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Module Extension Banach Algebras and (σ,τ)-amenability

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Abstract. In this paper among other things we find some necessary and sufficient conditions for a Banach algebra $\mathscr{A}$, to be (σ,τ)-amenable, where σ and τ are continuous homomorphisms on $\mathscr{A}$.

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1. Introduction.

Let $\mathscr{A}$ be a Banach algebra and $\mathscr{X}$ be a Banach $\mathscr{A}$-bimodule, that $\mathscr{X}$ is both a Banach space and an algebraic $\mathscr{A}$-bimodule, and the module operations $(a,x)\mapsto ax$ and $(a,x)\mapsto xa$ from $\mathscr{A}\times\mathscr{X}$ into $\mathscr{X}$ are (jointly) continuous. Then $\mathscr{X}^*$ is also a Banach $\mathscr{A}$-bimodule under the following module actions:

$$(a\cdot f)(x) = f(xa),$$

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$$(f \cdot a)(x) = f(ax),$$

$a \in \mathscr{A}, x \in \mathscr{X}, f \in \mathscr{X}^*$.

Let $\mathscr{A}$ be a Banach algebra. Given $f \in \mathscr{A}^*$ and $F \in \mathscr{A}^{**}$, then Ff and fF are defined in $\mathscr{A}^*$ by the following formulae

$$Ff(a) = F(f \cdot a), \qquad fF(a) = F(a \cdot f) \qquad (a \in \mathscr{A}).$$

Next, for $F, G \in \mathscr{A}^{**}$, FG is defined in $\mathscr{A}^{**}$ by the formulae

$$(FG)(f) = F(Gf),$$

this product is called first Arens product on $\mathscr{A}^{**}$ and $\mathscr{A}^{**}$ with the first Arens product is a Banach algebra.

Let $\mathscr{A}$ be a Banach algebra and $\mathscr{X}$ be a Banach $\mathscr{A}$-bimodule. The Banach space $\mathscr{X}^{**}$ is a Banach $\mathscr{A}^{**}$-bimodule under following actions

$$F \cdot G = w^* - \lim_i \lim_j a_i x_j, \qquad G \cdot F = w^* - \lim_j \lim_i x_j a_i$$

where $F = w^* - \lim_i a_i$, $G = w^* - \lim_j x_j$, (a_i) is a net in $\mathscr{A}$, (x_j) and is a net in X.

Suppose that $\varphi : \mathscr{A} \to \mathscr{B}$ is a Banach algebra homomorphism. The Banach algebra $\mathscr{B}$ is considered as a Banach $\mathscr{A}$- bimodule by the following module actions

$$a \cdot b = \varphi(a)b, \qquad b \cdot a = b\varphi(a) \qquad (a \in \mathscr{A}, b \in \mathscr{B})$$

we denote $\mathscr{B}_\varphi$ the above $\mathscr{A}$-bimodule.

Let $\mathscr{A}$ be a Banach algebra and σ, τ be continuous homomorphisms on $\mathscr{A}$. Suppose that $\mathscr{X}$ is a Banach $\mathscr{A}$-bimodule. A linear mapping $d : \mathscr{A} \to \mathscr{X}$ is called a (σ, τ)-derivation if

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b) \quad (a, b \in A).$$

For example every ordinary derivation of an algebra $\mathscr{A}$ into an $\mathscr{A}$-bimodule $\mathscr{X}$ is an $(id_{\mathscr{A}}, id_{\mathscr{A}})$-derivation, where $id_{\mathscr{A}}$ is the identity mapping on the algebra $\mathscr{A}$.

A linear mapping $d : \mathscr{A} \longrightarrow \mathscr{X}$ is called (σ,τ)-inner derivation if there exists $x \in \mathscr{X}$ such that $d(a) = \tau(a)x - x\sigma(a) \quad (a \in \mathscr{A})$. See also [3–6].

We denote the set of continuous (σ,τ)-derivations from $\mathscr{A}$ into $\mathscr{X}$ by $Z^1_{(\sigma,\tau)}(\mathscr{A},\mathscr{X})$ and the set of inner (σ,τ)-derivations by $B^1_{(\sigma,\tau)}(\mathscr{A},\mathscr{X})$. we define the space $H^1_{(\sigma,\tau)}(\mathscr{A},\mathscr{X})$ as the quotient space $Z^1_{(\sigma,\tau)}(\mathscr{A},\mathscr{X})/B^1_{(\sigma,\tau)}(\mathscr{A},\mathscr{X})$. The space $H^1_{(\sigma,\tau)}(\mathscr{A},\mathscr{X})$ is called the first (σ,τ)-cohomology group of $\mathscr{A}$ with coefficients in $\mathscr{X}$. $\mathscr{A}$ is called (σ,τ)-amenable if $H^1_{(\sigma,\tau)}(\mathscr{A},\mathscr{X}^*) = \{0\}$, for each Banach $\mathscr{A}$-bimodule $\mathscr{X}$.

