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Module over pseudo-valuation ring and domain

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Abstract. Modules over principal ideal rings, dedekind rings and valuation rings have been discussed in the literature. In this article, we introduce and discuss the module over pseudo-valuation ring(PVR) and its submodule. Also as a byproduct we introduce the module over pseudo-valuation domain(PVD).

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1. Introduction and Preliminaries

Number of research articles are available upon modules over different types of rings and domains, modules over principal ideal rings have been discussed in the literature, modules over dedekind rings and valuation rings have been introduced by I. Kaplansky in [6]. Similarly modules over dedekind prime rings introduced by David Eisenbud and J. C. Robin in [3]. In this connection we introduced module over pseudovaluation ring and pseudovaluation domain, and also their submodules.

We begin with the basics of module. R-module M over the ring R consists of an abelian group $(M, +)$ and an operation $R \times M \to M$ (called scalar multiplication, usually just written by juxtaposition, i.e. as rx for $r \in R$ and $x \in M$) such that for all r, s in R, x, y in M, we have $r(x+y) = rx+ry$, $(r+s)x = rx+sx$, $(rs)x = r(sx)$, $1_R x = x$ if R has identity 1_R. Suppose M is a left R-module and N is a subgroup of M. Then N is a submodule (or R-submodule, to be more explicit) if for any $n \in N$ and any $r \in R$, the product rn is in N (or nr for a right module). Let R be a ring. let M be a $R-$module. A subset N of M is a submodule of $M \Longleftrightarrow$ (1) $N \neq \phi$. (2) $x + ry \in N \; \forall \, r \in R \; and \; \forall x, y \in N$ [2, proposition 1]. Similarly, it is well known that the left R-module M is said to be finitely generated if there exist $m_1, m_2, ..., m_n \in M$ such that In this case, we say that $\{m_1, m_2, ..., m_n\}$ is a set of generators for M. The module M is called cyclic if there exists $m \in M$ such that $M = Rm$. A free module is a module with a free basis, a linearly independent generating set. For an R-module M, the set $E = \{e_1, e_2, ...e_n\}$ is a free basis for M such that (1) If E is a generating set for M; that is to say, every element of M is a finite sum of elements of

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E multiplied by coefficients in R; (2) E is a free set, that is, if $r_1e_1 + r_2e_2 + ... + r_ne_n = 0$, then $r_1 = r_2 = ... = r_n = 0$ (where 0 is the zero element of M and 0 is the zero element of R).

Let R be an integral domain with quotient field K. A prime ideal P of R is called strongly prime if x, $y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$ (alternatively P is strongly prime if and only if $x^{-1}P \subset P$ whenever $x \in K\backslash R$) [5, Definition, page2]. A domain R is called a pseudo-valuation domain if every prime ideal of R is a strongly prime [5, Definition, page2]. An integral domain R is a pseudo-valuation domain if and only if for every nonzero $x \in K$, either $x \in R$ or $ax^{-1} \in R$ for every nonunit $a \in R$ [5, Theorem 1.5(3)]. Every valuation domain is a pseudo-valuation domain [5, Proposition. 1.1] but converse is not true for example the valuation domain V of the form $K + M$, where K is a field and M is the maximal ideal of V. If F is a proper subfield of K, then $R = F + M$ is a pseudo-valuation domain which is not a valuation domain. Whereas R and V have the same quotient field L and that M is the maximal ideal of R [4, Theorem A, page 560]. A quasi-local domain (R, M) is a pseudo-valuation domain if and only if $x^{-1}M \subset M$ whenever $x \in K\backslash R$ [5, Theorem. 1.4].

Throughout we always mean by a ring R, a commutative ring having unity 1.

2. Module over pseudo-valuation ring

A commutative ring R with 1 is called a pseudo-valuation ring (PVR) if for every a, b in R, either a divides b or b divides ac for each nonunit c in R.

We begin with the following definition.

Definition 1. If R is a pseudo-valuation ring and M be an additive abelian group then M is said to be R-module if for all $r, s, t \in R$, where t is a nonunit of R and $m, n \in M$ satisfies:

(1) Either $r/s\ m \in M$ or $s/rt\ m \in M$, for all r, s, $t \in R$, where t is a nonunit of R and $m \in M$.

(2) $(r + s)m = rm + sm$

(3) $r(m + n) = rm + rn$

(4) $(rs)m = r(sm)$

If one writes the scalar action as f_r so that $f_r(x) = r/sm$, and f for the map which takes each r to its corresponding map f_r, then the second axiom states that every f_r is a group homomorphism of M, and the other third and fourth axioms asserts that f is a ring homomorphism from R to the endomorphism ring $End(M)$.

