EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 7, No. 3, 2014, 267-288
ISSN 1307-5543 – www.ejpam.com

Characterization Theorems for Scale Invariance Property of Insurance Premium Calculation Principles

Mykola Pratsiovytyi, Vitaliy Drozdenko*

Dragomanov National Pedagogical University, Department of Higher Mathematics, Institute of Mathematics and Physics, 9 Pyrogov Str., room 460, Kyiv, UA-01601, Ukraine.

Abstract. Characterization theorems for the scale invariance property of the insurance premium calculation principles are presented. Theorems formulated in a form of necessary and sufficient conditions for the mentioned property to be hold. Conditions are imposed on the auxiliary functions with the help of which several methods of pricing of insurance contracts are defined. Presented theorems cover cases of mean value, insurer equivalent/zero utility, customer equivalent/zero utility, and Swiss premium calculation principles.

2010 Mathematics Subject Classifications: 91B30, 62P20, 62P05

Key Words and Phrases: Characterization theorem, insurance premium, scale invariance property, mean value principle, insurer/customer equivalent/zero utility principle, Swiss principle

1. Introduction

Let us consider a random variable X representing size of the insurance compensation related to a particular insurance pact. Premium to be paid for the risk X will be denoted as $\pi[X]$.

In majority of the cases the random variable X is assumed to be a non-negative one, i.e., it takes vale zero if the contract will not produce a claim and will be equal to the claim size if there will be a claim. In some case, however, negative values of variable X are also allowed; such negative values are often interpreted as compensations which have to be paid by the customer to the insurance company.

Let us now define several insurance premium calculation principles which we would like to investigate.

Net premium is defined as expected value of the losses associated with the risk X, i.e.,

$$\pi_{\text{net}}[X] = \mathsf{E}[X].$$

*Corresponding author.

Email addresses: `prats4@yandex.ru` (M. Pratsiovytyi), `drozdenko@yandex.ru` (V. Drozdenko)

Mean value premium for the risk X, which in the article will be denoted as $\pi_{\text{m.v.}}[X]$, based on a function $v(x) \in C_2(\mathbb{R})$ such that $v'(x) > 0$ and $v''(x) \geq 0$ for $x \in \mathbb{R}$, is defined as a solution to the equation

$$v(\pi_{\text{m.v.}}[X]) = \mathsf{E}[v(X)]. \tag{1}$$

Insurer equivalent utility premium for the risk X, which we denote as $\pi_{\text{i.e.u.}}[X]$, is defined as a solution to the equation

$$U(W) = \mathsf{E}[U(W + \pi_{\text{i.e.u.}}[X] - X)], \tag{2}$$

where W is insurer's capital at the moment when the contract is initiated, and the function $U(x) \in C_2(\mathbb{R})$ is insurer's utility function satisfying conditions $U'(x) > 0$ and $U''(x) \leq 0$ for $x \in \mathbb{R}$.

In some cases insurer's utility function is selected in such a way that the value $U(0)$ represents insurer's utility at the moment when the contract is initiated. In such cases equation (2) for the risk X is replaced by the equation

$$U(0) = \mathsf{E}[U(\pi_{\text{i.z.u.}}[X] - X)] \tag{3}$$

and corresponding method of pricing of the insurance contracts is called *insurer zero utility premium calculation principle*; here the premium is denoted as $\pi_{\text{i.z.u.}}[X]$.

Customer equivalent utility premium for the risk X, which we denote as $\pi_{\text{c.e.u.}}[X]$, is defined as a solution to the equation

$$u(\omega - \pi_{\text{c.e.u.}}[X]) = \mathsf{E}[u(\omega - X)], \tag{4}$$

where ω is customer's capital at the moment when the contract is initiated, and the function $u(x) \in C_2(\mathbb{R})$ is customer's utility function satisfying conditions $u'(x) > 0$ and $u''(x) \leq 0$ for $x \in \mathbb{R}$.

In the cases when customer's utility function is selected in such a way that the value $u(0)$ represents customer's utility at the moment when the contract is initiated, equation (4) for the risk X is replaced by the equation

$$u(-\pi_{\text{c.z.u.}}[X]) = \mathsf{E}[u(-X)] \tag{5}$$

and corresponding method of pricing of the insurance contracts is called *customer zero utility premium calculation principle*; here the premium is denoted as $\pi_{\text{c.z.u.}}[X]$.

Swiss premium for the risk X, which in the article will be denoted as $\pi_{\text{Swiss}}[X]$, based on a parameter $\Delta \in [0,1]$ and a function $V(x) \in C_2(\mathbb{R})$ such that $V'(x) > 0$ and $V''(x) \geq 0$ for $x \in \mathbb{R}$, is defined as a solution to the equation

$$V((1-\Delta)\pi_{\text{Swiss}}[X]) = \mathsf{E}[V(X - \Delta\pi_{\text{Swiss}}[X])]. \tag{6}$$

Observe that, in the case of $\Delta = 0$, Swiss premium principle is equivalent to mean value premium principle with $v(x) := V(x)$.

Observe also that, in the case of $\Delta = 1$, equation (6) can be rewritten as

$$-V(0) = \mathsf{E}[-V(-(\pi[X] - X))].$$

Therefore, in the case of $\Delta = 1$, Swiss premium principle is equivalent to insurer zero utility premium principle with insurer's utility function $U(x) := -V(-x)$.

Sometimes described methods of pricing of the insurance contracts are applied to some special classes of risks: as an example of such a class one can mention the class of all non-negative risks, alternatively one could mention the class of all non-negative risks bounded from above by some fixed real value, etc. In such cases domains of the functions $v(x)$, $U(x)$, $u(x)$, and $V(x)$ could be subsets of $\mathbb{R}$ such that equations (1) – (6) (here choice of the equation depends on the chosen method of pricing) will preserve their correct mathematical meaning for all risks from the considered class, moreover, monotonicity and concavity-convexity properties of the functions $v(x)$, $U(x)$, $u(x)$, and $V(x)$ should also be preserved. It is interesting to see that in the case of subjecting of the describe premium principles to pricing of some special classes of risks, classes of the auxiliary functions defining scale invariant premiums can be larger than in the general case. Theorems 2 and 5 as well as Corollary 3 demonstrate examples of just such situations.

We will say that a premium calculation principle $\pi[X]$ possesses *scale invariance property* if for any admissible risk X and any positive real constant Θ the following equation holds

$$\pi[\Theta X] = \Theta \pi[X]. \tag{7}$$

More information about just defined methods of pricing of the insurance contracts as well as properties that can be possessed by an insurance premium calculation principles can be found, for example, in Asmussen and Albrecher [1], Boland [2], Bowers *et al.* [3], Bühlmann [4], Dickson [5], Gerber [6], de Vylder *et al.* [7], de Vylder *et al.* [8], Kaas *et al.* [9], Kremer [10], Rolski *et al.* [11], Straub [12].

We would like to emphasize that the research related to theorems of characterization type for properties possessed by certain insurance premium calculation principles was initiated by the Swiss mathematician Hans-Ulrich Gerber, see Gerber [6]. Corresponding theorems for the scale invariance property were still missing in the literature.

2. Mean Value Premium Principle

The following theorem describes necessary and sufficient conditions under which mean value premium calculation principle possesses scale invariance property.