Let $\mathscr{A}$ be a Banach algebra and let $\mathscr{X}$ be a Banach $\mathscr{A}$-bimodule. Define $\mathscr{A} \oplus_1 \mathscr{X}$ by actions:

$$\begin{aligned} (a,x)+(b,y) &= (a+b,x+y) \\ a(b,x)=(ab,ax) \quad &, \quad (b,x)a=(ba,xa) \\ (a,x)(b,y) &= (ab,ay+xb), \end{aligned}$$

for every $a,b \in \mathscr{A}$ and $x,y \in \mathscr{X}$.

It is clear $\mathscr{A} \oplus_1 \mathscr{X}$ is a Banach algebra with the following norm:

$$\|(a,x)\| = \|a\| + \|x\|.$$

This Banach algebra is called *module extension Banach algebra*.

We use some ideas and terminology of [2] to investigate (σ,τ)-amenability of Banach algebras.

2. (σ,τ)-amenability of Banach Algebras.

Let $\mathscr{A}$ be a Banach algebra and let σ,τ be continuous homomorphisms on $\mathscr{A}$. Suppose that $\mathscr{X}$ is a Banach $\mathscr{A}$-bimodule. Then $\mathscr{X}$ is a Banach $\mathscr{A}$-bimodule by the following module actions:

$$a \cdot x = \tau(a)b, \qquad x \cdot a = b\sigma(a) \qquad (a \in \mathscr{A}, x \in \mathscr{X}).$$

We denote $\mathscr{X}_{(\sigma,\tau)}$ for this $\mathscr{A}$-bimodule. It is easy to check that $(\mathscr{X}_{(\sigma,\tau)})^* = X^*_{(\tau,\sigma)}$, and that every (σ,τ)-derivation from $\mathscr{A}$ into $\mathscr{X}$ is a derivation from $\mathscr{A}$ into $\mathscr{X}_{(\sigma,\tau)}$. Thus we can show that $\mathscr{A}$ is amenable, if and only if $\mathscr{A}$ is (σ,τ)-amenable, for each $\sigma,\tau \in Hom(\mathscr{A})$. First we give the following examples for (σ,τ)-amenability of Banach algebras.

Example 2.1. *It is easy to see that ℓ^1 is a Banach algebra equipped with the following product [7]*

$$a \cdot b = a(1)b \qquad (a, b \in \ell^1),$$

and ℓ^1 has a left identity e defined by

$$e(n) = \begin{cases} 1 & if \quad n = 1 \\ 0 & if \quad n \neq 1. \end{cases}$$

The dual space $(\ell^1)^* = \ell^\infty$ is a ℓ^1-bimodule via the ordinary actions as follows

$$a \cdot f = f(a)e, \quad f \cdot a = a(1)f \qquad (a \in \ell^1, f \in \ell^\infty),$$

where e is regarded as an element of ℓ^∞.

Next let $\sigma : \ell^1 \longrightarrow \ell^1$ be a bounded homomorphism. We have $a(1)\sigma(b) = \sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) = \sigma(a)(1)\sigma(b)$ and so $\sigma(b)(a(1) - \sigma(a)(1)) = 0$ for all $a, b \in \mathbb{N}$. Since $\sigma \neq 0$, we have

$$\Big(\sigma(a)\Big)(1) = a(1) \quad (a \in \ell^1) \tag{2.1}$$

In [5] has been shown that ℓ^1 is (σ,τ)-weakly amenable for all homomorphisms σ,τ but for some homomorphisms σ and τ it is not (σ,τ)-amenable. In the following we prove if the Banach algebra ℓ^1 is (σ,τ)-amenable, then $\tau(a) = a(1)c$ where $c(1) = 1$.