Remark 1. Following *[1, Proposition 3(4)]*, an integral domain is a $PVD \Leftrightarrow$ for every $a, b \in R$ either a/b or b/ac for every non unit $c \in R$. An integral domain is a $PVR \Leftrightarrow$ it is a PVD

Theorem 1. *If R is a PVR, and M is an abelian group. then M is a module over $PVR \Leftrightarrow$ If $\forall\ r, s, t \in R$ where t is a nonunit and $m \in M$, either $r/s\ m \in M$ or $s/rt\ m \in M$.*

Proof. Let R is a PVR and M is a module then by definition for all $r, s, t \in R$ and $m \in M$, clearly either $r/s\ m \in M$ or $s/rt\ m \in M$. Conversely suppose M is an abelian group and R is a PVR, and either $r/s\ m \in M$ or $s/rt\ m \in M$, where $r, s, t \in R$ and $m \in M$. We have;

(1) Either $(r/s)\ m \in M$ or $s/rt\ m \in M$. If $s = 1$ then $rm \in M$, if $s \neq 1$ then $r/s \in R \Rightarrow R$ is a PVR and let $r/s = r_1 \Rightarrow r_1 m \in M$.

(2) It is followed by (1) that either $(r/s + a/b)m = (r/s\ m + a/b\ m) \in M$ or $(r/st + a/bc)m = (r/st\ m + a/bc\ m) \in M$.

(3) Either $r/s(m+n) = (r/s\ m + r/s\ n) \in M$ or $r/st\ (m+n) = (r/stm + r/stn) = M$, which is clear by (2).

(4) Either $(r/s \cdot a/b)m \in M$ or $(r/st \cdot a/bc)m \in M$ by definition $\forall\ r \in R$ and either $r/s\ m \in M$ or $s/rt\ m \in M$.

The above theorem reflects that M is a module. Thus we can conclude that the condition we put in definition1 is necessary and sufficient.

Example 1. Let $R = \mathbb{Q} + X\mathbb{Q}(y)[[X]]$, R is a PVR. Consider $\mathbb{R}[X]$ is an abelian group and define mapping $R \times \mathbb{R}[X] \to \mathbb{R}[X]$ i.e., $(a + xb(y))r \in \mathbb{R}[X]$ where $a \in \mathbb{Q}$, $b(y) \in \mathbb{Q}(y)[[X]]$ and $r = f(x) \in \mathbb{R}[X]$. For all $a + xb(y), c + xd(y)$ in R and r, s in $\mathbb{R}[X]$ we have;

(1) $[(a + xb(y))(c + xd(y))]r = a + xb(y)((c + xd(y))r)$.

(2) $[(a + xb(y) + c + xd(y)]r = [a + xb(y)]r + [c + xd(y)]r$.

(3) $a + xb(y)(r + s) = [a + xb(y)]r + [a + xb(y)]s$.

Definition 2. Let R be a pseudo-valuation ring and M be a module over PVR. A subset N of M is a submodule over pseudo-valuation ring if:

(1) $n - \acute{n} \in N\ \forall\ n, \acute{n} \in N$.

(2) $r \in R, n \in N$ either $r/s\ n \in N$ or $s/rt\ n \in N$.

Theorem 2. *Let R is a PVR and M is a module. A subset N of M is a submodule over $PVR \Leftrightarrow \forall\ r, s, t \in R$, $n \in N$ either $r/s\ n \in N$ or $s/rt\ n \in N$.*

Proof. If R is a PVR and N is a subset of M, where M is a module then by definition, we can write that $\forall\ r \in R$ and $n, \acute{n} \in N$, either $r/s\ n \in N$ or $s/rt\ n \in N$. Conversely, suppose that N is a submodule over PVR, since we have already define that $\forall\ r, s, t \in R$ where t is a non-unit, $n, n' \in N$ either $r/s\ n \in N$ or $s/rt\ n \in N$.Now we can easily conclude the properties of submodule as;

(1) $n - \acute{n} \in N\ \forall\ n, \acute{n} \in N$ (closure property holds).

(2) $r \in R, n \in N$ either $r/s\ n \in N$ or $s/rt\ n \in N$ (by definition).

which proves that N is a submodule.

Example 2. Let $R = \mathbb{Q} + X\mathbb{Q}(y)[[X]]$ is a PVR. Let $\mathbb{R}[X, Y]$ is a submodule over pseudo valuation ring if for all $a + xb(y), c + xd(y)$ in R and p, q in $\mathbb{R}[X, Y]$ where $a \in \mathbb{Q}$, $b(y) \in \mathbb{Q}(y)[[X]]$ and $p = f(x, y), q = g(x, y) \in \mathbb{R}[X, Y]$, we have

(1) $p - q \in \mathbb{R}[X, Y]\ \forall\ p, q \in \mathbb{R}[X, Y]$.

(2) $a + xb(y) \in R$, $p \in \mathbb{R}[X, Y]$, $a + xb(y)p \in \mathbb{R}[X, Y]$ and either $[a + xb(y)/c + xd(y)]p \in \mathbb{R}[X, Y]$ or $[a + xb(y)/(c + xd(y).m + n(y))]p \in \mathbb{R}[X, Y]$ (by definition).

