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# A Generalization of the Calderón Admissibility Condition

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**Abstract.** Many authors have been considered several conditions equivalent to the Calderón admissibility condition. In this paper, we review these results and give a characterization of generalized Calderón admissibility condition.

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**Key Words and Phrases**: Calderón admissibility condition, continuous wavelet transform, semidirect product

## 1. Introduction

For every $\psi \in L^2(\mathbb{R})$ the *continuous wavelet transform* (CWT) of $f \in L^2(\mathbb{R})$ is given by

$$(W_\psi f)(a,b) = |a|^{\frac{-1}{2}} \int_{\mathbb{R}} f(x)\overline{\psi(\frac{x-b}{a})}dx, \quad (a \in \mathbb{R}\setminus\{0\}, b \in \mathbb{R}).$$

The mapping $W_\psi$ is well-defined if $\psi$ satisfies the *Calderón admissibility condition*

$$\int_{\mathbb{R}\setminus\{0\}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi = 1. \tag{1}$$

The generalization of this construction, in particular to higher-dimensional Euclidean space, has been studied early on, see e.g. [4]. One class of groups and representations attracting particular attention are the semidirect products of the type $G = H \times_\tau \mathbb{R}^n$. Here $H$ is a closed matrix group, the so-called *dilation group*. To construct continuous wavelet transforms from quasi-regular representations of a semidirect product topological group we require a square integrable function whose Plancherel transform satisfies Calderón admissibility condition. The

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question then arises under what conditions such square-integrable functions exist. The CWT on these groups was discussed in [2, 7, 8, 11].

Many authors have been considered several conditions equivalent to the Calderón admissibility condition, [1, 2, 6, 8]. In this paper, we first review these results and then introduce a more general setting for admissible groups. Moreover, some necessary conditions are provided for a class of admissible groups. Let us shortly sketch the group-theoretic framework for the construction of continuous wavelet transforms on locally compact abelian groups. It is well known that for irreducible, square-integrable representations of a locally compact group, there exist so-called *admissible vectors* which allow the construction of generalized continuous wavelet transforms.

Let $G$ be a locally compact topological group with the left Haar measure $\mu_G$ and modular function $\Delta_G$. If $\pi$ is a unitary representation of $G$ on a Hilbert space $\mathscr{H}$, then a vector $\psi \in \mathscr{H}$ where

$$C_\psi := \frac{1}{\|\psi\|^2}\int_G |<\psi, \pi(x)\psi>|^2 \, d\mu_G(x) < \infty \tag{2}$$

is called an *admissible vector*. The existence of an admissible vector is not generally guaranteed [10]. Now for a fixed admissible vector $\psi$ in $\mathscr{H}$ the linear isometry $W_\psi : \mathscr{H} \to L^2(G)$ given by

$$(W_\psi \eta)(x) = C_\psi^{\frac{-1}{2}} < \eta, \pi(x)\psi >, \quad (\eta \in \mathscr{H}, x \in G)$$

is called the CWT on $G$. Also we refer to the inequality (2) as the admissibility condition.

Among the many useful aspects of wavelets, probably the most fundamental one is the wavelet inversion formula, usually given by

$$\int_G (W_\psi \eta)(x)\pi(x)\psi \, d\mu_G(x) = \eta, \quad (\eta \in \mathscr{H}).$$

For locally compact groups $H$ and $K$ where $K$ is also abelian, let $h \longmapsto \tau_h$ be a homomorphism of $H$ into the group of automorphisms of $K$ denoted by Aut(K). Also, assume that the mapping $(h, x) \longmapsto \tau_h(x)$ from $H \times K$ onto K is continuous. Then the set $H \times K$ endowed with the product topology and the operations:

$$(h, x).(h', x') = (hh', x.\tau_h(x')), \quad (h, x)^{-1} = (h^{-1}, \tau_{h^{-1}}(x^{-1}))$$

is a locally compact group. This group is called the *semidirect product* of $H$ and $K$, respectively, and is denoted by $H \times_\tau K$. Let $G = H \times_\tau K$. Then the left Haar measure of $G$ is $d\mu_G(h, x) = \delta(h)d\mu_H(h)d\mu_K(x)$ and $\Delta_G(h, x) = \delta(h)\Delta_H(h)$ is its modular function, in which $\delta$ is a positive continuous homomorphism on $H$ and is given by

$$\mu_K(E) = \delta(h)\mu_K(\tau_h(E)),$$

for all measurable subsets $E$ of $K$, for more details of these facts see [5].