Let $\mathscr{B} = \ell^1$ *by product* $a \bullet b = a(2)b$. *Then* $\mathscr{B}$ *is a Banach algebra and for each bounded homomorphism* $\psi : \mathscr{B} \longrightarrow \mathscr{B}$ *we have* $\Big(\psi(a)\Big)(2) = a(2)$. *Let* $a \in \ell^1$ *define* $a' \in \ell^1$ *by* $a' = \Big(a(2), a(1), a(3), \cdots\Big)$. *Let* $\varphi : \ell^1 \longrightarrow \mathscr{B}$ *defined by* $\varphi(a) = a'$. *It is clear that* φ *is a homomorphism. Consider the Banach* ℓ^1*-bimodule* $\mathscr{B}_\varphi$ *under actions* $a \circ b = \varphi(a) \bullet b = a' \bullet b = a'(2)b = a(1)b$ *and* $b \circ a = b \bullet \varphi(a) = b \bullet a' = b(2)a'$ *for each* $a \in \ell^1, b \in \mathscr{B}_\varphi$. *Let* $D : \ell^1 \longrightarrow \mathscr{B}_\varphi^*$ *be a bounded* (σ, τ)*-derivation. We have*

$$\begin{aligned}
\Big(D(a \cdot b)\Big)(c) &= D(a)\sigma(b)(c) + \tau(a)D(b)(c) \\
a(1)D(b)(c) &= D(a)(\sigma(b) \circ c) + D(b)(c \circ \tau(a)) \\
a(1)D(b)(c) &= b(1)D(a)(c) + c(2)D(b)(\tau(a))
\end{aligned}$$

for all $a, b \in \ell^1$ *and* $c \in B_\varphi$.

By taking $a = b$ *we obtain* $D(a)(\tau(a)) = 0$. *Also by taking* $c \in \mathscr{B}_\varphi$ *such that* $c(2) = 0$ *we can conclude* $a(1)D(b) = b(1)D(a)$.

If ℓ^1 *is* (σ, τ)*-amenable, then there exists* $f \in B_\varphi^*$ *such that* $D = D_f$ *is a* (σ, τ)*-inner derivation. So we have*

$$\begin{aligned}
a(1)D_f(b) &= b(1)D_f(a) \\
a(1)f(b(1)c - c(2)\tau(b)) &= b(1)f(a(1)c - c(2)\tau(a))
\end{aligned}$$

for all $a, b \in \ell^1$ *and* $c \in B_\varphi$.

Then $f(b(1)c(2)\tau(a) - a(1)c(2)\tau(b)) = 0$. *Since* $f \in B_\varphi^*$ *is arbitrary, immediately is conclude* $a(1)\tau(b) = b(1)\tau(a)$. *By taking* $b = e$ *we have* $\tau(a) = a(1)\tau(e)$*, where* $\tau(e)(1) = 1$.

So we have the following result.

Corollary 2.1. *Let* σ, τ *be two continuous homomorphisms on* ℓ^1 *(by above product). If* ℓ^1 *is* (σ, τ)*-amenable then there is* $c \in \ell^1$ *such that* $\tau(a) = a(1)c$*, and* $c(1) = 1$.

Example 2.2. *Let $\mathscr{A}$ be a Banach algebra. Then $\mathscr{A}$ has a bounded approximate identity if and only if $\mathscr{A}$ is $(id,0)$ and $(0,id)$-amenable.*

Corollary 2.2. *Let $\mathscr{A}$ be a C^*-algebra or $\mathscr{A} = L^1(G)$ for a locally compact topological group G. Then $\mathscr{A}$ is $(id,0)$ and $(0,id)$-amenable.*

Let $T : \mathscr{A} \to \mathscr{B}$ be a continuous linear map between Banach algebras. Two continuous linear maps $T' : \mathscr{B}^* \to \mathscr{A}^*$ and $T'' : \mathscr{A}^{**} \to \mathscr{B}^{**}$ are known, that are defined by the following formula

$$\Big(T'(f)\Big)(a) = f\Big(T(a)\Big), \qquad \Big(T''(G)\Big)(f) = G\Big(T'(f)\Big)$$

where $a \in \mathscr{A}, f \in \mathscr{B}^*$ and $G \in \mathscr{A}^{**}$.

