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New Types of Generalized Difference Double A-sequence Spaces Defined by Orlicz Function

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Abstract. In this paper we introduce some new generalized difference double sequence spaces defined by Orlicz function and study different topological properties of these spaces and also establish some inclusion results among them.

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1. Introduction

In 1971 Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to construct the sequence space for single sequences as follows:

$$l_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) \le 1 \right\}.$$

Definition 1. *An Orlicz function is a function* $M : [0, \infty) \to [0, \infty)$ *which is continuous, non-decreasing and convex with* $M(0) = 0$, $M(x) > 0$ *for* $x > 0$ *and* $M(x) \to \infty$ *as* $x \to \infty$.

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An Orlicz function M is said to satisfy the Δ_2-*condition* for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)$, $u \geq 0$. Note that, if $0 < \lambda < 1$, then $M(\lambda x) \leq \lambda M(x)$, for all $x \geq 0$.

In the later stage different Orlicz sequence spaces were introduced and studied by Parashar and Choudhary [8], Et and Colak [2] and many others.

Kizmaz [5] introduced the notion of difference sequence spaces as follows:

$$X(\Delta) = \left\{x = (x_k) : (\Delta x_k) \in X\right\}$$

for $X = l_\infty, c$ and c_o. Later on, the notion was generalized by Et and Colak [2] as follows:

$$X(\Delta^m) = \left\{x = (x_k) : (\Delta^m x_k) \in X\right\}$$

for $X = l_\infty, c$ and c_o, where $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}), \Delta^0 x = x$ and also this generalized difference notion has the following binomial representation:

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} \text{ for all } k \in \mathbb{N}.$$

Definition 2 ([9]). *A double sequence $x = (x_{k,l})$ has a* Pringsheim limit *L (denoted by $P-\lim x = L$) provided that given an $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k, l > N$. We shall describe such an $x = (x_{k,l})$ more briefly as* "P-convergent".

The four dimensional matrix A is said to be *RH-regular* if it maps every bounded *P-convergent* sequence into a *P-convergent* sequence with the same *P-limit*. The assumption of boundedness was made because a double sequence which is P-convergent is not necessarily bounded. Using this definition Robison and Hamilton, independently, both presented the following Silverman-Toeplitz type characterization of *RH-regularity*.

Lemma 1 ([4, 10]). *The four dimensional matrix A is RH*-regular *if and only if*

RH_1: $P-\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH_2: $P-\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;

RH_3: $P-\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 0$ *for each l;*

RH_4: $P-\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| = 0$, *for each k;*

RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P*-convergent;

RH_6: *There exist finite positive integers E and F such that $\sum_{k,l>F} |a_{m,n,k,l}| < E$.*

2. New Generalized Difference Double Sequence Spaces

Let M be an Orlicz function, $p=\left(p_{k,l}\right)$ be a factorable double sequence of strictly positive real numbers and $A=\left(a_{m,n,k,l}\right)$ be a nonnegative *RH-regular* summability matrix method. We now define the following new difference double sequence spaces (for some $\rho>0$ and L):

$$w_o^2\left[A,M,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}=0,\right\},$$

$$w^2\left[A,M,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}-L\right|}{\rho}\right)\right]^{p_{k,l}}=0,\right\},$$

$$w_\infty^2\left[A,M,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}<\infty,\right\}.$$

When $M(x)=x$, we have the following difference sequence spaces:

$$w_o^2\left[A,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left|\Delta^r x_{k,l}\right|^{p_{k,l}}=0\right\},$$

$$w^2\left[A,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left|\Delta^r x_{k,l}-L\right|^{p_{k,l}}=0,\text{ for some } L\right\},$$

$$w_\infty^2\left[A,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left|\Delta^r x_{k,l}\right|^{p_{k,l}}<\infty\right\}.$$

Some spaces are defined by specializing A,M,r and $p=\left(p_{k,l}\right)$. For example, if $A=(C,1,1)$ the difference sequence spaces defined above become $w_o^2\left[M,p\right](\Delta^r)$, $w^2\left[M,p\right](\Delta^r)$ and $w_\infty^2\left[M,p\right](\Delta^r)$ which are as follows (for some $\rho>0$ and L):

$$w_o^2\left[M,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right)\in w^2:P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left[M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}=0,\right\},$$

$$w^2\left[M,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left[M\left(\frac{\left|\Delta^r x_{k,l}-L\right|}{\rho}\right)\right]^{p_{k,l}}=0,\right\},$$

$$w_\infty^2\left[M,p\right](\Delta^r)=\left\{x=\left(x_{k,l}\right):\sup_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left[M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}<\infty,\right\}.$$

