



Osculating Spheres of a Semi Real Quaternionic Curve in \mathbb{E}_2^4

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Abstract. In this study, we define the osculating spheres of a semi real quaternionic curve in semi-Euclidean spaces \mathbb{E}_1^3 and \mathbb{E}_2^4 . We give the equation of the osculating spheres with respect to Frenet frames $\{\mathbf{t}_0, \mathbf{n}_{1_0}, \mathbf{n}_{2_0}\}$ and $\{\mathbf{T}_0, \mathbf{N}_{1_0}, \mathbf{N}_{2_0}, \mathbf{N}_{3_0}\}$.

2010 Mathematics Subject Classifications: 14Q05, 53A17, 53C50.

Key Words and Phrases: Osculating sphere, semi real quaternionic curve, semi-Euclidean space.

1. Introduction

Quaternion algebra was introduced by Hamilton in 1843. Important precursors to this work included Euler's four square identity and Olinde Rodrigues parametrization of general rotations by four parameters, but neither of these writers treated the four parameter rotations as an algebra. Carl Friedrich Gauss had also discovered quaternions in 1819, but this work was not published until 1900. Hamilton knew that the complex numbers could be interpreted as points in a plane, and he was looking for a way to do the same for points in three-dimensional space. Points in space can be represented by their coordinates, which are triples of numbers, and for many years Hamilton had known how to add and subtract triples of numbers. However, Hamilton had been stuck on the problem of multiplication and division for a long time. He could not figure out how to calculate the quotient of the coordinates of two points in space.

The great breakthrough in quaternions finally came on Monday 16 October 1843 in Dublin, when Hamilton was on his way to the Royal Irish Academy where he was going to preside at a council meeting. While walking along the towpath of the Royal Canal with

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his wife, the concepts behind quaternions were taking shape in his mind. When the answer dawned on him, Hamilton could not resist the urge to carve the formula for the quaternions,

$$e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1$$

into the stone of Brougham Bridge as he paused on it.

Hamilton called a quadruple with these rules of multiplication a quaternion, and he devoted most of the remainder of his life to studying and teaching them. Hamilton's treatment is more geometric than the modern approach, which emphasizes quaternions' algebraic properties. He founded a school of "quaternionists", and he tried to popularize quaternions in several books. The last and longest of his books, *Elements of Quaternions*, was 800 pages long; it was published shortly after his death. After Hamilton's death, his student Peter Tait continued promoting quaternions. At this time, quaternions were a mandatory examination topic in Dublin. Topics in physics and geometry that would now be described using vectors, such as kinematics in space and Maxwell's equations, were described entirely in terms of quaternions. There was even a professional research association, the Quaternion Society, devoted to the study of quaternions and other hypercomplex number systems.

From the mid-1880s, quaternions began to be displaced by vector analysis, which had been developed by Josiah Willard Gibbs, Oliver Heaviside, and Hermann von Helmholtz. Vector analysis described the same phenomena as quaternions, so it borrowed some ideas and terminology liberally from the literature of quaternions. However, vector analysis was conceptually simpler and notationally cleaner, and eventually quaternions were relegated to a minor role in mathematics and physics. A side effect of this transition is that Hamilton's work is difficult to comprehend for many modern readers. Hamilton's original definitions are unfamiliar and his writing style was wordy and difficult to understand. However, quaternions have had a revival since the late 20th Century, primarily due to their utility in describing spatial rotations. The representations of rotations by quaternions are more compact and quicker to compute than the representations by matrices. In addition, unlike Euler angles they are not susceptible to gimbal lock. For this reason, quaternions are used in computer graphics, computer vision, robotics, control theory, signal processing, attitude control, physics, bioinformatics, molecular dynamics, computer simulations, and orbital mechanics. For example, it is common for the attitude-control systems of spacecraft to be commanded in terms of quaternions. Quaternions have received another boost from number theory because of their relationships with the quadratic forms [1].