Theorem 1. *Mean value premium calculation principle possesses scale invariance property if and only if $v(x) = ax + b$, for $a > 0$, i.e., only in the case when it coincides with net premium principle.*

Proof. Let us at the beginning prove the sufficiency of the statement. From equation (1) in the case of $v(x) = ax + b$, for $a > 0$, it follows

$$a\pi_{\text{m.v.}}[X] + b = \mathsf{E}[aX + b] = a\mathsf{E}[X] + b,$$

thus

$$\pi_{\text{m.v.}}[X] = \mathsf{E}[X] = \pi_{\text{net}}[X].$$

On the other hand, again from equation (1) for any $\Theta > 0$ it follows

$$a\pi_{\text{m.v.}}[\Theta X] + b = \mathsf{E}[a\Theta X + b] = a\Theta\mathsf{E}[X] + b,$$

so we get

$$\pi_{\text{m.v.}}[\Theta X] = \Theta\mathsf{E}[X] = \Theta\pi_{\text{m.v.}}[X],$$

and we see that scale invariance property holds in this particular case. This completes the proof of the sufficiency. Let us now check the necessity.

Note that, by the definition, scale invariance property for a particular premium calculation principle holds if equation (7) holds for any admissible risk X.

To show that mean value premium calculation principle with non-linear functions $v(x)$ will not possess scale invariance property, we will consider a risk X which takes only two possible values namely 0 and t with probabilities $1-p$ and p respectively. The risk X in this case can be viewed as a random function of two parameters, namely p and t, and therefore within the proof of Theorem 1 will be denoted as X_p^t.

For the described risk X_p^t, equation (1) will take the following form

$$v(\pi_{\text{m.v.}}[X_p^t]) = pv(t) + (1-p)v(0). \tag{8}$$

Substituting $p = 0$ into (8), obtain

$$v(\pi_{\text{m.v.}}[X_0^t]) = v(0). \tag{9}$$

Since the function $v(\cdot)$ is a strictly increasing function, then from the equation (9) it follows

$$\pi_{\text{m.v.}}[X_0^t] = 0. \tag{10}$$

Let us now calculate partial derivatives with respect to the parameter p from both sides of the equation (8)

$$v'(\pi_{\text{m.v.}}[X_p^t]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] = v(t) - v(0). \tag{11}$$

Substitution of the value $p = 0$ into (11) yields

$$v'(\pi_{\text{m.v.}}[X_0^t]) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} \right) = v(t) - v(0). \tag{12}$$

Since the function $v(\cdot)$ is a strictly increasing function, then $v'(0) > 0$. Therefore, taking into account identity (10) we get from (12) a representation for the partial derivative of the premium with respect to the parameter p at the point $p = 0$, namely,

$$\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} = \frac{v(t) - v(0)}{v'(0)}. \tag{13}$$

For any $\Theta > 0$ equation (1) for the risk ΘX_p^t in the case of scale invariant mean value principle will have the following form

$$v(\Theta\pi_{\text{m.v.}}[X_p^t]) = pv(\Theta t) + (1-p)v(0). \tag{14}$$

Taking partial derivatives with respect to the parameter p from both sides of the equation (14), obtain

$$v'(\Theta\pi_{\text{m.v.}}[X_p^t]) \cdot \Theta \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] = v(\Theta t) - v(0). \tag{15}$$

Substituting $p = 0$ into (15), using identity (10) as well as inequality $v'(0) > 0$, we again get a representation for the partial derivative of the premium with respect to the parameter p at the point $p = 0$, namely,

$$\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} = \frac{v(\Theta t) - v(0)}{v'(0) \cdot \Theta}. \tag{16}$$

Observe that equations (13) and (16) have equal left-hand sides, this means that their right-hand sides also have to be equal. In this way we finally get an equation which function $v(\cdot)$ has to satisfy in the case of scale invariant mean value premium calculation principle, namely,

$$\frac{v(t) - v(0)}{v'(0)} = \frac{v(\Theta t) - v(0)}{v'(0) \cdot \Theta}. \tag{17}$$

Equation (17) can be simplified to the following one

$$v(\Theta t) - v(0) = \Theta \cdot (v(t) - v(0)). \tag{18}$$

Calculating partial derivatives with respect to the parameter t from both sides of the equation (18), we get

$$\Theta v'(\Theta t) = \Theta v'(t)$$

or equivalently, after cancelation of Θ factor,

$$v'(\Theta t) = v'(t). \tag{19}$$

By fixing the parameter t in equation (19) to a positive value and varying values of the parameter Θ we will make $v'(\Theta t)$ a function of changing variable defined on $\mathbb{R}_+$ while the value $v'(t)$ will be fixed to a constant. By doing this we will see that the function $v'(x)$ will take for all $x > 0$ one and the same value, let us denote this value by a_1. In a very similar way, by fixing the parameter t in equation (19) to a negative value and varying values of the parameter Θ we will make $v'(\Theta t)$ a function of changing variable defined on $\mathbb{R}_-$ while the value $v'(t)$ will be fixed to a constant. In this way we will see that the function $v'(x)$ will take for all $x < 0$ one and the same value, let us denote this value by a_2. Since the function $v(x)$ was twice differentiable, then the function $v'(x)$ must be continuous, this yields

$$a_1 = a_2 = v'(0) =: a,$$

hence

$$v'(x) = a, \text{ for } x \in \mathbb{R}.$$

Integrating function $v'(x)$, we get

$$v(x) = ax + b, \text{ for a constant } b,$$

Initial assumption of positivity of first derivative of the function $v(x)$ gives us additional restriction on parameter a: parameter a must be a strictly positive constant. This completes the proof of Theorem 1. □

From the proof of Theorem 1 it follows that $v(x) = ax + b$, for $a > 0$, is the only case when mean value premium calculation principle coincides with net premium principle. Indeed, let us assume that for some function $v(x)$, different from the linear function, mean value premium calculation principle will be equivalent to net premium principle. Then, due to linearity property of the expectation, such method of pricing must be scale invariant, however, in the proof of Theorem 1 was shown that mean value premium calculation principle possesses scale invariance property if and only if $v(x) = ax + b$, for $a > 0$, so we come to a contradiction.

Using similar argumentations one can conclude that: $U(x) = ax + b$, for $a > 0$, is the only case when insurer's equivalent/zero utility premium calculation principle coincides with net premium principle (see Theorem 3); $u(x) = ax + b$, for $a > 0$, is the only case when customer's equivalent/zero utility premium calculation principle coincides with net premium principle (see Theorem 4); and $V(x) = ax + b$, for $a > 0$, is the only case when Swiss premium calculation principle coincides with net premium principle (see Theorem 6).

As was already mentioned, in the case when mean value premium principle is applied to a special class of risks, it is enough to define the function $v(x)$ on a subset $A \subset \mathbb{R}$ preserving monotonicity and convexity properties, i.e., $v(x)$ must be such that $v'(x) > 0$ and $v''(x) \geq 0$ for all $x \in A$, and, moreover, equation (1) must preserve its correct mathematical meaning for all risks from the mentioned class. It is interesting to see that in the case of subjecting of mean value principle to pricing of only strictly positive risks, the class of functions $v(x)$ producing scale invariant premiums is larger than in the general case. We believe that this observation deserves to be formulated in a form of theorem.

Theorem 2. *Mean value premium calculation principle subjected to consideration of only strictly positive risks possesses scale invariance property if and only if* $v(x) = ax^{\kappa} + b$, *for* $a > 0$ *and* $\kappa \geq 1$, *defined for* $x \in (0, +\infty)$.