Let $A=(C,1,1)$, $p_{k,l}=1$, for all $k,l\in\mathbb{N}$ and $M(x)=x$, we obtain the following difference sequence spaces:

$$w_o^2(\Delta^r)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left|\Delta^r x_{k,l}\right|=0\right\},$$

$$w^2(\Delta^r)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left|\Delta^r x_{k,l}-L\right|=0,\text{ for some }L\right\},$$

$$w^2_\infty(\Delta^r)=\left\{x=\left(x_{k,l}\right):\sup_{m,n}\frac{1}{mn}\sum_{k,l=0,0}^{m-1,n-1}\left|\Delta^r x_{k,l}\right|<\infty\right\}.$$

If $r=1$ the we obtain the following difference sequence spaces (for some $\rho>0$ and L):

$$w^2_o\left[A,M,p\right](\Delta)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}=0,\right\},$$

$$w^2\left[A,M,p\right](\Delta)=\left\{x=\left(x_{k,l}\right):P-\lim_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta x_{k,l}-L\right|}{\rho}\right)\right]^{p_{k,l}}=0,\right\},$$

$$w^2_\infty\left[A,M,p\right](\Delta)=\left\{x=\left(x_{k,l}\right):\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}<\infty,\right\}$$

which were defined and studied by Esi [1].

3. Main Results

In this section we shall establish some basic properties for the difference sequence spaces defined above.

Theorem 1. *Let $p=\left(p_{k,l}\right)$ be bounded. The classes of sequences $w^2_o\left[A,M,p\right](\Delta^r)$, $w^2\left[A,M,p\right](\Delta^r)$ and $w^2_\infty\left[A,M,p\right](\Delta^r)$ are linear spaces.*

Proof. The proof of the theorem is easy, so omitted. □

Theorem 2. *If $0<h=\inf p_{k,l}\le\sup p_{k,l}=H<\infty$, then for any Orlicz function M and a nonnegative RH*-regular *summability matrix method A, then $w^2\left[A,p\right](\Delta^r)\subset w^2\left[A,M,p\right](\Delta^r)$.*

Proof. Let $0<h=\inf p_{k,l}\le\sup p_{k,l}=H<\infty$ and $x=\left(x_{k,l}\right)\in w^2\left[A,p\right](\Delta^r)$ and let $0<\varepsilon<1$ and δ with $0<\delta<1$ such that $M(t)<\varepsilon$ for $0\le t<\delta$. We can write for each m and n

$$\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}-L\right|}{\rho}\right)\right]^{p_{k,l}}=\sum_{k,l=0,0\ \&\ \left|\Delta^r x_{k,l}-L\right|\le\delta}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}-L\right|}{\rho}\right)\right]^{p_{k,l}}$$
$$+\sum_{k,l=0,0\ \&\ \left|\Delta^r x_{k,l}-L\right|>\delta}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}-L\right|}{\rho}\right)\right]^{p_{k,l}}.$$

Then

$$\sum_{k,l=0,0\ \&\ \left|\Delta^r x_{k,l}-L\right|\le\delta}^{\infty,\infty}a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}-L\right|}{\rho}\right)\right]^{p_{k,l}}\le\varepsilon^h\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}. \tag{1}$$

On the other hand, we use the fact that

$$\left|\Delta^r x_{k,l} - L\right| < 1 + \left[\left|\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right|\right]$$

where $[|t|]$ denotes the integer part of t. Since M is Orlicz function we have

$$M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right) \geq M(1).$$

Now, let us consider the second part where the sum is taken over $\left|\Delta^r x_{k,l} - L\right| > \delta$. Thus

$$\begin{aligned}
&\sum_{k,l=0,0\ \&\ \left|\Delta^r x_{k,l}-L\right|>\delta}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right)\right]^{p_{k,l}} \\
&\leq \sum_{k,l=0,0\ \&\ \left|\Delta^r x_{k,l}-L\right|>\delta}^{\infty,\infty} a_{m,n,k,l}\left[M\left(1 + \left[\left|\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right|\right]\right)\right]^{p_{k,l}} \\
&\leq \left(2M(1)\delta^{-1}\right)^H \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right)^{p_{k,l}}
\end{aligned}$$