Lemma 2.1. *Let $\mathscr{A}$ be a Banach algebra, $\mathscr{X}$ be a Banach $\mathscr{A}$-bimodule, and let σ and τ be two continuous homomorphisms on $\mathscr{A}$. Suppose that $D : \mathscr{A} \longrightarrow \mathscr{X}$ is (σ,τ)-derivation. Then $D'' : \mathscr{A}^{**} \longrightarrow \mathscr{X}^{**}$ is a (σ'',τ'')-derivation.*

Proof. Let $F, G \in \mathscr{A}^{**}$ and let $F = w^* - \lim_\alpha a_\alpha, G = w^* - \lim_\beta b_\beta$ in $\mathscr{A}^{**}$, where $(a_\alpha), (b_\beta)$ are nets in $\mathscr{A}$ with $||a_\alpha|| \leq ||F||, ||b_\beta|| \leq ||G||$. Then

$$\begin{aligned} D''(FG) &= D''\Big(w^* - \lim_\alpha w^* - \lim_\beta a_\alpha b_\beta\Big) \\ &= w^* - \lim_\alpha w^* - \lim_\beta D''(a_\alpha b_\beta) \\ &= w^* - \lim_\alpha w^* - \lim_\beta \Big(\tau(a_\alpha)D(b_\beta) + D(a_\alpha)\sigma(b_\beta)\Big) \\ &= \tau''(F)D''(G) + D''(F)\sigma''(G) \end{aligned}$$

and so D'' is a (σ'',τ'')-derivation.

Now we are ready to state some equivalent conditions by (σ,τ)-amenability of Banach algebras.

Theorem 2.1. *Let σ and τ be two continuous homomorphisms on Banach algebra $\mathscr{A}$. The following statements are equivalent:*

1. *$\mathscr{A}$ is (σ,τ)-amenable.*
2. *For each Banach algebra $\mathscr{B}$ and every homomorphism $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$, $H^1_{(\sigma,\tau)}(\mathscr{A}, \mathscr{B}^*_\varphi) = 0$.*
3. *For each Banach algebra $\mathscr{B}$ and every injective homomorphism $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$, $H^1_{(\sigma,\tau)}(\mathscr{A}, \mathscr{B}^*_\varphi) = 0$.*
4. *For each Banach algebra $\mathscr{B}$ and every injective homomorphism $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$, if $d : \mathscr{A} \longrightarrow {\mathscr{B}_\varphi}^*$ is a (σ,τ)-derivation satisfies*

$$(d(a))(\varphi(b)) + (d(b))(\varphi(a)) = 0 \qquad (a, b \in \mathscr{A}),$$

then d is (σ,τ)-inner derivation.

Proof. Clearly $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$. It is sufficient to show that $(4) \Rightarrow (1)$. Let $\mathscr{X}$ be a Banach $\mathscr{A}$-bimodule and $D : \mathscr{A} \longrightarrow \mathscr{X}^*$ be a (σ,τ)-derivation. Set $\mathscr{B} = \mathscr{A} \oplus_1 \mathscr{X}$ and define injective homomorphism $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$ by $\varphi(a) = (a, 0)$ and so we can assume that $\mathscr{A}$ is a subalgebra of $\mathscr{B}$. Define $d : \mathscr{A} \longrightarrow \mathscr{B}^*_\varphi$ by $d(a) = (0, D(a))$. The map d is (σ,τ)-derivation, since

$$\begin{aligned} d(ab) &= (0, D(ab)) = (0, D(a)\sigma(b) + \tau(a)D(b)) \\ &= (0, D(a))(0, \sigma(b)) + (0, \tau(a))(0, D(b)) \\ &= d(a)\varphi(\sigma(b)) + \varphi(\tau(a))d(b) \\ &= d(a) \cdot \sigma(b) + \tau(a) \cdot d(b) \qquad (a, b \in \mathscr{A}). \end{aligned}$$

Since $(d(a))(\varphi(b)) + (d(b))(\varphi(a)) = (0, D(a))((b, 0)) + (0, D(b))((a, 0)) = 0$, we have $(d(a))(\varphi(b)) + (d(b))(\varphi(a)) = 0$.

It follows from our assumption that d is a (σ,τ)-inner derivation. Hence there are $f \in \mathscr{A}^*$ and $g \in \mathscr{X}^*$ such that

$$\begin{aligned}(0, D(a)) = d(a) &= (\sigma(a), 0)(f, g) - (f, g)(\tau(a), 0) \\ &= (\sigma(a)f - f\tau(a), \sigma(a)g - g\tau(a)).\end{aligned}$$

Thus $D(a) = \sigma(a)g - g\tau(a)$, hence D is (σ,τ)-inner derivation.