Observe that for the function $v(x) = ax^{\kappa} + b$ with $a > 0$ and $\kappa > 1$ condition $v'(x) > 0$ violates at the point $x = 0$, therefore, statement of Theorem 2 does not contradict statement of Theorem 1.

Proof. Since in the case of strictly positive risk X we get $\mathsf{E}[X] > 0$, then, combining Jensen inequality

$$v(\mathsf{E}[X]) \leq \mathsf{E}[v(X)]$$

with definition equation (1), we see that mean value premium calculation principle will be well-defined if the function $v(x)$ will be defined just for $x \in (0, +\infty)$ with preservation of monotonicity and convexity assumptions, i.e., the function $v(x)$ must be defined on $(0,+\infty)$ such that $v'(x) > 0$ and $v''(x) \geq 0$ for all $x \in (0,+\infty)$.

Let us from the beginning prove the sufficiency of the statement. Indeed in the case of $v(x) = ax^\kappa + b$, with $a > 0$ and $\kappa \geq 1$, for any strictly positive risk X equation (1) will have the following form

$$a(\pi_{\text{m.v.}}[X])^\kappa + b = \mathsf{E}[aX^\kappa + b] = a\mathsf{E}[X^\kappa] + b,$$

therefore, in the considered case

$$\pi_{\text{m.v.}}[X] = (\mathsf{E}[X^\kappa])^{1/\kappa}.$$

On the other hand, for the same function $v(x)$, the same risk X, and any $\Theta > 0$, from equation (1) it follows

$$a(\pi_{\text{m.v.}}[\Theta X])^\kappa + b = \mathsf{E}[a(\Theta X)^\kappa + b] = a\Theta^\kappa \mathsf{E}[X^\kappa] + b$$

so, here we get

$$\pi_{\text{m.v.}}[\Theta X] = \Theta(\mathsf{E}[X^\kappa])^{1/\kappa} = \Theta\pi_{\text{m.v.}}[X],$$

and as we see, mean value premium calculation principle subjected to consideration of only strictly positive risks possesses scale invariance property in the case of $v(x) = ax^\kappa + b$, for $a > 0$ and $\kappa \geq 1$, defined for $x \in (0,+\infty)$.

Let us now switch to the statement of the necessity. In order to show that mean value premium calculation principle subjected to consideration of only strictly positive risks with all other types of function $v(x)$ will not possess scale invariance property, we will consider a risk X taking values $\varepsilon > 0$ and 1 with probabilities p and $1-p$ respectively. Being a random function of the parameters ε and p, the risk X within the proof of Theorem 2 will be denoted as X_p^ε.

For the described risk X_p^ε equation (1) will have the following form

$$v(\pi_{\text{m.v.}}[X_p^\varepsilon]) = pv(\varepsilon) + (1-p)v(1). \tag{20}$$

From the equation (20) it follows

$$v(\pi_{\text{m.v.}}[X_0^\varepsilon]) = 0 \cdot v(\varepsilon) + 1 \cdot v(1),$$

moreover, since $v(x)$ is a strictly increasing function, then

$$\pi_{\text{m.v.}}[X_0^\varepsilon] = 1. \tag{21}$$

Calculating partial derivatives with respect to the parameter p from both sides of the equation (20), obtain

$$v'(\pi_{\text{m.v.}}[X_p^\varepsilon]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon] = v(\varepsilon) - v(1). \tag{22}$$

Substituting $p = 0$ into equation (22), we get

$$v'(\pi_{\text{m.v.}}[X_0^\varepsilon]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0} = v(\varepsilon) - v(1). \tag{23}$$

Using (21) equation (23) can be rewritten as

$$v'(1)\cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0} = v(\varepsilon)-v(1). \tag{24}$$

Let us now calculate partial derivatives with respect to the parameter p from both sides of the equation (22)

$$v''(\pi_{\text{m.v.}}[X_p^\varepsilon])\cdot\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\right)^2 + v'(\pi_{\text{m.v.}}[X_p^\varepsilon])\cdot\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]=0. \tag{25}$$

Substituting $p=0$ into the equation (25), and using identity (21), obtain

$$v''(1)\cdot\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2 + v'(1)\cdot\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)=0. \tag{26}$$

Taking ε small enough, namely $\varepsilon<1$, and taking into account strict monotonicity of the function $v(x)$, without of loss of generality, using (24), we may conclude that

$$\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\neq 0, \tag{27}$$

hence, equation (26) can be rewritten as

$$\frac{v''(1)}{v'(1)} = -\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)\Bigg/\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2. \tag{28}$$

For any $\Theta>0$, equation (1) for the risk ΘX_p^ε will take the following form

$$v(\pi_{\text{m.v.}}[\Theta X_p^\varepsilon]) = pv(\Theta\varepsilon)+(1-p)v(\Theta). \tag{29}$$

In the case of scale invariant mean value premium principle equation (29) can be rewritten as

$$v(\Theta\pi_{\text{m.v.}}[X_p^\varepsilon]) = pv(\Theta\varepsilon)+(1-p)v(\Theta). \tag{30}$$

Calculating second partial derivative with respect to p from both sides of the equation (30), obtain

$$v''(\Theta\pi_{\text{m.v.}}[X_p^\varepsilon])\cdot\Theta^2\cdot\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\right)^2 + v'(\Theta\pi_{\text{m.v.}}[X_p^\varepsilon])\cdot\Theta\cdot\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]=0. \tag{31}$$

Substituting $p=0$ into the equation (31), canceling Θ factor, and using identity (21), we get

$$v''(\Theta)\cdot\Theta\cdot\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2 + v'(\Theta)\cdot\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)=0. \tag{32}$$

Since $v'(\Theta) > 0$, then using relation (27), equation (32) can be rewritten as

$$\frac{v''(\Theta)\cdot\Theta}{v'(\Theta)} = -\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)\bigg/\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2. \tag{33}$$

Observe that equations (28) and (33) have equal right-hand sides, this means that their left-hand sides also have to be equal, in this way we finally get an equation which the function $v(x)$ has to satisfy in the case of scale invariant mean value premium calculation principle subjected to consideration of only strictly positive risks, namely,

$$\frac{v''(\Theta)\cdot\Theta}{v'(\Theta)} = \frac{v''(1)}{v'(1)}, \text{ for all } \Theta > 0. \tag{34}$$

Assigning $v''(1)/v'(1) =: \varkappa$ (since $v''(1) \geq 0$ and $v'(1) > 0$ then $\varkappa \geq 0$) and making substitution $z(\Theta) := v'(\Theta)$ equation (34) can be rewritten in the following equivalent form

$$\frac{dz}{z} = \varkappa\frac{d\Theta}{\Theta},$$

therefore

$$\log(z(\Theta)) = \varkappa\log(\Theta) + \log(C_1), \text{ for some constant } C_1 > 0,$$

and the function $z(\Theta)$ itself will have a form

$$z(\Theta) = C_1\Theta^\varkappa.$$

Switching back to the function $v'(\cdot)$, and switching to the original parameter $x \in (0, +\infty)$, obtain

$$v'(x) = C_1 x^\varkappa.$$

Taking antiderivative, we get

$$v(x) = \frac{C_1}{\varkappa+1}x^{\varkappa+1} + C_2,$$

therefore, the function $v(x)$ must be a function of the form

$$v(x) = ax^\kappa + b, \text{ for some real constants } a,\ b, \text{ and } \kappa.$$

Moreover, since $C_1 > 0$ and $\varkappa \geq 0$ then $a > 0$, and since $\varkappa \geq 0$ then $\kappa \geq 1$.