This inequality and from (1) and *RH-regularity of A,* we are granted that $x = \left(x_{k,l}\right) \in w^2\left[A, M, p\right](\Delta^r)$ and this completes the proof. □

Theorem 3. *$w_o^2\left[A, M, p\right](\Delta^r)$, $w^2\left[A, M, p\right](\Delta^r)$ and $w_\infty^2\left[A, M, p\right](\Delta^r)$ are complete linear topological spaces with the paranorm*

$$\begin{aligned}
g\left(\left(x_{k,l}\right)\right) = &\sum_{k=1}^{r} |x_{k,1}| + \sum_{l=1}^{r} |x_{1,l}| \\
&+ \inf\left\{\rho^{\frac{p_{k,l}}{T}} > 0 : \sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{|\Delta^r x_{k,l}|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \leq 1\right\}.
\end{aligned}$$

where $T = \max(1, H)$, $H = \sup_{k,l} p_{k,l}$.

Proof. Clearly $g(0) = 0$, $g(-x) = g(x)$. Let $x = \left(x_{k,l}\right)$, $y = \left(y_{k,l}\right) \in w_\infty^2\left[A, M, p\right](\Delta^r)$. Then there exist some ρ_1 and ρ_2 such that

$$\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho_1}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \leq 1$$

and

$$\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r y_{k,l}\right|}{\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \le 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\begin{aligned}
&\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r\left(x_{k,l}+y_{k,l}\right)\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\\
&\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}+\Delta^r y_{k,l}\right|}{\rho_1+\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\\
&\le \sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[\frac{\rho_1}{\rho_1+\rho_2}M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho_1}\right)+\frac{\rho_2}{\rho_1+\rho_2}M\left(\frac{\left|\Delta^r y_{k,l}\right|}{\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}
\end{aligned}$$

By Minkowsky's inequality

$$\begin{aligned}
&\le\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho_1}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\\
&+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r y_{k,l}\right|}{\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \le 1.
\end{aligned}$$

Now

$$\begin{aligned}
g\left(\left(x_{k,l}\right)+\left(y_{k,l}\right)\right) =& \sum_{k=1}^{r}\left(|x_{k,1}|+|y_{k,1}|\right)+\sum_{l=1}^{r}\left(|x_{1,l}|+|y_{1,l}|\right)\\
&+\inf\left\{\rho^{\frac{p_{k,l}}{T}}>0:\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r\left(x_{k,l}+y_{k,l}\right)\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\le 1\right\}\\
\le& \sum_{k=1}^{r}|x_{k,1}|+\sum_{k=1}^{r}|y_{k,1}|+\sum_{l=1}^{r}|x_{1,l}|+\sum_{l=1}^{r}|y_{1,l}|\\
&+\inf\left\{\rho_1^{\frac{p_{k,l}}{T}}>0:\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho_1}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\right\}\\
&+\inf\left\{\rho_2^{\frac{p_{k,l}}{T}}>0:\sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^r y_{k,l}\right|}{\rho_2}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}}\right\}\\
=& g\left(\left(x_{k,l}\right)\right)+g\left(\left(y_{k,l}\right)\right).
\end{aligned}$$

Let $\lambda \in \mathbb{C}$, then the continuity of the product follows from the following equality:

$$\begin{aligned} g\left(\lambda\left(x_{k,l}\right)\right) =& \sum_{k=1}^{r}|\lambda x_{k,1}| + \sum_{l=1}^{r}|\lambda x_{1,l}| \\ &+ \inf\left\{\rho^{\frac{p_{k,l}}{T}} > 0 : \sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\lambda\Delta^{r}x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \leq 1, \rho > 0\right\} \\ =& |\lambda|\sum_{k=1}^{r}|x_{k,1}| + |\lambda|\sum_{l=1}^{r}|x_{1,l}| \\ &+ \inf\left\{(|\lambda| r)^{\frac{p_{k,l}}{T}} > 0 : \sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\lambda\Delta^{r}x_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} \leq 1, r > 0\right\} \\ =& |\lambda|\, g\left(\left(x_{k,l}\right)\right) \end{aligned}$$

where $\frac{1}{r} = \frac{|\lambda|}{\rho}$.