The set of quaternions is denoted by \mathcal{H} . While the quaternions are not commutative, they are associative, and they form a group known as the quaternion group. By analogy with the complex numbers being representable as a sum of real and imaginary parts, $a_01 + a_1e_1$, a quaternion can also be written as a linear combination

$$a_01 + a_1e_1 + a_2e_2 + a_3e_3.$$

As a set, the quaternions \mathcal{H} are equal to \mathbb{E}^4 , a four dimensional vector space over the real numbers. \mathcal{H} has three operations: addition, scalar multiplication and quaternion multiplication. The sum of two elements of \mathcal{H} is defined to be their sum as elements of \mathbb{E}^4 . Similarly

the product of an element of \mathcal{H} by a real number is defined to be same as the product in \mathbb{E}^4 . To define the product of two elements in \mathcal{H} requires a choice of basis for \mathbb{E}^4 . The elements of this basis are customarily denoted as $1, i, j$, and k . Every element of \mathcal{H} can be uniquely written as a linear combination of these basis elements, that is, as $a_0 1 + a_1 e_1 + a_2 e_2 + a_3 e_3$, where a_0, a_1, a_2 , and a_3 are real numbers. The basis element 1 will be the identity element of \mathcal{H} , meaning that multiplication by 1 does nothing, and for this reason, elements of \mathcal{H} are usually written $a_0 1 + a_1 e_1 + a_2 e_2 + a_3 e_3$, suppressing the basis element 1 . Given this basis, associative quaternion multiplication is defined by first defining the products of basis elements and then defining all other products using the distributive law, [2–4].

In [5], Baharathi and Nagaraj studied the differential geometry of a smooth quaternionic curve in \mathbb{E}^3 and \mathbb{E}^4 . Elements of \mathbb{E}^4 were identified with quaternions in a natural way. The Serret-Frenet formulae for a quaternionic curves in \mathbb{E}^3 and \mathbb{E}^4 were given by them. Then Serret Frenet formulae for quaternionic curves in semi-Euclidean Space are given by [6]. By using of these formulas, new definitions and new characterizations of quaternionic curves are studied in [2, 3].

In Euclidean space \mathbb{E}^3 , there is a unique sphere for a curve α which contacts α at the third order at $\alpha(0)$. The intersection of the sphere with the osculating plane is a circle which contacts at the second order at $\alpha(0)$, [7–9]. This concept studied in [10] in terms of quaternionic curves in \mathbb{E}^4 . In [11], the osculating sphere and the osculating circle of the curve are investigated in semi-Euclidean spaces $\mathbb{E}_1^3, \mathbb{E}_1^4$, and \mathbb{E}_2^4 .

In this study, we give definition of the osculating spheres for semi quaternionic curve in semi-Euclidean spaces \mathbb{E}_1^3 and \mathbb{E}_2^4 with respect to Frenet frames $\{t_0, n_{1_0}, n_{2_0}\}$ and $\{T_0, N_{1_0}, N_{2_0}, N_{3_0}\}$, respectively.

2. Preliminaries

A semi real quaternion is defined by $q = ae_1 + be_2 + ce_3 + de_4$ such that

$$\begin{aligned} e_i \times e_i &= -\varepsilon_{e_i}, \quad 1 \leq i \leq 3 \\ e_i \times e_j &= \varepsilon_{e_i} \varepsilon_{e_j} e_k, \quad \text{in } \mathbb{E}_1^3 \\ e_i \times e_j &= -\varepsilon_{e_i} \varepsilon_{e_j} e_k, \quad \text{in } \mathbb{E}_2^4, \end{aligned}$$

where (ijk) is an even permutation of (123) . Notice here that we denote the set of all spatial semi real quaternions by \mathcal{H}_p and all semi real quaternions by \mathcal{H} . It is defined by

$$\mathcal{H} = \{q = ae_1 + be_2 + ce_3 + de_4 : a, b, c, d \in \mathbb{R}, e_1, e_2, e_3 \text{ in } \mathbb{E}_1^3, \langle e_i, e_i \rangle = \varepsilon_{e_i}, 1 \leq i \leq 3\}$$

where index = 1, 2. If e_i is a spacelike or timelike vector, then $\varepsilon_{e_i} = +1$ or -1 , respectively. For $p = S_p + \mathbf{V}_p$ and $q = S_q + \mathbf{V}_q$, the multiplication of two semi real quaternions p and q is defined as follows:

$$p \times q = S_p S_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p \mathbf{V}_q + S_q \mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q, \quad \forall p, q \in \mathcal{H},$$

where we have used the scalar and cross products in \mathbb{E}_1^3 . Let \bar{q} denotes the conjugate of a quaternion, $\bar{q} = -ae_1 - be_2 - ce_3 + d$ for every $q \in \mathcal{H}$. This helps to define the symmetric, non-degenerate, bilinear form as follows:

$$\begin{aligned} \langle , \rangle : \mathcal{H} \times \mathcal{H} &\rightarrow \mathbb{R}, \\ (p, q) \rightarrow \langle p, q \rangle_p &= \frac{1}{2} \left[\varepsilon_p \varepsilon_{\bar{q}} (p \times \bar{q}) + \varepsilon_q \varepsilon_{\bar{p}} (q \times \bar{p}) \right] \text{ for } \mathbb{E}_1^3 \\ (p, q) \rightarrow \langle p, q \rangle &= -\frac{1}{2} \left[\varepsilon_p \varepsilon_{\bar{q}} (p \times \bar{q}) + \varepsilon_q \varepsilon_{\bar{p}} (q \times \bar{p}) \right] \text{ for } \mathbb{E}_2^4. \end{aligned}$$