This completes the proof of Theorem 2. □

3. Insurer Equivalent Utility Premium Principle

Conditions under which scale invariance property will be satisfied by insurer equivalent utility premium calculation principle are described by the following theorem.

Theorem 3. *Insurer equivalent utility premium calculation principle possesses scale invariance property if and only if $U(x) = ax + b$, for $a > 0$, i.e., only in the case when it coincides with net premium principle.*

Proof. We start from the sufficiency. In the case of $U(x) = ax + b$, for $a > 0$, and any initial capital W, from the equation (2) it follows

$$aW + b = \mathsf{E}[aW + a\pi_{\text{i.e.u.}}[X] - aX + b] = aW + a\pi_{\text{i.e.u.}}[X] - a\mathsf{E}[X] + b,$$

thus

$$\pi_{\text{i.e.u.}}[X] = \mathsf{E}[X] = \pi_{\text{net}}[X].$$

From the equation (2), for any $\Theta > 0$, any initial capital W, and the same utility function, it follows

$$aW + b = \mathsf{E}[aW + a\pi_{\text{i.e.u.}}[\Theta X] - a\Theta X + b] = aW + a\pi_{\text{i.e.u.}}[\Theta X] - a\Theta\mathsf{E}[X] + b,$$

hence

$$\pi_{\text{i.e.u.}}[\Theta X] = \Theta\mathsf{E}[X] = \Theta\pi_{\text{i.e.u.}}[X],$$

and we see that scale invariance property holds in this particular case.

The proof of sufficiency was completed, so we switch to the necessity.

To show that insurer equivalent utility premium calculation principle with non-linear insurer's utility function $U(x)$ will not possess scale invariance property, we will choose a risk X which takes only two possible values, namely, 0 and t with probabilities $1 - p$ and p respectively. The risk X can in this case be considered as a random function of two parameters, namely p and t, and, therefore, within the proof of Theorem 3 it will be denoted by X_p^t.

For any insurer's initial capital W, equivalent utility equation (2) for the risk X_p^t will take the following form

$$U(W) = U(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \cdot p + U(W + \pi_{\text{i.e.u.}}[X_p^t]) \cdot (1 - p). \tag{35}$$

Substituting $p = 1$ into equation (35), we get

$$\begin{aligned} U(W) =& U(W + \pi_{\text{i.e.u.}}[X_1^t] - t) \cdot 1 + U(W + \pi_{\text{i.e.u.}}[X_1^t]) \cdot 0 \\ =& U(W + \pi_{\text{i.e.u.}}[X_1^t] - t). \end{aligned} \tag{36}$$

Since $U(x)$ is a strictly increasing function, then identity (36) yields

$$\pi_{\text{i.e.u.}}[X_1^t] = t. \tag{37}$$

Taking partial derivative with respect to p from both sides of the equation (35), obtain

$$\begin{aligned} 0 =& U'(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] \cdot p + U(W + \pi_{\text{i.e.u.}}[X_p^t] - t) \\ &+ U'(W + \pi_{\text{i.e.u.}}[X_p^t]) \cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] \cdot (1 - p) - U(W + \pi_{\text{i.e.u.}}[X_p^t]). \end{aligned} \tag{38}$$

Substituting $p = 1$ into the equation (38), and using identity (37), obtain an equation

$$U'(W) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\bigg|_{p=1} \right) = U(W + t) - U(W). \tag{39}$$

Since we a searching for the conditions when the premium calculation principle will be scale invariant, then for any insurer's initial capital W, and any positive constant Θ, insurer's equivalent utility equation (2) for the risk ΘX_p^t can be written in the following way

$$U(W) = U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t] - \Theta t) \cdot p + U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t]) \cdot (1 - p). \tag{40}$$

Calculating partial derivatives with respect to p from both sides of equation (40), obtain

$$\begin{aligned} 0 =& U'(W + \Theta\pi_{\text{i.e.u.}}[X_p^t] - \Theta t) \cdot \Theta \cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] \cdot p + U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t] - \Theta t) \\ &+ U'(W + \Theta\pi_{\text{i.e.u.}}[X_p^t]) \cdot \Theta \cdot \frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t] \cdot (1-p) - U(W + \Theta\pi_{\text{i.e.u.}}[X_p^t]). \end{aligned} \tag{41}$$

Substituting $p = 1$ into the equation (41), and using identity (37), we get an equation

$$U'(W) \cdot \Theta \cdot \left(\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\bigg|_{p=1} \right) = U(W + \Theta t) - U(W). \tag{42}$$

Since $\Theta > 0$, then equation (42) can be rewritten as

$$U'(W) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{i.e.u.}}[X_p^t]\bigg|_{p=1} \right) = \frac{U(W + \Theta t) - U(W)}{\Theta}. \tag{43}$$

Observe that equations (39) and (43) have equal left-hand sides, hence their right-hand sides also have to be equal; this finally gives us an equation which insurer's utility function has to satisfy for the premium calculation principle to be scale invariant, namely

$$U(W + t) - U(W) = \frac{U(W + \Theta t) - U(W)}{\Theta}. \tag{44}$$

Taking partial derivatives with respect to parameter t from both sides of (44) yields

$$U'(W + t) = U'(W + \Theta t). \tag{45}$$

By fixing values of the parameters W and t, and changing values of the parameter Θ, we will make $U'(W + \Theta t)$ a function of changing variable while the value $U'(W + t)$ will be a fixed constant. Using this technique and taking into account monotonicity of function $U(\cdot)$ and continuity of function $U'(\cdot)$, since $U(\cdot) \in C_2(\mathbb{R})$, using equation (45) we conclude that

$$U'(x) = a > 0, \text{ for } x \in \mathbb{R}.$$

Integration yields

$$U(x) = ax + b, \text{ for } x \in \mathbb{R}, \text{ and constant } a > 0.$$

Let us give also a geometrical interpretation showing that non-linear insurer's utility functions will not satisfy equation (44). Let us consider two triangles: the first one will be formed by the points $(W, U(W))$, $(W + t, U(W))$, $(W + t, U(W + t))$ and the second one will be formed by the points $(W, U(W))$, $(W+\Theta t, U(W))$, $(W+\Theta t, U(W+\Theta t))$. Observe that both triangles are right-angled triangles, they have a common vertex at the point $(W, U(W))$, and, moreover, the points $(W, U(W))$, $(W+t, U(W))$, and $(W+\Theta t, U(W))$ lie on the same straight line. With out of loss of generality equation (44) can be rewritten in the following way

$$\frac{U(W+t)-U(W)}{(W+t)-W} = \frac{U(W+\Theta t)-U(W)}{(W+\Theta t)-W}. \tag{46}$$

Geometrically, equation (46) can be interpreted as follows: ratio of the catheti in one of the triangles is equal to ratio of the catheti in the other triangle, hence our two considered triangles are similar triangles. Due to the common vertex, catheti which lie on a common straight line, and the vertexes which lie on the same half plane with respect to the mentioned line, we conclude that the hypotenuses will also lie on a common straight line; in other words, the points $(W+\Theta t, U(W+\Theta t))$, for any initial capital W, all non-zero t, and all $\Theta > 0$, will form a straight line. So, we can conclude that insurer's utility function $U(x)$ is a linear function, i.e., a function of the form $U(x) = ax + b$. Initial assumption of positivity of first derivative of the function $U(x)$ gives us additional restriction on the parameter a: parameter a must be a strictly positive constant. This completes the proof of Theorem 3. □

Since initial capital used in the proof of Theorem 3 was chosen arbitrary and no additional restrictions on insurer's initial capital were involved, then we can formulate the following corollary to Theorem 3.