From the canonical action of $G$ on $K$ arises a natural unitary representation, which is called the quasi regular representation on the semidirect product group $G$.

**Definition 1.** *The quasi regular representation* $(U, L^2(K))$ *on* $G = H \times_\tau K$ *is defined by*

$$U(h,x)f(y) = \delta(h)^{\frac{1}{2}} f(\tau_{h^{-1}}(yx^{-1})), \quad (f \in L^2(K)).$$

$U$ is not generally irreducible [10]. An element $\psi \in L^2(K)$ is admissible if satisfies the generalized Calderón admissibility condition

$$\int_H \int_K | < \psi, U(h,x)\psi > |^2 \delta(h) d\mu_H(h) d\mu_K(x) < \infty, \quad (f \in L^2(K)). \tag{3}$$

Consider $\widehat{K}$ as the dual group of the LCA group $K$ and denote its left Haar measure by $d\omega$. Then one can define a continuous action from $H$ on $\widehat{K}$ by $(h,\omega) \longmapsto \omega \circ \tau_{h^{-1}}$. Now for each $\omega \in \widehat{K}$, the stabilizer and the orbit of $\omega$, that play a key role in our discussion are defined by

$$H^\omega := \{h \in H\ ;\ ;\ \omega \circ \tau_h = \omega\}, \quad O_\omega := \{\omega \circ \tau_h;\ h \in H\},$$

respectively. The set $H^\omega$ is a closed subgroup of $H$ and $O_\omega$ is an $H$-invariant subset in $\widehat{K}$.

## 2. Admissible Subgroups of $GL(n, \mathbb{R})$

Let $H \leq GL(n,\mathbb{R})$ be the group consisting of diagonal matrices, also let $H \times_\tau \mathbb{R}^n$ be the semidirect product of $H$ and $\mathbb{R}^n$, with the usual action of $H$ on $\mathbb{R}^n$. In [3] it is shown that $\psi \in L^2(\mathbb{R}^n)$ is admissible if and only if

$$\int_H \frac{|\widehat{\psi}(\xi)|^2}{|\xi_1 \xi_2 \ldots \xi_n|} d\xi < \infty.$$

A subgroup $H$ of $GL(n,\mathbb{R})$ is said to be *admissible* if the quasi-regular representation on the semidirect product group $H \times_\tau \mathbb{R}^n$, with the natural action of $H$ on $\mathbb{R}^n$, has an admissible vector $\psi \in L^2(\mathbb{R}^n)$. In [12] it is shown that a subgroup $H$ of $GL(n,\mathbb{R})$ is admissible if and only if there exists $\psi \in L^2(\mathbb{R}^n)$ such that

$$\int_H |\widehat{\psi}(\omega h)|^2 d\mu_H(h) = 1 \quad \text{for a.e. } \omega \in \mathbb{R}^n.$$

A straightforward calculation gives that $H \leq GL(n,\mathbb{R})$ is admissible if and only if there exists a Borel measurable function $g \in L^1(\mathbb{R}^n)$ such that $g \geq 0$ and

$$\int_H g(h^t x) d\mu_H(h) = 1 \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{4}$$

in which $h^t$ is the transpose of $h$. For example, with the natural action on $\mathbb{R}^2$ the Affine group $\mathbb{R} \setminus \{0\} \times_\tau \mathbb{R}$ is admissible. The best results are due to Laugesen *et al.* proved in [11] is a characterization of those admissible groups which admit an inversion formula;

**Theorem 1** ([11]). *Let $H$ be a $\sigma$-compact, locally compact group, and $h \longmapsto \tau_h$ from $H$ to $GL(n,\mathbb{R})$ be a continuous homomorphism. Then*

*(i) If $H$ is admissible, then $\Delta_H \not\equiv \delta^{-1}$ and $H^\omega$ is compact for a.e. $\omega \in \mathbb{R}^n$.*

*(ii) If $\Delta_H \not\equiv \delta^{-1}$ and for a.e. $\omega \in \mathbb{R}^n$ there exists an $\epsilon > 0$ such that*

$$H^\omega_\epsilon = \{h \in H;\ \|\omega \circ \tau_h - \omega\| \leq \epsilon\},$$