3. Module over pseudo-valuation domain

In this section we defined module over pseudo-valuation domain and its submodule. By [5] let R be a domain with quotient field K. The following statements are equivalent.

(1) R is a pseudo-valuation domain.

(2) For each $x \in K - R$ and for each nonunit a of R, we have $(x + a)R = xR$.

(3) If module over pseudo-valuation domain R consists of an abelian group $(M, +)$ and an operation $R \times M \to M$ scalar multiplication, usually just written by juxtaposition.

Definition 3. Let K be the quotient field of R, $x \in K - R$, $a, b, r, s \in R$ and m $\in M$*(an additive abelian group)* be a module over R if;

(1) For $r \in R$, $m \in M$ and for each nonunit a of R, $x^{-1}am \in M$.

(2) $(rs)m = r(sm) = (x^{-1}ax^{-1}b)m = x^{-1}a(x^{-1}bm)$

(3) $(r + s)m = rm + sm$

(4) $r(m + n) = rm + rn$

Example 3. Let $R = \mathbb{Z}+x\mathbb{Q}[[X]]$ be a pseudo-valuation domain where $\mathbb{Q}[[X]]$ is a maximal ideal and take $\mathbb{Q}[X]$ be an additive abelian group. $\mathbb{Q}[X]$ is a module over pseudo-valuation domain, clearly $(a + bx)r \in \mathbb{Q}[X]$ where $a \in \mathbb{Z}$, $b \in \mathbb{Q}[[X]]$ and $r = f(x) \in \mathbb{Q}[X]$, $\forall$ $a + xb$, $c + xd$ is in R and r, s in $\mathbb{Q}[X]$ we have;

(1) $[(a + xb)(c + xd)]r = a + xb((c + xd)r)$

(2) $(a + xb + c + xd)r = a + xbr + c + xdr$

(3) $a + xb(r + s) = a + xbr + a + xbs$

Theorem 3. *If R is pseudo-valuation domain and M is an abelian group then M is module over $R \Leftrightarrow$ $(a + xb)m \in M$.*

Proof. Assume if M is a module over pseudo-valuation domain R then for each non-unit element a in R, $x^{-1}am \in M$ where $x \in k - R$.

Conversely, let $x^{-1}am \in M$, where $x \in k - R$, we have to prove that M is a module over pseudo-valuation domain R. As $x^{-1}am \in M \implies (x^{-1}ax^{-1}b)m \in M$, so we may write $(x^{-1}ax^{-1}b)m = x^{-1}a(x^{-1}bm)$.Also, if $x^{-1}am \in M$ then $(x^{-1}a + x^{-1}b)m \in M \implies (x^{-1}a + x^{-1}b)m = x^{-1}am + x^{-1}bm$.Finally $x^{-1}a\ (m + n) = x^{-1}am + x^{-1}an$

for all $x^{-1}a$, $x^{-1}b \in R$ and $m, n \in M$ where a and b are non-unit elements of R. As the scaler multiplication of module is given so we can write the other properties of a module.

Suppose M is R-module and N is a subgroup of M. N is said to be a submodule (or R-submodule, to be more explicit) if, for any $n \in N$, $r \in R$, the product rn is in N.

Definition 4. Let R be a pseudo-valuation domain and M be a $R-$module. If N is a subset of $R-$module M then N is a submodule of M if;

(1) $n - \acute{n} \in N$ for all n, $\acute{n} \in N$.

(2) For each $x \in K - R$, where K be a quotient field and for each nonunit a of R, $x^{-1}a \in R$, $n \in N$, $x^{-1}an \in N$.

Example 4. Suppose $M = \mathbb{C}[X]$ is a $R - module$ where $R = \mathbb{Z} + X\mathbb{Q}[[X]]$ is pseudo-valuation domain and $N = \mathbb{R}[X]$ is a subgroup of M. N is a submodule (or $R-submodule$, to be more explicit) since for any $n = f(x) \in N$ and any $a+xb$ in R, the product $n(a+xb)$ is in N where $a \in \mathbb{Z}, b = f(x) \in \mathbb{Q}[[X]]$.

Theorem 4. *If R is pseudo-valuation domain, M is an abelian group and N is a subgroup of M then N is $R - submodule$ of $M \iff x^{-1}an \in N$ where $x \in K - R$.*

Proof. Assume that N is $R-$submodule of M therefore N is closed under the action of ring elements, so we have $x^{-1}an \in N$ where $x \in K - R$ and a non-unit element a in R.Conversely, assume $x^{-1}an \in N$, where $x \in K - R$ we have to prove that N is $R-$submodule of M. As $x^{-1}an \in N$ we have;

(1) $x^{-1}an \in N \ \forall \ x^{-1}a \in R, \ n \in N$.

Given that N is a subgroup of M so it is closed under subtraction then 2nd condition of $R - submodule$ is satisfied so;

(2) $n - n' \in N \ \forall \ n, \ n' \in N$

Thus N is a $R-$submodule of M.

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