Corollary 1. *Insurer zero utility premium calculation principle possesses scale invariance property if and only if $U(x) = ax + b$, for $a > 0$, i.e., only in the case when it coincides with net premium principle.*

Observe that analog of Theorems 2 and 5 does not exist for insurer equivalent/zero utility premium calculation principle.

4. Customer Equivalent Utility Premium Principle

The following theorem describes conditions under which scale invariance property will be satisfied by customer equivalent utility premium calculation principle.

Theorem 4. *Customer equivalent utility premium calculation principle possesses scale invariance property if and only if $u(x) = ax + b$, for $a > 0$, i.e., only in the case when it coincides with net premium principle.*

Proof. We begin again from the sufficiency. From the definition equation (4), for $u(x) = ax + b$ with $a > 0$, and any initial capital ω, it follows

$$a\omega - a\pi_{\text{c.e.u.}}[X] + b = \mathsf{E}[a\omega - aX + b] = a\omega - a\mathsf{E}[X] + b,$$

hence

$$\pi_{\text{c.e.u.}}[X] = \mathsf{E}[X] = \pi_{\text{net}}[X].$$

On the other hand, from the equation (4), for any $\Theta > 0$, any initial capital ω, and the same utility function, it follows

$$a\omega - a\pi_{\text{c.e.u.}}[\Theta X] + b = \mathsf{E}[a\omega - a\Theta X + b] = a\omega - a\Theta\mathsf{E}[X] + b,$$

thus

$$\pi_{\text{c.e.u.}}[\Theta X] = \Theta\mathsf{E}[X] = \Theta\pi_{\text{c.e.u.}}[X],$$

and we see that scale invariance property holds in this particular case.

The proof of the sufficiency was finished, so we can start to prove the necessity.

To show that customer equivalent utility premium calculation principle with non-linear customer utility function $u(x)$ will not possess scale invariance property, we will choose a risk X which takes only two possible values, namely 0 and t with probabilities $1-p$ and p respectively. The risk X can in this case be considered as a random function of two parameters, namely p and t, and, therefore, within the proof of Theorem 4 it will be denoted as X_p^t.

For any customer's initial capital ω, customer's equivalent utility equation (4) for the risk X_p^t will take the following form

$$u(\omega - \pi_{\text{c.e.u.}}[X_p^t]) = u(\omega - t) \cdot p + u(\omega) \cdot (1-p). \tag{47}$$

Substituting $p = 0$ into the equation (47), obtain

$$u(\omega - \pi_{\text{c.e.u.}}[X_0^t]) = u(\omega). \tag{48}$$

Since $u(x)$ is a strictly increasing function, then the equation (48) yields

$$\pi_{\text{c.e.u.}}[X_0^t] = 0. \tag{49}$$

Let us now take partial derivatives with respect to p from both sides of the equation (47)

$$-u'(\omega - \pi_{\text{c.e.u.}}[X_p^t]) \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] = u(\omega - t) - u(\omega). \tag{50}$$

Substituting $p = 0$ into the equation (50) and using identity (49), we get

$$-u'(\omega) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} \right) = u(\omega - t) - u(\omega). \tag{51}$$

Since we are searching for the conditions when the premium calculation principle will be scale invariant, then for any customer's initial capital ω and any positive constant Θ, customer's equivalent utility equation (4) for the risk ΘX_p^t can be written in the following way

$$u(\omega - \Theta\pi_{\text{c.e.u.}}[X_p^t]) = u(\omega - \Theta t) \cdot p + u(\omega) \cdot (1-p). \tag{52}$$

We are now taking partial derivative with respect to p from both sides of the equation (52)

$$-u'(\omega - \pi_{\text{c.e.u.}}[X_p^t]) \cdot \Theta \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] = u(\omega - \Theta t) - u(\omega). \tag{53}$$

Substituting $p = 0$ into the equation (53), and using identity (49), we get

$$-u'(\omega)\cdot\Theta\cdot\left(\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0}\right) = u(\omega-\Theta t) - u(\omega). \tag{54}$$

Since $\Theta > 0$, then the equation (54) can be rewritten as follows

$$-u'(\omega)\cdot\left(\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0}\right) = \frac{u(\omega-\Theta t) - u(\omega)}{\Theta}. \tag{55}$$

Observe that equations (51) and (55) have equal left-hand sides, this means that their right-hand sides also have to be equal, and we finally get an equation which customer's utility function $u(x)$ has to satisfy for the premium calculation principle to be scale invariant, namely

$$u(\omega-t) - u(\omega) = \frac{u(\omega-\Theta t) - u(\omega)}{\Theta}. \tag{56}$$

Taking partial derivatives with respect to the parameter t from both sides of (56), we get

$$u'(\omega-t) = u'(\omega-\Theta t). \tag{57}$$

By fixing values of the parameters ω and t, and changing values of the parameter Θ, we will make $u'(\omega-\Theta t)$ a function of changing variable while the value $u'(\omega-t)$ will be a fixed constant. Using this technique and taking into account monotonicity of function $u(\cdot)$ and continuity of function $u'(\cdot)$, since $u(\cdot)\in C_2(\mathbb{R})$, using the equation (57) we may conclude that

$$u'(x) = a > 0, \text{ for } x\in\mathbb{R}.$$

Integration yields

$$u(x) = ax + b, \text{ for } x\in\mathbb{R}, \text{ and constant } a > 0.$$

Let us give also a geometrical interpretation showing that non-linear customer utility functions will not satisfy equation (56). Let us consider two triangles: the first one will be formed by the points $(\omega-t, u(\omega-t))$, $(\omega, u(\omega-t))$, $(\omega, u(\omega))$ and the second one will be formed by the points $(\omega-\Theta t, u(\omega-\Theta t))$, $(\omega, u(\omega-\Theta t))$, $(\omega, u(\omega))$. Observe that both triangles are right-angled triangles, they have a common vertex at the point $(\omega, u(\omega))$, and, moreover, the points $(\omega, u(\omega))$, $(\omega, u(\omega-t))$, and $(\omega, u(\omega-\Theta t))$ lie on the same straight line. With out of loss of generality equation (56) can be rewritten in the following way

$$\frac{u(\omega)-u(\omega-t)}{\omega-(\omega-t)} = \frac{u(\omega)-u(\omega-\Theta t)}{\omega-(\omega-\Theta t)}. \tag{58}$$

Geometrically, equation (58) can be interpreted as follows: ratio of the catheti in one of the triangles is equal to ratio of the catheti in the other triangle, and hence our two considered triangles are similar triangles. Due to the common vertex, catheti which lie on a common straight line, and vertexes which lie on the same half plane with respect to the mentioned line, we conclude that hypotenuses will also lie on a common straight line; in other words, the points $(\omega - \Theta t, u(\omega - \Theta t))$, for any initial capital ω, any non-zero t, and every $\Theta > 0$, will form a straight line. So, we can conclude that the customer's utility function $u(x)$ is a linear function, i.e., a function of the form $u(x) = ax + b$. Initial assumption of positivity of first derivative of the function $u(x)$ gives us additional restriction on the parameter a: parameter a must be a strictly positive constant. This completes the proof of Theorem 4. □

Since initial capital used in the proof of Theorem 4 was chosen arbitrary and no restrictions on customer's initial capital were imposed, then we can formulate the following corollary to Theorem 4.