The norm of semi real quaternion q is denoted by

$$\|q\| = |\langle q, q \rangle| = |\bar{p} (q \times \bar{q})| = |-a^2 - b^2 + c^2 + d^2|$$

for $p, q \in \mathcal{H}$. If $\langle p, q \rangle = 0$, then p and q are called orthogonal. q is called a spatial semi quaternion whenever $q + \bar{q} = 0$ [6]. The Serret Frenet formulae for semi-real quaternionic curves in \mathbb{E}_1^3 and \mathbb{E}_2^4 are as follows:

The three-dimensional semi-Euclidean space \mathbb{E}_1^3 is identified with the space of spatial quaternions $\mathcal{H}_p = \{\gamma \in \mathcal{H} \mid \gamma + \bar{\gamma} = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in the real line \mathbb{R} and

$$\begin{aligned} \gamma : I \subset \mathbb{R} &\rightarrow \mathcal{H}_p \\ s \rightarrow \gamma(s) &= \sum_{i=1}^3 \gamma_i(s) e_i, \quad (1 \leq i \leq 3) \end{aligned}$$

be an arc length curve with nonzero curvatures $\{k, r\}$. Let $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2\}$ denote the Frenet frame of the curve γ . Then Frenet formulae are given by

$$\begin{aligned} \mathbf{t}'(s) &= \varepsilon_{\mathbf{n}_1} k \mathbf{n}_1(s) \\ \mathbf{n}_1'(s) &= -\varepsilon_{\mathbf{t}} k \mathbf{t}(s) + \varepsilon_{\mathbf{n}_1} r \mathbf{n}_2(s), \\ \mathbf{n}_2'(s) &= -\varepsilon_{\mathbf{n}_2} r \mathbf{n}_1(s) \end{aligned} \tag{1}$$

where $\langle \mathbf{t}, \mathbf{t} \rangle_p = \varepsilon_{\mathbf{t}}$, $\langle \mathbf{n}_1, \mathbf{n}_1 \rangle_p = \varepsilon_{\mathbf{n}_1}$, $\langle \mathbf{n}_2, \mathbf{n}_2 \rangle_p = \varepsilon_{\mathbf{n}_2}$ [6].

The four dimensional semi-Euclidean space \mathbb{E}_2^4 is identified with the space of unit quaternions \mathcal{H} . Let $I = [0, 1]$ be an unit interval in the real line \mathbb{R} and

$$\begin{aligned} \beta : I \subset \mathbb{R} &\rightarrow \mathcal{H} \\ s \rightarrow \beta(s) &= \sum_{i=1}^4 \gamma_i(s) e_i, e_4 = +1 \end{aligned}$$

be a smooth curve in \mathbb{E}_2^4 with nonzero curvatures $\{K, k, r - K\}$. Let $\{\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3\}$ be Frenet Frame of β . Then Frenet formulae are given by

$$\begin{aligned} \mathbf{T}'(s) &= \varepsilon_{\mathbf{N}_1} K \mathbf{N}_1(s) \\ \mathbf{N}'_1(s) &= -\varepsilon_{\mathbf{N}_1} \varepsilon_t K \mathbf{T}(s) + \varepsilon_{\mathbf{n}_1} k \mathbf{N}_2(s) \\ \mathbf{N}'_2(s) &= -\varepsilon_t k \mathbf{N}_1(s) + \varepsilon_{\mathbf{n}_1} (r - K \varepsilon_{\mathbf{T}} \varepsilon_t \varepsilon_{\mathbf{N}_1}) \mathbf{N}_3(s) \\ \mathbf{N}'_3(s) &= -\varepsilon_{\mathbf{n}_2} (r - K \varepsilon_{\mathbf{T}} \varepsilon_t \varepsilon_{\mathbf{N}_1}) \mathbf{N}_2(s), \end{aligned} \tag{2}$$

where $\langle \mathbf{T}, \mathbf{T} \rangle = \varepsilon_{\mathbf{T}}$, $\langle \mathbf{N}_1, \mathbf{N}_1 \rangle = \varepsilon_{\mathbf{N}_1}$, $\langle \mathbf{N}_2, \mathbf{N}_2 \rangle = \varepsilon_{\mathbf{N}_2}$ and $K = \varepsilon_{\mathbf{N}_1} \left\| \mathbf{T}'(s) \right\|$, [6].