Now let $\left(x^{s}_{k,l}\right)$ be a Cauchy sequence in $w^{2}_{\infty}\left[A, M, p\right](\Delta^{r})$. Then

$$g\left(\left(x^{s}_{k,l} - x^{t}_{k,l}\right)\right) \to 0 \text{ as } s, t \to \infty.$$

For given $\varepsilon > 0$, choose $b > 0$ and $x_o > 0$ be such that $\frac{\varepsilon}{bx_o} > 0$ and $M\left(\frac{bx_o}{2}\right) \geq 1$. Now $g\left(\left(x^{s}_{k,l} - x^{t}_{k,l}\right)\right) \to 0$ as $s, t \to \infty$ implies that there exists $n_o \in \mathbb{N}$ such that

$$\begin{aligned} g\left(\left(x^{s}_{k,l} - x^{t}_{k,l}\right)\right) <& \frac{\varepsilon}{bx_o} \text{ for all } s, t \geq n_o. \\ &\Rightarrow \sum_{k=1}^{r}\left|x^{s}_{k,1} - x^{t}_{k,1}\right| + \sum_{l=1}^{r}\left|x^{s}_{1,l} - x^{t}_{1,l}\right| \\ &+ \inf\left\{\rho^{\frac{p_{k,l}}{T}} > 0 : \sup_{m,n}\left(\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\left|\Delta^{r}x^{s}_{k,l} - \Delta^{r}x^{t}_{k,l}\right|}{\rho}\right)\right]^{p_{k,l}}\right)^{\frac{1}{T}} leq1\right\} < \frac{\varepsilon}{bx_o} \end{aligned} \tag{2}$$

This implies that

$$\sum_{k=1}^{r}\left|x^{s}_{k,1} - x^{t}_{k,1}\right| + \sum_{l=1}^{r}\left|x^{s}_{1,l} - x^{t}_{1,l}\right| < \varepsilon,\ for\ all\ s, t \geq n_0.$$

This shows that $\left(x^{s}_{k,1}\right)$, $\left(x^{t}_{1,l}\right)$ are Cauchy sequences of real numbers. As the set of real numbers is complete so there exists real numbers $x_{k,1}, x_{1,l}$ such that

$$\lim_{s\to\infty} x^{s}_{k,1} = x_{k,1} \text{ and } \lim_{t\to\infty} x^{t}_{1,l} = x_{1,l}\ .$$

Now from 0(2) we have,

$$M\left(\frac{\left|\Delta^r x^s_{k,l}-\Delta^r x^t_{k,l}\right|}{\rho}\right)\le 1\le M\left(\frac{bx_o}{2}\right)$$
$$\Rightarrow\frac{\left|\Delta^r x^s_{k,l}-\Delta^r x^t_{k,l}\right|}{g\left(\left(x^s_{k,l}-x^t_{k,l}\right)\right)}\le\frac{bx_o}{2}$$
$$\Rightarrow\left|\Delta^r x^s_{k,l}-\Delta^r x^t_{k,l}\right|<\frac{bx_o}{2}.\frac{\varepsilon}{bx_o}=\frac{\varepsilon}{2}.$$

This implies that $\left(\Delta^r x^s_{k,l}\right)$ is a Cauchy sequence of real numbers. Let $\lim_{s\to\infty}\Delta^r x^s_{k,l}=z_{k,l}$ for all $k,l\in N$.

Let $k,l=1$, we have $\lim_{s\to\infty}\Delta^r x^s_{1,1}=\lim_{s\to\infty}\sum_{i=0}^{r}\sum_{j=0}^{r}(-1)^{i+j}\binom{r}{i}\binom{r}{j}x_{1+i,1+j}=z_{1,1}$.

Similarly we have $\lim_{s\to\infty}\Delta^r x^s_{k,l}=\lim_{s\to\infty}x^s_{k,l}=z_{k,l}$ for $k,l=1,2,\dots,r$.