Corollary 2. *Customer zero utility premium calculation principle possesses scale invariance property if and only if* $u(x) = ax+b$*, for* $a > 0$*, i.e., only in the case when it coincides with net premium principle.*

In the case when customer zero utility premium calculation principle is applied to a special class of risks, it is enough to define the utility function $u(x)$ on a subset $A \subset \mathbb{R}$ preserving monotonicity and concavity properties, i.e., $u(x)$ must be such that $u'(x) > 0$ and $u''(x) \leq 0$ for all $x \in A$, and, moreover, equation (5) must preserve its correct mathematical meaning for all risks from the mentioned class. It is interesting to see that in the case of subjecting of customer zero utility premium calculation principle to pricing of only strictly positive risks the class of the functions $u(x)$ producing scale invariant premiums is larger than in the general case. We believe that this observation deserves to be formulated in a form of theorem.

The following theorem is valid only for customer zero utility principle and not for customer equivalent utility principle.

Theorem 5. *Customer zero utility premium calculation principle subjected to consideration of only strictly positive risks possesses scale invariance property if and only if* $u(x) = -a(-x)^{\kappa} + b$*, for* $a > 0$ *and* $\kappa \geq 1$*, defined for* $x \in (-\infty, 0)$*.*

Observe that for the function $u(x) = -a(-x)^{\kappa}+b$ with $a > 0$ and $\kappa > 1$ condition $u'(x) > 0$ violates at the point $x = 0$, therefore, statement of Theorem 5 does not contradict statement of Theorem 4.

Proof. Since in the case of strictly positive risk X we get $\mathsf{E}[X] > 0$, then, combining Jensen inequality

$$u(-\mathsf{E}[X]) \geq \mathsf{E}[u(-X)]$$

with definition equation (5), we see that customer zero utility premium calculation principle will be well-defined if the function $u(x)$ will be defined just for $x \in (-\infty, 0)$ with preservation of monotonicity and concavity assumptions, i.e., the function $u(x)$ must be defined on $(-\infty, 0)$ such that $u'(x) > 0$ and $u''(x) \leq 0$ for all $x \in (-\infty, 0)$.

Let us at the beginning prove the sufficiency of the statement. Indeed in the case of $u(x) = -a(-x)^{\kappa} + b$, with $a > 0$ and $\kappa \geq 1$, for any strictly positive risk X equation (5) will have the following form

$$-a(\pi_{\text{c.z.u.}}[X])^{\kappa} + b = \mathsf{E}[-aX^{\kappa} + b] = -a\mathsf{E}[X^{\kappa}] + b,$$

therefore, in the considered case

$$\pi_{\text{c.z.u.}}[X] = (\mathsf{E}[X^\kappa])^{1/\kappa}.$$

On the other hand, for the same function $u(x)$, the same risk X, and any $\Theta > 0$, from the equation (5) it follows

$$-a(\pi_{\text{c.z.u.}}[\Theta X])^\kappa + b = \mathsf{E}[-a(\Theta X)^\kappa + b] = -a\Theta^\kappa \mathsf{E}[X^\kappa] + b$$

so, here we get

$$\pi_{\text{c.z.u.}}[\Theta X] = \Theta(\mathsf{E}[X^\kappa])^{1/\kappa} = \Theta\pi_{\text{c.z.u.}}[X],$$

and as we see, customer zero utility premium calculation principle subjected to consideration of only strictly positive risks possesses scale invariance property in the case of $u(x) = -a(-x)^\kappa + b$, for $a > 0$ and $\kappa \geq 1$, defined for $x \in (-\infty, 0)$.

Let us now switch to the statement of the necessity. In order to show that customer zero utility premium calculation principle subjected to consideration of only strictly positive risks with all other types of function $u(x)$ will not possess scale invariance property, we will consider a risk X taking values $\varepsilon > 0$ and 1 with probabilities p and $1-p$ respectively. Being a random function of the parameters ε and p, the risk X within the proof of Theorem 5 will be denoted as X_p^ε.

For the described risk X_p^ε definition equation (5) will have the following form

$$u(-\pi_{\text{c.z.u.}}[X_p^\varepsilon]) = u(-\varepsilon)\cdot p + u(-1)\cdot(1-p). \tag{59}$$

From the equation (59) it follows

$$u(-\pi_{\text{c.z.u.}}[X_0^\varepsilon]) = u(-1),$$

moreover, since $u(x)$ is a strictly increasing function, then

$$\pi_{\text{c.z.u.}}[X_0^\varepsilon] = 1. \tag{60}$$

Calculating partial derivatives with respect to the parameter p from both sides of the equation (59), obtain

$$-u'(-\pi_{\text{c.z.u.}}[X_p^\varepsilon])\cdot\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon] = u(-\varepsilon) - u(-1). \tag{61}$$

Substituting $p = 0$ into the equation (61), obtain

$$-u'(-\pi_{\text{c.z.u.}}[X_0^\varepsilon])\cdot\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0} = u(-\varepsilon) - u(-1). \tag{62}$$

Using (60), equation (62) can be rewritten in the following way

$$-u'(-1)\cdot\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0} = u(-\varepsilon) - u(-1). \tag{63}$$

Let us now calculate partial derivatives with respect to the parameter p from both sides of the equation (61)

$$u''(-\pi_{\text{c.z.u.}}[X_p^\varepsilon])\cdot\left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\right)^2 - u'(-\pi_{\text{c.z.u.}}[X_p^\varepsilon])\cdot\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon] = 0. \tag{64}$$

Substituting $p = 0$ into the equation (64), and using identity (60), obtain

$$u''(-1)\cdot\left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2 - u'(-1)\cdot\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right) = 0. \tag{65}$$

Taking ε small enough, namely $\varepsilon < 1$, and taking into account strict monotonicity of the function $u(x)$, without of loss of generality, using (63), we may conclude that

$$\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0} \neq 0, \tag{66}$$

and hence, the equation (65) can be rewritten in the following way

$$\frac{u''(-1)}{u'(-1)} = \left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)\Bigg/\left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2. \tag{67}$$

For any $\Theta > 0$, definition equation (5) for the risk ΘX_p^ε will take the following form

$$u(-\pi_{\text{c.z.u.}}[\Theta X_p^\varepsilon]) = u(-\Theta\varepsilon)\cdot p + u(-\Theta)\cdot(1-p). \tag{68}$$

In the case of scale invariant customer zero utility premium principle equation (68) can be rewritten in the following way

$$u(-\Theta\pi_{\text{c.z.u.}}[X_p^\varepsilon]) = u(-\Theta\varepsilon)\cdot p + u(-\Theta)\cdot(1-p). \tag{69}$$

Calculating second partial derivatives with respect to p from both sides of the equation (69), obtain

$$\begin{aligned} &u''(-\Theta\pi_{\text{c.z.u.}}[X_p^\varepsilon])\cdot\Theta^2\cdot\left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\right)^2 \\ &\quad - u'(-\Theta\pi_{\text{c.z.u.}}[X_p^\varepsilon])\cdot\Theta\cdot\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon] = 0. \end{aligned} \tag{70}$$

Substituting $p = 0$ into the equation (70), canceling Θ factor, and using identity (60), we get

$$u''(-\Theta)\cdot\Theta\cdot\left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2 - u'(-\Theta)\cdot\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right) = 0. \tag{71}$$