3. Spatial Semi Quaternionic Osculating Spheres of a Spatial Semi Quaternionic Curve in \mathbb{E}_1^3

Definition 1. Let

$$\begin{aligned} \gamma : I \subset \mathbb{R} &\rightarrow \mathcal{H}_p \\ s &\rightarrow \gamma(s) = \sum_{i=1}^3 \gamma_i(s) e_i, \end{aligned}$$

be a spatial semi quaternionic curve in \mathbb{E}_1^3 identified with \mathcal{H}_p in an obvious manner. Let $I = [0, 1]$ be an interval in the real line \mathbb{R} and s be an arc-length parameter. So we say that

$\left\| \gamma'(s) \right\|_p = \|\mathbf{t}(s)\|_p = 1$. We assume that $l = (l_1, l_2, l_3)$ be a rectangular coordinate system of \mathbb{E}_1^3 . We take a sphere $\langle l - m, l - m \rangle_p = r^2$ with origin m and radius r . We define a function $f(s) = \langle \gamma(s) - m, \gamma(s) - m \rangle_p - r^2$ holds the following equations

$$f(0) = f'(0) = f''(0) = f'''(0) = 0, f^{(4)} \neq 0.$$

Then we called that the sphere contacts at third order to the curve γ at $\gamma(0)$. The sphere is called spatial semi real quaternionic osculating sphere for spatial semi quaternionic curves in \mathbb{E}_1^3 .

Theorem 1. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathcal{H}_p$ be a spatial semi quaternionic curve with nonzero curvatures $k(0)$ and $r(0)$ at $\gamma(0)$. Then there exists a sphere which contacts at the third order to the curve γ at $\gamma(0)$ and the equation of the spatial semi quaternionic osculating sphere according to the Frenet frame $\{\mathbf{t}_0, \mathbf{n}_{1_0}, \mathbf{n}_{2_0}\}$ as follows:

$$\varepsilon_{\mathbf{t}_0} x_1^2 + \varepsilon_{\mathbf{n}_{1_0}} (x_2 - \varepsilon_{\mathbf{t}_0} \rho_0)^2 + \varepsilon_{\mathbf{n}_{2_0}} \left(x_3 - \frac{\varepsilon_{\mathbf{t}_0}}{\varepsilon_{\mathbf{n}_{2_0}}} \rho'_0 \sigma_0 \right)^2 = \varepsilon_{\mathbf{n}_{1_0}} \rho_0^2 + \varepsilon_{\mathbf{n}_{2_0}} (\rho'_0 \sigma_0)^2$$

where $\rho_0 = \frac{1}{k_0}$ and $\sigma_0 = \frac{1}{r_0}$

Proof. If $f(0) = 0$ then $\langle \gamma(0) - m, \gamma(0) - m \rangle_p = r^2$. By differentiating this equation, we get $f'(0) = 0$ and $f' = 2 \langle \gamma', \gamma - m \rangle_p = 0$. Because of this, we can write

$$\langle \mathbf{t}_0, \gamma(0) - m \rangle_p = 0. \tag{3}$$

In a similar way we can get

$$f'' = 2 \left[\langle \gamma'', \gamma - m \rangle_p + \langle \gamma', \gamma' \rangle_p \right] = 0 \text{ and } f''(0) = 0.$$

If we use the equation (1) we have

$$\langle \varepsilon_{\mathbf{n}_{1_0}} k_0 \mathbf{n}_{1_0}(s), \gamma(0) - m \rangle_p + \langle \mathbf{t}_0, \mathbf{t}_0 \rangle_p = 0$$

and

$$\langle \mathbf{n}_{1_0}(s), \gamma(0) - m \rangle_p = -\frac{\varepsilon_{\mathbf{t}_0}}{\varepsilon_{\mathbf{n}_{1_0}} k_0} = -\frac{\varepsilon_{\mathbf{t}_0}}{\varepsilon_{\mathbf{n}_{1_0}}} \rho_0 \tag{4}$$

where $\frac{1}{k_0} = \rho_0$. By considering

$$f''' = 2 \left[\langle \gamma''', \gamma - m \rangle_p + 3 \langle \gamma'', \gamma' \rangle_p \right] = 0 \text{ and } f'''(0) = 0$$

and using the equations (3) and (4), we have

$$\langle \mathbf{n}_{2_0}(s), \gamma(0) - m \rangle_p = -\varepsilon_{\mathbf{t}_0} \frac{k'_0}{k_0^2 r_0} = -\varepsilon_{\mathbf{t}_0} \rho'_0 \sigma_0. \tag{5}$$