Thus we have $\lim_{s\to\infty}x^s_{1+r,1+r}$ exists. Let $\lim_{s\to\infty}x^s_{1+r,1+r}=x_{1+r,1+r}$. Proceeding in this way inductively we conclude that $\lim_{s\to\infty}x^s_{k,l}=x_{k,l}$ exists for each $k,l\in\mathbf{N}$. Using continuity of M, we have

$$\lim_{t\to\infty}M\left(\frac{\left|\Delta^r x^s_{k,l}-\Delta^r x^t_{k,l}\right|}{\rho}\right)\le 1$$
$$\Rightarrow M\left(\frac{\left|\Delta^r x^s_{k,l}-\Delta x_{k,l}\right|}{\rho}\right)\le 1.$$

Let $s\ge n_o$, then taking the infimum of such $\rho's$ we have $g\left(\left(x^s_{k,l}-x_{k,l}\right)\right)<\varepsilon$. Thus $(x^s_{k,l}-x_{k,l})\in w^2_\infty\left[A,M,p\right](\Delta^r)$. By linearity of the space $w^2_\infty\left[A,M,p\right](\Delta^r)$ we have $\left(x_{k,l}\right)\in w^2_\infty\left[A,M,p\right](\Delta^r)$. Hence $w^2_\infty\left[A,M,p\right](\Delta^r)$ is complete space. □

Proposition 1.

(a) $w^2\left[A,M,p\right](\Delta^r)\subset w^2_\infty\left[A,M,p\right](\Delta^r)$,

(b) $w^2_o\left[A,M,p\right](\Delta^r)\subset w^2_\infty\left[A,M,p\right](\Delta^r)$.

Proof. The proof is easy. □

Theorem 4. *The spaces* $w^2_o\left[A,M,p\right](\Delta^r)$ *and* $w^2\left[A,M,p\right](\Delta^r)$ *are nowhere dense subsets of* $w^2_\infty\left[A,M,p\right](\Delta^r)$.

Proof. The proof is clear in view of Theorem 3 and Proposition 1. □

Theorem 5.

(a) If $0 < h = \inf p_{k,l} < p_{k,l} \leq 1$*, then*

$$w^2\left[A, M, p\right](\Delta^r) \subset w^2[A, M](\Delta^r).$$

(b) If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$*, then*

$$w^2[A, M](\Delta^r) \subset w^2\left[A, M, p\right](\Delta^r).$$

Proof. **(a)** Let $x = \left(x_{k,l}\right) \in w^2\left[A, M, p\right](\Delta^r)$, since $0 < h = \inf p_{k,l} < p_{k,l} \leq 1$, we obtain the following:

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right) \leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right)\right]^{p_{k,l}}$$

thus $x = \left(x_{k,l}\right) \in w^2[A, M](\Delta^r)$.

(b) Let $p_{k,l} \geq 1$ for each k , l and $\sup p_{k,l} < \infty$ and let $x = \left(x_{k,l}\right) \in w^2[A, M](\Delta^r)$. Then for each $0 < \varepsilon < 1$ there exists a positive integer K such that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right) \leq \varepsilon < 1$$

for all $n, m \geq K$. This implies that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right)\right]^{p_{k,l}} \leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right).$$

Thus $x = \left(x_{k,l}\right) \in w^2\left[A, M, p\right](\Delta^r)$. This completes the proof. □

Corollary 1. *Let* $A = (C, 1, 1)$*. Then*

(a) If $0 < h = \inf p_{k,l} < p_{k,l} \leq 1$*, then*

$$w^2\left[M, p\right](\Delta^r) \subset w^2[M](\Delta^r).$$

(b) If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$*, then*

$$w^2[M](\Delta^r) \subset w^2\left[M, p\right](\Delta^r).$$

Theorem 6. *If* $\sup \frac{p_{k,l}}{p_{i,j}} < \infty$ *for all* $k \geq i$, $l \geq j$, *then* $w^2\left[A, M, p\right] \subset w^2\left[A, M, p\right](\Delta^r)$ *and the inclusion is strict, where*

$$w^2\left[A, M, p\right] = \left\{ x = \left(x_{k,l}\right) \in w^2 : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k,l} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } \rho > 0 \text{ and } L \right\}.$$

Proof. Let $x = \left(x_{k,l}\right) \in w^2\left[A, M, p\right]$. Then

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k,l} - L\right|}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \text{ and } L. \tag{3}$$

Since $\sup \frac{p_{k,l}}{p_{i,j}} < \infty$, so there exists $C > 0$ such that $p_{k,l} < Cp_{i,j}$ for all $k \geq i$, $l \geq j$. Thus from (3) we have,