Since $u'(-\Theta) > 0$, then using relation (66), equation (71) can be rewritten in the following way

$$\frac{u''(-\Theta)\cdot\Theta}{u'(-\Theta)} = \left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)\Bigg/\left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2. \tag{72}$$

Observe that equations (67) and (72) have equal right-hand sides, this means that their left-hand sides also have to be equal, in this way we finally get an equation which the function $u(x)$ has to satisfy in the case of scale invariant customer zero utility premium calculation principle subjected to consideration of only strictly positive risks, namely,

$$\frac{u''(-\Theta)\cdot\Theta}{u'(-\Theta)} = \frac{u''(-1)}{u'(-1)}, \text{ for all } \Theta > 0. \tag{73}$$

Assigning $-u''(-1)/u'(-1) =: \varkappa$ (since $u''(-1) \leq 0$ and $u'(-1) > 0$ then $\varkappa \geq 0$) and making substitution $z(\Theta) := u'(-\Theta)$ equation (73) can be rewritten in the following equivalent form

$$\frac{dz}{z} = \varkappa \frac{d\Theta}{\Theta},$$

therefore

$$\log(z(\Theta)) = \varkappa \log(\Theta) + \log(C_1), \text{ for some constant } C_1 > 0,$$

and the function $z(\Theta)$ itself will have a form

$$z(\Theta) = C_1 \Theta^{\varkappa}.$$

Switching back to the function $u'(-\Theta)$, obtain

$$u'(-\Theta) = C_1 \Theta^{\varkappa}. \tag{74}$$

Switching back to the original parameter $x \in (-\infty, 0)$ representation (74) can be rewritten in the following way

$$u'(x) = C_1(-x)^{\varkappa}.$$

Taking antiderivative, obtain

$$u(x) = -\frac{C_1}{\varkappa+1}(-x)^{\varkappa+1} + C_2,$$

therefore the function $u(x)$ must be a function of the form

$$u(x) = -a(-x)^{\kappa} + b, \text{ for some real constants } a,\ b, \text{ and } \kappa.$$

Moreover, since $C_1 > 0$ and $\varkappa > 0$ then $a > 0$, and since $\varkappa \geq 0$ then $\kappa \geq 1$.

This completes the proof of Theorem 5. □

5. Swiss Premium Principle

The following theorem describes conditions of attainment of scale invariance property by Swiss insurance premium calculation principle.

Theorem 6. *For any $\Delta \in [0, 1]$, Swiss premium calculation principle possesses scale invariance property if and only if $V(x) = ax + b$, for $a > 0$, i.e., only in the case when it coincides with net premium principle.*

Proof. Let us at the beginning prove the sufficiency of the statement. From definition equation (6) for any risk X, any $\Delta \in [0,1]$, and function $V(x) = ax + b$, for $a > 0$, it follows

$$a(1-\Delta)\pi_{\text{Swiss}}[X] + b = \mathsf{E}[a(X - \Delta\pi_{\text{Swiss}}[X]) + b] = a\mathsf{E}[X] - a\Delta\pi_{\text{Swiss}}[X] + b,$$

so, here we get

$$\pi_{\text{Swiss}}[X] = \mathsf{E}[X] = \pi_{\text{net}}[X].$$

On the other hand, for the same risk X, any $\Delta \in [0,1]$, the same function $V(x)$, and any $\Theta > 0$ from definition equation (6) it follows

$$a(1-\Delta)\pi_{\text{Swiss}}[\Theta X] + b = \mathsf{E}[a(\Theta X - \Delta\pi_{\text{Swiss}}[\Theta X]) + b] = a\Theta\mathsf{E}[X] - a\Delta\pi_{\text{Swiss}}[\Theta X] + b,$$

therefore, in the considered case,

$$\pi_{\text{Swiss}}[\Theta X] = \Theta\mathsf{E}[X] = \Theta\pi_{\text{Swiss}}[X],$$

and as we see, Swiss premium calculation principle possesses scale invariance property in the case of increasing linear function $V(x)$.

The proof of the sufficiency was finished, so we can start to prove the necessity.

In order to show that Swiss premium calculation principle with non-linear functions $V(x)$ will not possess scale invariance property let us consider a risk X taking value t (here parameter t takes non-zero real values) and 0 with probabilities p and $1-p$ respectively. Being a random function of the parameters p and t the risk X within the proof of Theorem 6 will be denoted as X_p^t.

Observe that Swiss premium calculation principle is invariant with respect to linear transformations of the function $V(x)$, i.e., principle based on a function $V(x)$ and principle based on the function $\overline{V}(x) := l_1 V(x) + l_2$, for $l_1 > 0$, will produce the same premiums for the same risks. Here condition $l_1 > 0$ is imposed because otherwise the assumption of positivity of first derivative of the function $\overline{V}(x)$ will vanish.

In order to simplify the computations, we will first obtain all possible representations (in the case when Swiss premium calculation principle is scale invariant) for the scaled function $\overline{V}(x)$ with

$$l_1 = 1/V'(0) \quad \text{and} \quad l_2 = -V(0)/V'(0),$$

and then we will switch back to the original function $V(x)$.

Observe that the just defined function $\overline{V}(x)$ satisfies the following boundary conditions

$$\overline{V}(0) = 0, \text{ and } \overline{V}'(0) = 1. \tag{75}$$

Definition equation (6) based on the function $\overline{V}(x)$ for the risk X_p^t and $\Delta \in [0,1]$ will take the following form

$$\overline{V}((1-\Delta)\pi_{\text{Swiss}}[X_p^t]) = \overline{V}(t - \Delta\pi_{\text{Swiss}}[X_p^t])p + \overline{V}(-\Delta\pi_{\text{Swiss}}[X_p^t])(1-p). \tag{76}$$

Putting $p = 0$ into the equation (76), obtain

$$\overline{V}((1-\Delta)\pi_{\text{Swiss}}[X_0^t]) = \overline{V}(-\Delta\pi_{\text{Swiss}}[X_0^t]). \tag{77}$$

Since $\overline{V}'(x) > 0$ for all x, then from the equation (77) it follows

$$(1-\Delta)\pi_{\text{Swiss}}[X_0^t] = -\Delta\pi_{\text{Swiss}}[X_0^t],$$

which yields

$$\pi_{\text{Swiss}}[X_0^t] = 0. \tag{78}$$

Let us calculate partial derivatives with respect to p from both sides of the equation (76)

$$\begin{aligned}&\overline{V}'((1-\Delta)\pi_{\text{Swiss}}[X_p^t]) \cdot (1-\Delta) \cdot \frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\\ =&\overline{V}(t-\Delta\pi_{\text{Swiss}}[X_p^t]) - \overline{V}(-\Delta\pi_{\text{Swiss}}[X_p^t])\\ &-\Delta\cdot\overline{V}'(t-\Delta\pi_{\text{Swiss}}[X_p^t])\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\cdot p\\ &-\Delta\cdot\overline{V}'(-\Delta\pi_{\text{Swiss}}[X_p^t])\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\cdot(1-p).\end{aligned} \tag{79}$$