Definition 2.1. *Let $\mathscr{A}$ be a Banach algebra and σ be a continuous homomorphisms on $\mathscr{A}$. The Banach algebra $\mathscr{A}$ is called approximately σ-contractible, if for each Banach $\mathscr{A}$-bimodule $\mathscr{X}$ and σ-derivation $D : \mathscr{A} \longrightarrow \mathscr{X}$, there exists a bounded net $(x_\alpha) \subseteq \mathscr{X}$ such that*

$$D(a) = \lim_{\alpha} \Big(\sigma(a)x_\alpha - x_\alpha\sigma(a)\Big) \qquad (a \in \mathscr{A}).$$

In the following theorem we follow the structure of Proposition 2.8.59 [1].

Theorem 2.2. *Let $\mathscr{A}$ be a Banach algebra and σ be a bounded homomorphism on $\mathscr{A}$. Then the following assertion are equivalent:*

1. *$\mathscr{A}$ is σ-amenable.*

2. *For every $\mathscr{A}$-bimodule $\mathscr{X}$, $H^1_{(\sigma,\sigma)}(\mathscr{A}, \mathscr{X}^{**}) = 0$*

3. *$\mathscr{A}$ is approximately σ-contractible.*

Proof. (1) $\Rightarrow$ (2) is trivially. (2) $\Rightarrow$ (3): Let $D : \mathscr{A} \longrightarrow \mathscr{X}$ be a σ-derivation from $\mathscr{A}$ into $\mathscr{A}$-bimodule $\mathscr{X}$ and let $J_{\mathscr{X}} : \mathscr{X} \longrightarrow \mathscr{X}^{**}$ be the canonical embedding, then for each $a, b \in \mathscr{A}$ we have

$$\widetilde{D}(ab) = (J_{\mathscr{X}} \circ D)(ab) \;=\; J_{\mathscr{X}}\Big(\sigma(a)D(b) + D(a)\sigma(b)\Big)$$

$$= \sigma(a)\widetilde{D}(b) + \widetilde{D}(a)\sigma(b).$$

Thus $\widetilde{D}$ is a σ-derivation. Then by (2) there exists $\Lambda \in \mathscr{X}^{**}$ such that $\widetilde{D}(a) = \sigma(a)\Lambda - \Lambda\sigma(a) \qquad (a \in \mathscr{A})$. Set $m = ||\Lambda||, \mathscr{U} = \mathscr{X}_{[m]}$. Then $\Lambda \in \overline{J_{\mathscr{X}}(\mathscr{U})}^{w^*}$. Let $a_1, a_2, a_3, \ldots, a_n \in \mathscr{A}$, then $\mathscr{V} = \Pi_{j=1}^n \Big(\sigma(a_j)\mathscr{U} - \mathscr{U}\sigma(a_j)\Big)$ is a convex subset of $\mathscr{X}^{(n)}$ and $(D(a_1), D(a_2), \ldots, D(a_n)) \in \overline{\mathscr{V}}^{weak}$. Thus for each finite subset F of $\mathscr{A}$, and $\varepsilon > 0$, there exists $x_{(F,\varepsilon)} \in \mathscr{U}$ such that

$$||D(a) - (\sigma(a)x_{(F,\varepsilon)} - x_{(F,\varepsilon)}\sigma(a))|| < \varepsilon \qquad (a \in F).$$

The family of such pairs (F, ε) is a directed if order $\leq$ given by

$$(F_1, \varepsilon_1) \leq (F_2, \varepsilon_2) \Leftrightarrow F_1 \subseteq F_2, \varepsilon_1 \leq \varepsilon_2.$$

Also we have

$$D(a) = \lim_{(F,\varepsilon)} \Big(\sigma(a)x_{(F,\varepsilon)} - x_{(F,\varepsilon)}\sigma(a)\Big).$$

(3) $\Rightarrow$ (1): Let $D : \mathscr{A} \longrightarrow \mathscr{X}^*$ be a σ-derivation. Then there exists a net $(x'_\alpha) \subseteq \mathscr{X}^*$ such that $D(a) = \lim_\alpha \Big(\sigma(a)x'_\alpha - x'_\alpha\sigma(a)\Big) \qquad (a \in \mathscr{A})$. By passing to a subnet we may assume that $w^* - \lim x'_\alpha = x'$ in $\mathscr{X}^*$ and then $D(a) = \sigma(a)x' - x'\sigma(a)$. Thus $\mathscr{A}$ is σ-amenable.