When all is said and done, presently we study to find the numbers $\vartheta_1, \vartheta_2,$ and ϑ_3 such that

$$\gamma(0) - m = \vartheta_1 \mathbf{t}_0 + \vartheta_2 \mathbf{n}_{1_0} + \vartheta_3 \mathbf{n}_{2_0}. \tag{6}$$

From $\langle \mathbf{t}_0, \gamma(0) - m \rangle_p = \vartheta_1$ and by using the equation (3) we obtain $\vartheta_1 = 0$. In the same vein, by using the equations (4) and (5), we get $\vartheta_2 = -\varepsilon_{\mathbf{t}_0} \rho_0$ and $\vartheta_3 = -\frac{\varepsilon_{\mathbf{t}_0}}{\varepsilon_{\mathbf{n}_{2_0}}} \rho'_0 \sigma_0$. Also the origin of the sphere that contacts at the third order to the curve at the point $\gamma(0)$ is

$$m = \gamma(0) - \vartheta_1 \mathbf{t}_0 - \vartheta_2 \mathbf{n}_{1_0} - \vartheta_3 \mathbf{n}_{2_0}.$$

Let Q be a spatial semi quaternionic variable on spatial semi quaternionic osculating sphere, assume

$$Q = \gamma(0) + x_1 \mathbf{t}_0 + x_2 \mathbf{n}_{1_0} + x_3 \mathbf{n}_{2_0}.$$

From the above discussions, we have the following statement

$$Q - m = x_1 \mathbf{t}_0 + \left(x_2 - \varepsilon_{\mathbf{t}_0} \rho_0 \right) \mathbf{n}_{1_0} + \left(x_3 - \frac{\varepsilon_{\mathbf{t}_0}}{\varepsilon_{\mathbf{n}_{2_0}}} \rho'_0 \sigma_0 \right) \mathbf{n}_{2_0}$$

and

$$\langle Q - m, Q - m \rangle_p = \varepsilon_{t_0} x_1^2 + \varepsilon_{n_{1_0}} (x_2 - \varepsilon_{t_0} \rho_0)^2 + \varepsilon_{n_{2_0}} \left(x_3 - \frac{\varepsilon_{t_0}}{\varepsilon_{n_{2_0}}} \rho'_0 \sigma_0 \right)^2.$$

Finally, with the aim of the equation (6) we get

$$r^2 = \langle \gamma(0) - m, \gamma(0) - m \rangle_p = \varepsilon_{n_{1_0}} \rho_0^2 + \varepsilon_{n_{2_0}} (\rho'_0 \sigma_0)^2$$

which is the equation of the spatial semi quaternionic osculating sphere in \mathbb{E}_1^3 . This completes proof. \square

4. Semi Quaternionic Osculating Spheres of a Semi Quaternionic Curve in \mathbb{E}_2^4

Definition 2.

$$\beta : I \subset \mathbb{R} \rightarrow \mathcal{H}$$

$$s \rightarrow \beta(s) = \sum_{i=0}^3 \gamma_i(s) e_i,$$

be a semi quaternionic curve in \mathbb{E}_2^4 identified with the space of semi quaternions \mathcal{H} . Let $I = [0, 1]$ be an interval in the real line \mathbb{R} and s be an arc-length parameter. In this position we say that $\|\beta'(s)\| = \|\mathbf{T}(s)\| = 1$. We assume that $L = (L_1, L_2, L_3, L_4)$ be a rectangular coordinate system of \mathbb{E}_2^4 . We take a sphere $\langle L - M, L - M \rangle = R^2$ with origin M and radius R . We define a function $g(s) = \langle \beta(s) - M, \beta(s) - M \rangle - R^2$ satisfies the following equations

$$g(0) = g'(0) = g''(0) = g'''(0) = g^{(4)}(0) = 0, g^{(5)} \neq 0.$$

Then we called that the sphere contacts at fourth order to the curve β at $\beta(0)$. The sphere is called semi real quaternionic osculating sphere for semi quaternionic curves in \mathbb{E}_2^4 .