*the $\epsilon$-stabilizer of $\omega$, is compact, then $H$ is admissible.*

**Proposition 1.** *For any $n > 1$ the group $GL(n,\mathbb{R})$ is not admissible.*

*Proof.* Assume that $H = GL(n,\mathbb{R})$. It is sufficient to show that the stabilizers $H^\omega$, for all $\omega$ in a positive Lebesgue measure subset of $\mathbb{R}^n$, are not compact. Let $E = \{\omega \in \mathbb{R}^n; \omega_i > 0\}$. Then the equation

$$h^t \omega = \omega$$

can be solved with respect to any $\omega \in E$. In fact, we may find solutions $h \in H$ whose some arrays are arbitrary large. So that $H^\omega$ is not compact for all $\omega \in E$. □

The structure of stabilizers of a group and its subgroup are almost the same. Let $H$ be an admissible group and $L \leq H$. Then $L^\omega$ is compact for a.e. $\omega \in \mathbb{R}^n$ since it is a closed subgroup of $H^\omega$. But the condition $\Delta_H \not\equiv |det|$ about $H$ and $L$ may be different. For example, the group $SL(2,\mathbb{R})$ and its subgroup

$$K = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, y \in \mathbb{R} \right\}$$

are not admissible by Theorem 1. Although the subgroup

$$H = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} x \neq 0, y \in \mathbb{R} \right\}$$

of $SL(2,\mathbb{R})$ is admissible [11].

The following theorem is about the admissibility of $H$ and $H^t$.

**Theorem 2.** *A closed unimodular subgroup $H$ of $GL(n,\mathbb{R})$ is admissible if and only if $H^t$ is admissible.*

*Proof.* First we would like to compute $\Delta_{H^t}$, the modular function of $H^t$. Clearly $\nu(E) = \mu_{H^t}(E^t)$ defines a right Haar measure on $H$, so $\mu_H(E^{-1}) = c\mu_{H^t}(E^t)$ for all Borel sets $E$ of $H$ and for some $c > 0$. This implies that

$$\Delta_{H^t}(h^t) = \frac{\mu_{H^t}(Eh^t)}{\mu_{H^t}(E)} = \frac{\mu_H((hE^t)^{-1})}{\mu_H((E^t)^{-1})} = \Delta_H(h^{-1})$$

i.e. $\Delta_{H^t} = \Delta_H^{-1}$. Now assume that $H$ is admissible, then there exists a non-negative $g \in L^1(\mathbb{R}^n)$ such that (4) holds. This implies that

$$\int_{H^t} g(hx) d\mu_{H^t}(h^t) = \int_H g(h^t x) d\mu_H(h^{-1})$$

$$= \int_H g(h^t x)\Delta_H(h^{-1})d\mu_H(h)$$
$$= \int_H g(h^t x)d\mu_H(h) = 1, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Therefore, $H^t$ is admissible. □

The following example shows that the condition $H$ is unimodular can not be removed;

**Example 1.** *Let $H = (\mathbb{R} \setminus \{0\}) \times_\tau \mathbb{R}$ be the affine group. As we have seen before, $d\mu_H(h,x) = h^{-2}dhdx$ is the left Haar measure of $H$. This shows that every non-negative normalized function $g \in L^1(\mathbb{R}^2)$ such that $x_1^2 g(x_1,x_2)$ is integrable satisfies in (4). Hence, $H$ is admissible but $\Delta_{H^t} \equiv |det|$ and so $H^t$ is not admissible by Theorem 1.*

## 3. Admissibility of Arbitrary Topological Groups

A more general family of admissible groups was studied by Grochenig, Kaniuth and Taylor [9], who focused on certain one-parameter groups; in particular all of the aforementioned examples fall under the class described in [8, 11]. Consider the semidirect product group $G \times_\tau \mathbb{R}^n$ where $G$ is an arbitrary topological group and $\tau : G \to GL(n,\mathbb{R}); a \longmapsto \tau_a$ is a homomorphism such that $(a,x) \longmapsto \tau_a(x)$ is continuous. The topological group $G$ is called *admissible* if the quasi-regular representation on the semidirect group $G \times_\tau \mathbb{R}^n$ has an admissible vector.

A further extension, replacing $\mathbb{R}^n$ by a general locally compact abelian group $K$. In this case, we can also modify (4) to describe admissible groups. A characterization of such admissible groups which extends (1) can be found in [2, 8].