Theorem 2.3. *Let $\mathscr{A}$ be a Banach algebra and σ be a continuous homomorphism on $\mathscr{A}$. If $\mathscr{A}^{**}$ is σ''-amenable, then $\mathscr{A}$ is σ-amenable.*

Proof. Let $\mathscr{X}$ be a Banach $\mathscr{A}$-bimodule, and $D : \mathscr{A} \longrightarrow \mathscr{X}^{**}$ be a σ-derivation. Then by Lemma 2.1, $D'' : \mathscr{A}^{**} \longrightarrow \mathscr{X}^{****}$ is a σ''-derivation. Since $\mathscr{A}^{**}$ is σ''-amenable, then there exists $x^{(4)} \in \mathscr{X}^{****}$ such that $D''(a'') = \sigma''(a'')x^{(4)} - x^{(4)}\sigma''(a'')$, $(a'' \in \mathscr{A}^{**})$. We have $\mathscr{X}^{****} = \mathscr{X}^{**} \oplus (\mathscr{X}^*)^\perp$ (as $\mathscr{A}^{**}$-bimodules). Let $P : \mathscr{X}^{****} \longrightarrow \mathscr{X}^{**}$ be the natural projection. Then for each $a \in \mathscr{A}$, we have $D(a) = \sigma(a)P(x^{(4)}) - P(x^{(4)})\sigma(a)$, and so $D \in N^1_{(\sigma,\sigma)}(\mathscr{A}, \mathscr{X}^{**})$. Thus by above theorem, $\mathscr{A}$ is σ-amenable.

In the following we fined an easy equivalent condition for σ-amenability of a Banach algebra.

Proposition 2.1. *Let $\mathscr{A}$ be a Banach algebra and let σ be a continuous homomorphism on $\mathscr{A}$. Then $\mathscr{A}$ is a σ-amenable if and only if for every Banach algebra $\mathscr{B}$ and every injective homomorphism $\varphi : \mathscr{A} \longrightarrow \mathscr{B}$, $H^1_{(\sigma,\sigma)}(\mathscr{A}, B^{**}_{\varphi}) = 0$.*

Proof. One side is clear, so we prove the other side. Let $\mathscr{X}$ be a Banach $\mathscr{A}$-bimodule and $D : \mathscr{A} \longrightarrow \mathscr{X}^{**}$ be a σ-derivation. If $\phi : \mathscr{A} \longrightarrow \mathscr{A} \oplus_1 \mathscr{X}$ is defined by $\varphi(a) = (a, 0)$. Then φ is injective and $\varphi^{**} : \mathscr{A}^{**} \longrightarrow (\mathscr{A} \oplus_1 \mathscr{X})^{**}$ the second transpose of φ is a Banach algebra homomorphism and $((\mathscr{A} \oplus_1 \mathscr{X})_{\varphi})^{**} \simeq (\mathscr{A}^{**} \oplus_1 \mathscr{X}^{**})_{\varphi^{**}}$ as $\mathscr{A}^{**}$-bimodules. Then

$$H^1_{(\sigma,\sigma)}(\mathscr{A}, (\mathscr{A}^{**} \oplus_1 \mathscr{X}^{**})_{\varphi^{**}}) = H^1_{(\sigma,\sigma)}(\mathscr{A}, ((\mathscr{A} \oplus_1 \mathscr{X})_{\varphi})^{**}) = \{0\}. \tag{2.2}$$

Now we define $D_1 : \mathscr{A} \longrightarrow \mathscr{A}^{**} \oplus_1 \mathscr{X}^{**}$ by $D_1(a) = (0, D(a))$. For $a, b \in \mathscr{A}$ we have $D_1(ab) = D_1(a)\varphi^{**}(\widehat{b}) + \varphi^{**}(\widehat{a})D_1(b)$. Thus D_1 is a σ-derivation from $\mathscr{A}$ into $(\mathscr{A}^{**} \oplus_1 \mathscr{X}^{**})_{\varphi^{**}}$. By (2.2), D_1 is σ-inner. Therefore there exist $a'' \in \mathscr{A}^{**}, x'' \in \mathscr{X}^{**}$ such that

$$(0, D(a)) = D_1(a) = (a'', x'')(0, \sigma(a)) - (0, \sigma(a))(a'', x''),$$

Thus D is σ-inner. Therefore $H^1_{(\sigma,\sigma)}(\mathscr{A}, \mathscr{X}^{**}) = 0$, and by Theorem 2.2, $\mathscr{A}$ is σ-amenable.

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