Theorem 2. Let $\beta : I \subset \mathbb{R} \rightarrow \mathcal{H}$ be a semi quaternionic curve with nonzero curvatures $K(0)$, $r(0)$, and $(r - K)(0)$ at $\beta(0)$. Then there exists a sphere which contacts at the fourth order to the curve β at $\beta(0)$ and the equation of the semi quaternionic osculating sphere according to the Frenet frame $\{\mathbf{T}_0, \mathbf{N}_{1_0}, \mathbf{N}_{2_0}, \mathbf{N}_{3_0}\}$ such that

$$\begin{aligned} & \varepsilon_{T_0} X_1^2 + \varepsilon_{N_{1_0}} \left(X_2 - \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \varsigma_0 \right)^2 + \varepsilon_{N_{2_0}} \left(X_3 - \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}} \varepsilon_{n_{1_0}}} \varsigma'_0 \rho_0 \right)^2 \\ & + \varepsilon_{N_{3_0}} \left(X_4 - \Omega_0 \left\{ \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left[(\varsigma'_0 \rho_0)' + \varepsilon_{t_0} \left(\varepsilon_{n_{1_0}} \left(\frac{\varsigma_0}{\rho_0} \right) - \varepsilon_{N_{1_0}} \rho_0 \right. \right. \right. \right. \right. \\ & \left. \left. \left. \left. - 4 \left(\frac{\rho_0}{\varsigma_0} \right) \right) + 3 \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left(\frac{\rho_0}{\varsigma_0} \right) \right] \right\} \right) \right)^2 \end{aligned}$$

$$= \varepsilon_{N_{1_0}} \zeta_0^2 + \varepsilon_{N_{2_0}} (\zeta_0' \rho_0)^2 + \varepsilon_{N_{3_0}} \left\{ \Omega_0^2 \left[(\zeta_0' \rho_0)' + \varepsilon_{t_0} (\varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0} \right) - \varepsilon_{N_{1_0}} \rho_0 - 4 \left(\frac{\rho_0}{\zeta_0} \right)) + 3 \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left(\frac{\rho_0}{\zeta_0} \right) \right] \right\}$$

where $\zeta_0 = \frac{1}{K_0}$, $\rho_0 = \frac{1}{k_0}$, and $\Omega_0 = \frac{1}{r_0 - \varepsilon_{t_0} \varepsilon_{T_0} \varepsilon_{N_{1_0}} k_0}$.

Proof. If $g(0) = 0$ then $\langle \beta(0) - M, \beta(0) - M \rangle = R^2$. By differentiating this equation, we get $g'(0) = 0$ and $g' = 2 [\langle \beta'(0), \beta(0) - M \rangle] = 0$. Because of this, we can write

$$\langle T_0, \beta(0) - M \rangle = 0. \tag{7}$$

Similarly we can get

$$g''(0) = 2 [\langle \beta''(0), \beta(0) - M \rangle + \langle \beta'(0), \beta'(0) \rangle] = 0 \text{ and } g''(0).$$

By taking into account the equations (2) and using the last equation we have

$$\begin{aligned} \langle \varepsilon_{N_{1_0}} K_0 N_{1_0}, \beta(0) - M \rangle + \langle T_0, T_0 \rangle &= 0 \\ \langle N_{1_0}, \beta(0) - M \rangle &= -\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}} K_0} = -\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \zeta_0 \end{aligned} \tag{8}$$

By considering

$$g'''(0) = 2 [\langle \beta'''(0), \beta(0) - M \rangle + 3 \langle \beta''(0), \beta'(0) \rangle] = 0 \text{ and } g'''(0) = 0$$

and from equations (7) and (8), we get

$$\langle N_{2_0}(s), \beta(0) - M \rangle = \frac{\varepsilon_{T_0} K_0'}{\varepsilon_{N_{1_0}} \varepsilon_{n_{1_0}} k_0 K_0^2} = -\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}} \varepsilon_{n_{1_0}}} \zeta_0' \rho_0. \tag{9}$$