Putting $p = 0$ into the equation (79), we get

$$\begin{aligned}&\overline{V}'((1-\Delta)\pi_{\text{Swiss}}[X_0^t]) \cdot (1-\Delta) \cdot \frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\bigg|_{p=0}\\ =&\overline{V}(t-\Delta\pi_{\text{Swiss}}[X_0^t]) - \overline{V}(-\Delta\pi_{\text{Swiss}}[X_0^t])\\ &-\Delta\cdot\overline{V}'(-\Delta\pi_{\text{Swiss}}[X_0^t])\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\bigg|_{p=0}.\end{aligned} \tag{80}$$

Combination of (80) and (78) yields

$$\overline{V}'(0)\cdot(1-\Delta)\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\bigg|_{p=0} = \overline{V}(t) - \overline{V}(0) - \Delta\cdot\overline{V}'(0)\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\bigg|_{p=0}. \tag{81}$$

Substituting boundary conditions $\overline{V}(0) = 0$ and $\overline{V}'(0) = 1$ into the equation (81) we obtain a representation for the partial derivative with respect to the parameter p of the premium at the point $p = 0$, namely,

$$\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\bigg|_{p=0} = \overline{V}(t). \tag{82}$$

On the other hand, equation (6) for the risk ΘX_p^t based on $\overline{V}(x)$ for any $\Theta > 0$ and any $\Delta \in [0,1]$, will have the following form

$$\overline{V}((1-\Delta)\pi_{\text{Swiss}}[\Theta X_p^t]) = \overline{V}(t\Theta - \Delta\pi_{\text{Swiss}}[\Theta X_p^t])\cdot p + \overline{V}(-\Delta\pi_{\text{Swiss}}[\Theta X_p^t])\cdot(1-p). \tag{83}$$

Since in the case of scale invariant Swiss premium calculation principle for any $\Theta > 0$ the following identity must hold

$$\pi_{\text{Swiss}}[\Theta X_p^t] = \Theta\pi_{\text{Swiss}}[X_p^t],$$

then, in the case of scale invariant Swiss principle, equation (83) can be rewritten in the following equivalent form

$$\overline{V}((1-\Delta)\Theta\pi_{\text{Swiss}}[X_p^t]) = \overline{V}(t\Theta - \Delta\Theta\pi_{\text{Swiss}}[X_p^t])\cdot p + \overline{V}(-\Delta\Theta\pi_{\text{Swiss}}[X_p^t])\cdot(1-p). \quad (84)$$

Let us now calculate partial derivatives with respect to the parameter p from both sides of the equation (84)

$$\begin{aligned} &\overline{V}'((1-\Delta)\Theta\pi_{\text{Swiss}}[X_p^t])\cdot(1-\Delta)\cdot\Theta\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t] \\ =&\overline{V}(\Theta t - \Delta\Theta\pi_{\text{Swiss}}[X_p^t]) - \overline{V}(-\Delta\Theta\pi_{\text{Swiss}}[X_p^t]) \\ &-\Delta\cdot\Theta\cdot\overline{V}'(\Theta t - \Delta\Theta\pi_{\text{Swiss}}[X_p^t])\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\cdot p \\ &-\Delta\cdot\Theta\cdot\overline{V}'(-\Delta\Theta\pi_{\text{Swiss}}[X_p^t])\cdot\frac{\partial}{\partial p}\pi_{\text{Swiss}}[X_p^t]\cdot(1-p). \end{aligned} \quad (85)$$

Substituting $p = 0$ into the equation (85) and applying identities (78) and (82), obtain

$$\overline{V}'(0)\cdot(1-\Delta)\cdot\Theta\cdot V(t) = \overline{V}(t\Theta) - V(0) - \Delta\cdot\Theta\cdot\overline{V}'(0)\cdot\overline{V}(t). \quad (86)$$

Using boundary conditions $\overline{V}(0) = 0$ and $\overline{V}'(0) = 1$ equation (86) will be simplified to the following one

$$\overline{V}(t\Theta) = \Theta\overline{V}(t). \quad (87)$$

Taking partial derivatives with respect to the parameter t from both sides of the equation (87), obtain

$$\Theta\overline{V}'(\Theta t) = \Theta\overline{V}'(t), \text{ or equivalently, } \overline{V}'(\Theta t) = \overline{V}'(t). \quad (88)$$

By fixing value of the parameter t in (88) to a constant from $\mathbb{R}\setminus\{0\}$ and varying values of the parameter Θ we will make $\overline{V}'(\Theta t)$ a function of changing parameter while the value $\overline{V}'(t)$ will be fixed to a constant. Taking into account this fact as well as continuity of function $\overline{V}'(\cdot)$ (which follows from differentiability) and using boundary condition $\overline{V}'(0) = 1$ we conclude that

$$\overline{V}'(x) = 1, \text{ for } x \in \mathbb{R}.$$

Taking antiderivative of the function $\overline{V}'(x)$ and using boundary condition $\overline{V}(0) = 0$ we find the admissible representation for the function $\overline{V}(x)$, namely,

$$\overline{V}(x) = x, \text{ for } x \in \mathbb{R}.$$

Switching back to the original function $V(x)$ obtain

$$V(x) = ax + b, \text{ for } x \in \mathbb{R}, \text{ and some real constants } a \text{ and } b.$$

Initial assumption concerning positivity of first derivative of the function $V(x)$ gives us additional restriction on the parameter a, namely, parameter a must be a strictly positive constant. □

Since, in the case of $\Delta = 0$, Swiss premium principle entirely coincides with mean value premium principle then we can formulate the following corollary to Theorem 2.

Corollary 3. *In the case of* $\Delta = 0$*, Swiss premium calculation principle subjected to consideration of only strictly positive risks possesses scale invariance property if and only if* $V(x) = ax^{\kappa} + b$*, for* $a > 0$ *and* $\kappa \geq 1$*, defined for* $x \in (0, +\infty)$.

References

[1] S. Asmussen and H. Albrecher. *Ruin Probabilities (second edition)*, World Sientific, Singapore, 2010.

[2] P.J. Boland. *Statistical and Probabilistic Methods in Actuarial Science*, Chapman & Hall, Boca Raton, 2007.

[3] N.L. Bowers, H.-U. Gerber, J.C. Hickman, D.A. Jones, and C.J. Nesbit. *Actuarial Mathematics (second edition)*, The Society of Actuaries, Illinois, 1997.

[4] H. Bühlmann. *Mathematical Methods in Risk Theory*, Springer, Berlin, 1970.

[5] D.C.M. Dickson. *Insurance Risk and Ruin*, Cambridge University Press, Cambridge, 2005.

[6] H.-U. Gerber. *An Introduction to Mathematical Risk Theory*, S.S. Huebner Foundation for Insurance Education, Philadelpia, 1979.

[7] F.E. de Vylder, M. Goovaerts, and J. Haezendonck (editors). *Premium Calculation in Insurance (collection of articles)*, Kluwer Academic Publishers, Boston, 1984.

[8] F.E. de Vylder, M. Goovaerts, and J. Haezendonck (editors). *Insurance and Risk Theory (collection of articles)*, Kluwer Academic Publishers, Boston, 1986.

[9] R. Kaas, M. Goovaerts, J. Dhaene, and M. Denuit. *Modern Actuarial Risk Theory using R*, Springer, Berlin, 2008.

[10] E. Kremer. *Applied Risk Theory*, Shaker, Aachen, 1999.

[11] T. Rolski, H. Schmidli, V. Schmidt, and J. Teugels. *Stochastic Processes for Insurance and Finance*, John Wiley & Sons, Chichester, 1999.

[12] E. Straub. *Non-Life Insurance Mathematics*, Springer, Berlin, 1988.