(a)_q = H_a$. So, we have

$$B_a = (a)_b \subseteq (a)_q \subseteq H_a.$$

By Theorem 4, H_a is a Γ–subgroup, therefore $H_a \subseteq B_a$ and consequently $(a)_b = (a)_q$.

Since the principal quasi–ideal $(a)_q$ is minimal, then the Theorem 7 implies that the principal quasi–ideal $(c)_q$ is minimal. Now from the Theorem 5 we have

$$B_c \subseteq (c)_b \subseteq (c)_q = H_c.$$

By the Theorem 4, H_c is a Γ–group, therefore $H_c \subseteq B_c$ and consequently there are true the equalities

$$B_c = (c)_b = (c)_q = H_c.$$

So, $(c)_b = (c)_q$. □

At the end of this paper, we raise the following open problem:

Problem. If the element a,c of a Γ–semigroup M without zero are such that $a\mathcal{D}c$, then is it the principal bi–ideal $(a)_b$ minimal if and only if the same holds for $(c)_b$?

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