Additionally, we obtain

$$\begin{aligned} g^{(4)}(0) &= 2 [\langle \beta^{(4)}(0), \beta(0) - M \rangle + 4 \langle \beta'''(0), \beta'(0) \rangle + 3 \langle \beta''(0), \beta''(0) \rangle] = 0 \text{ and} \\ g^{(4)}(0) &= 0 \end{aligned}$$

where $\langle \beta'''(0), \beta'(0) \rangle = -\varepsilon_{t_0} \varepsilon_{T_0} K_0^2$, $\langle \beta''(0), \beta''(0) \rangle = \varepsilon_{N_{1_0}} K_0^2$. So we have

$$\langle N_{3_0}(s), \beta(0) - M \rangle = -\Omega_0 \left\{ -\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left[K_0'' \zeta_0^2 \rho_0 - \varepsilon_{t_0} \varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0} \right) - \varepsilon_{t_0} \varepsilon_{N_{1_0}} \rho_0 + 2K_0' \zeta_0' \rho_0 \zeta_0 + k_0' \zeta_0' \rho_0^2 + 4\varepsilon_{t_0} \left(\frac{\rho_0}{\zeta_0} \right) \right] + 3 \frac{\rho_0}{\zeta_0} \right\}.$$

The following two statements are found such that

$$\langle \mathbf{N}_{3_0}(s), \beta(0) - M \rangle = -\Omega_0 \left\{ -\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left[\begin{array}{c} 2\left(\zeta'_0\right)^2 \frac{\rho_0}{\zeta_0} - \zeta''_0 \rho_0 - \varepsilon_{t_0} \varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0}\right) - \varepsilon_{t_0} \varepsilon_{N_{1_0}} \rho_0 \\ -2\left(\zeta'_0\right)^2 \frac{\rho_0}{\zeta_0} - \zeta'_0 \rho'_0 + 4\varepsilon_{t_0} \left(\frac{\rho_0}{\zeta_0}\right) \end{array} \right] + 3\frac{\rho_0}{\zeta_0} \right\}$$

and

$$\langle \mathbf{N}_{3_0}(s), \beta(0) - M \rangle = -\Omega_0 \left\{ \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left[\left(\zeta'_0 \rho_0\right)' + \varepsilon_{t_0} \left(\varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0}\right) - \varepsilon_{N_{1_0}} \rho_0 - 4\left(\frac{\rho_0}{\zeta_0}\right)\right) + 3\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left(\frac{\rho_0}{\zeta_0}\right) \right] \right\} \tag{10}$$

Furthermore, let take the numbers $\omega_1, \omega_2, \omega_3$ and ω_4 such that

$$\beta(0) - M = \omega_1 \mathbf{T}_0 + \omega_2 \mathbf{N}_{1_0} + \omega_3 \mathbf{N}_{2_0} + \omega_4 \mathbf{N}_{3_0}.$$

From $\langle \mathbf{T}_0, \beta(0) - M \rangle = \omega_1$ and by using the equation (7) we obtain $\omega_1 = 0$. Similarly, using the equations (8), (9), and (10) we get $\omega_2 = -\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \zeta_0, \omega_3 = -\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}} \varepsilon_{n_{1_0}}} \zeta'_0 \rho_0$, and

$\omega_4 = -\Omega_0 \left\{ \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left[\left(\zeta'_0 \rho_0\right)' + \varepsilon_{t_0} \left(\varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0}\right) - \varepsilon_{N_{1_0}} \rho_0 - 4\left(\frac{\rho_0}{\zeta_0}\right)\right) + 3\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left(\frac{\rho_0}{\zeta_0}\right) \right] \right\}$. Also the origin of the sphere that contacts at the fourth order to the curve at the point $\beta(0)$ is

$$M = \beta(0) - \omega_1 \mathbf{T}_0 - \omega_2 \mathbf{N}_{1_0} - \omega_3 \mathbf{N}_{2_0} - \omega_4 \mathbf{N}_{3_0}.$$

Let P be a semi quaternionic variable on semi quaternionic osculating sphere, assume

$$P = \beta(0) + X_1 \mathbf{T}_0 + X_2 \mathbf{N}_{1_0} + X_3 \mathbf{N}_{2_0} + X_4 \mathbf{N}_{3_0}.$$

Then we can write

$$\begin{aligned} P - M = & X_1 \mathbf{T}_0 + \left(X_2 - \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \zeta_0 \right) \mathbf{N}_{1_0} + \left(X_3 - \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}} \varepsilon_{n_{1_0}}} \zeta'_0 \rho_0 \right) \mathbf{N}_{2_0} \\ & + \left(X_4 - \Omega_0 \left\{ \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left[\left(\zeta'_0 \rho_0\right)' + \varepsilon_{t_0} \left(\varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0}\right) - \varepsilon_{N_{1_0}} \rho_0 - 4\left(\frac{\rho_0}{\zeta_0}\right)\right) + 3\frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left(\frac{\rho_0}{\zeta_0}\right) \right] \right\} \right) \mathbf{N}_{3_0}. \end{aligned}$$