**Theorem 3.** *[2] Equality (3) is valid for $\psi \in L^2(K)$ if*

$$\int_H |\widehat{\psi}(\omega \circ \tau_h)|^2 d\mu_H(h) = 1 \quad \text{for a.e. } \omega \in \widehat{K}. \tag{5}$$

*Moreover, the converse is also true by more assumptions.*

Let $H$ be a closed subgroup of $G$. We consider the left multiplication as the usual action of $G$ on quotient space $G/H$. A Radon measure $\mu$ on $G/H$ is called invariant if $\mu(a\mathscr{B}) = \mu(\mathscr{B})$ for every $g \in G$ and Borel set $\mathscr{B}$ of $G/H$. There is an invariant measure on $G/H$ if and only if $\Delta_G \mid_H = \Delta_H$, for more details see 2.49 of [5]. The following theorem shows that the admissibility can be extended from a subgroup to own group.

**Theorem 4.** *Let $H$ be a closed subgroup of a $\sigma$−compact group $G$ such that $G/H$ is compact. If $H$ is admissible and $G/H$ has an invariant measure, then is $G$ also admissible.*

*Proof.* Suppose $\mu$ is an invariant measure on $G/H$, then we have

$$\int_G f(a)d\mu_G(a) = \int_{G/H}\int_H f(ah)d\mu_H(h)d\mu(aH), \quad (f \in L^1(G)). \tag{6}$$

This identity is known as Weil's formula and holds also for any $f \geq 0$ that vanishes outside a finite set [5]. Now let $H$ be admissible and $g \in L^1(\mathbb{R})$ a non-negative measurable function such that

$$\int_H g((\tau_h)^t x) d\mu_H(h) = 1, \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Since $a \longmapsto g((\tau_a)^t x)$ on $G$ is positive by using (6) with the fact that $\mu$ is finite we obtain

$$\begin{aligned}
\int_G g((\tau_a)^t x) d\mu_G(a) &= \int_{G/H} \int_H g((\tau_a h)^t x) d\mu_H(h) d\mu(aH) \\
&= \int_{G/H} \int_H g((\tau_h)^t (\tau_a)^t x) d\mu_H(h) d\mu(aH) \\
&= \int_{G/H} d\mu(aH) < \infty, \quad \text{for a.e. } x \in \mathbb{R}.
\end{aligned}$$

Therefore, $G$ is admissible. □

**Theorem 5.** *If $G_1$ and $G_2$ are admissible groups, then so is $G_1 \times G_2$.*

*Proof.* Since $G_i$ is admissible there exists a measurable function $g_i \in L^1(\mathbb{R}^{n_i})$ such that $g_i \geq 0$ and

$$\int_G g_i((\tau_a^i)^t x) d\mu_{G_i}(a) = 1, \quad \text{for a.e. } x \in \mathbb{R}^{n_i}$$

where $n_i \in \mathbb{N}$ and $\tau^i : G_i \to GL(n_i, \mathbb{R})$ is a continuous homomorphism ($i = 1, 2$). Consider $G = G_1 \times G_2$ and define the continuous homomorphism $\tau : G \to GL(n_1 + n_2, \mathbb{R})$ by

$$\tau_{(a_1,a_2)} = \begin{pmatrix} \tau_{a_1}^1 & 0 \\ 0 & \tau_{a_2}^2 \end{pmatrix}. \tag{7}$$

Then the semidirect product group $(G_1 \times G_2) \times_\tau \mathbb{R}^{n_1+n_2}$ is well defined and $g : \mathbb{R}^{n_1+n_2} \to \mathbb{C}$ given by $g(x_1, x_2) = g_1(x_1) g_2(x_2)$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ is positive and belongs to $L^1(\mathbb{R}^{n_1+n_2})$. Moreover

$$\int_G g((\tau_{(a_1,a_2)})^t (x_1, x_2)) d\mu_G(a_1, a_2) = 1, \quad \text{for a.e. } x \in \mathbb{R}^{n_1}, x \in \mathbb{R}^{n_2}.$$

Therefore, $G$ is admissible. □

The above theorem help us to construct admissible groups from a given admissible group. For example if $G$ is an admissible group, then $G \times G^\omega$ for a.e. $\omega \in \mathbb{R}^n$ is also admissible.

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