From the last equation, we get

$$\langle P - M, P - M \rangle = \varepsilon_{T_0} X_1^2 + \varepsilon_{N_{1_0}} \left(X_2 - \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \zeta_0 \right)^2 + \varepsilon_{N_{2_0}} \left(X_3 - \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}} \varepsilon_{n_{1_0}}} \zeta'_0 \rho_0 \right)^2$$

$$\begin{aligned}
& + \varepsilon_{N_{3_0}} \left(X_4 - \Omega_0 \left\{ \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left[(\zeta'_0 \rho_0)' + \varepsilon_{t_0} \left(\varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0} \right) - \varepsilon_{N_{1_0}} \rho_0 \right. \right. \right. \right. \\
& \left. \left. \left. - 4 \left(\frac{\rho_0}{\zeta_0} \right) \right) + 3 \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left(\frac{\rho_0}{\zeta_0} \right) \right] \right\} \right)^2
\end{aligned}$$

and from the equation (10) we finally calculate the equation of the semi real quaternionic osculating sphere in \mathbb{E}_2^4 such that

$$\begin{aligned}
R^2 = \langle \beta(0) - M, \beta(0) - M \rangle & = \varepsilon_{N_{1_0}} \zeta_0^2 + \varepsilon_{N_{2_0}} (\zeta'_0 \rho_0)^2 \\
& + \varepsilon_{N_{3_0}} \left\{ \Omega_0^2 \left[(\zeta'_0 \rho_0)' + \varepsilon_{t_0} \left(\varepsilon_{n_{1_0}} \left(\frac{\zeta_0}{\rho_0} \right) - \varepsilon_{N_{1_0}} \rho_0 - 4 \left(\frac{\rho_0}{\zeta_0} \right) \right) + 3 \frac{\varepsilon_{T_0}}{\varepsilon_{N_{1_0}}} \left(\frac{\rho_0}{\zeta_0} \right) \right] \right\}.
\end{aligned}$$

This completes proof. \square

ACKNOWLEDGEMENTS The authors thank the readers of European Journal of Pure and Applied Mathematics, for making our journal successful.

References

- [1] W. K. Clifford. Preliminary sketch of biquaternions, Proceedings of the London Mathematical Society, 4, 361-395, 1873.
- [2] A. C. Çöken and A. Tuna. On the quaternionic inclined curves in the semi-Euclidean space \mathbb{E}_2^4 , Appl. Math. Comput. 155, 373-389, 2004.
- [3] F. Kahraman, İ. Gök, and H. H. Hacısalihoğlu. On the quaternionic B_2 slant helices in the semi- Euclidean space \mathbb{E}_2^4 , Applied Mathematics and Computation, 218,6391-6400, 2012.
- [4] J. P. Ward. Quaternions and Cayley Numbers, Kluwer Academic Publishers, Boston/London, 1997.
- [5] K. Bharathi and M. Nagaraj. Quaternion Valued Function of a Real Variable Serret-Frenet Formulae, Indian Journal of Pure and Applied Mathematics, 18(6), 507-511, 1987.
- [6] A. Tuna. Serret Frenet formulae for Quaternionic Curves in semi-Euclidean Space, Master Thesis, Süleyman Demirel University, Graduate School of Natural and Applied Science, Department of Mathematics, Isparta, Turkey, 2002.
- [7] H. H. Hacısalihoğlu. Diferensiyel Geometri, Faculty of Sciences, University of Ankara Press, 1993.
- [8] A. Sabuncuoğlu. Diferensiyel Geometri, Nobel Press, 2010.

- [9] D. J. Struik. Differential Geometry, second ed., Addison-Wesley, Reading, Massachusetts. 1961.
- [10] D. Sağlam. On the Osculating Spheres of a Real Quaternionic Curve in the Euclidean Space \mathbb{E}^4 , International Journal of Mathematical Combinatorics, 3, 46-53, 2012.
- [11] E. Soytürk, K. İlarıslan, and D. Sağlam. Osculating spheres and osculating circles of a curve in semi-Reimannian space, Communications, Faculty of Science, University of Ankara Series A1, 54 (2) ,39-48, 2005.