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k,l} - L\right|}{\rho} \right) \right]^{p_{k,l+1}} = 0,$$
$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k,l} - L\right|}{\rho} \right) \right]^{p_{k+1,l}} = 0,$$
$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|x_{k,l} - L\right|}{\rho} \right) \right]^{p_{k+1,l+1}} = 0.$$

Now for $\left|\Delta^r x_{k,l}\right| = \left|\Delta^{r-1} x_{k,l} - \Delta^{r-1} x_{k,l+1} - \Delta^{r-1} x_{k+1,l} + \Delta^{r-1} x_{k+1,l+1} + L - L + L - L\right|$ we have,

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho} \right) \right]^{p_{k,l}}$$
$$\leq P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{\left|\Delta^{r-1} x_{k,l} - L\right|}{\rho} + \frac{\left|\Delta^{r-1} x_{k+1,l} - L\right|}{\rho} + \frac{\left|\Delta^{r-1} x_{k,l+1} - L\right|}{\rho} \right.\right.$$
$$\left.\left. + \frac{\left|\Delta^{r-1} x_{k+1,l+1} - L\right|}{\rho} \right) \right]^{p_{k,l}}$$
$$\leq D^2 P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[\left(M\left(\frac{\left|\Delta^{r-1} x_{k,l} - L\right|}{\rho} \right) \right)^{p_{k,l}} + \left(M\left(\frac{\left|\Delta^{r-1} x_{k+1,l} - L\right|}{\rho} \right) \right)^{p_{k,l}} \right.$$
$$\left. + \left(M\left(\frac{\left|\Delta^{r-1} x_{k,l+1} - L\right|}{\rho} \right) \right)^{p_{k,l}} + \left(M\left(\frac{\left|\Delta^{r-1} x_{k+1,l+1} - L\right|}{\rho} \right) \right)^{p_{k,l}} \right]$$

$$\leq D^2 P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[\left(M \left(\frac{\left|\Delta^{r-1} x_{k,l} - L\right|}{\rho} \right) \right)^{p_{k,l}} + \left(M \left(\frac{\left|\Delta^{r-1} x_{k+1,l} - L\right|}{\rho} \right) \right)^{p_{k+1,l}} + \left(M \left(\frac{\left|\Delta^{r-1} x_{k,l+1} - L\right|}{\rho} \right) \right)^{p_{k,l+1}} + \left(M \left(\frac{\left|\Delta^{r-1} x_{k+1,l+1} - L\right|}{\rho} \right) \right)^{p_{k+1,l+1}} \right] = 0$$

where $D = \max\left(1, 2^{H-1}\right)$. Thus $x = \left(x_{k,l}\right) \in w^2\left[A, M, p\right](\Delta^r)$. This completes the proof. □

The inclusion is strict follows from the following example.

Example 1. *Let $A = (C, 1, 1)$, $M(x) = x^p$, $p_{k,l} = 1$ for all k odd and for all $l \in \mathbb{N}$ and $p_{k,l} = 2$ otherwise. Consider the sequence $x = \left(x_{k,l}\right)$ defined by $x_{k,l} = (k+l)^r$ for all $k, l \in \mathbb{N}$. We have $\Delta^r x_{k,l} = 0$ for all $k, l \in \mathbb{N}$. Hence $x = \left(x_{k,l}\right) \in w^2\left[A, M, p\right](\Delta^r)$ but $x = \left(x_{k,l}\right) \notin w^2\left[A, M, p\right]$.*

Let E be a sequence space. Then E is called

(a) solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbf{N}$;

(b) monotone provided E contains the canonical preimages of all its step spaces.

It is a well known result that if E is normal then it is monotone.

Theorem 7. *The spaces $w_o^2\left[A, M, p\right](\Delta^r)$ and $w_\infty^2\left[A, M, p\right](\Delta^r)$ are normal as well as monotone.*

Proof. Let $(\alpha_{k,l})$ be a double sequences of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbf{N}$. Since M is monotone, we get for some $\rho > 0$

$$\begin{aligned} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|\Delta^r(\alpha_{k,l} x_{k,l})|}{\rho} \right) \right]^{p_{k,l}} &\leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\sup |\alpha_{k,l}| \frac{|\Delta^r x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} \\ &\leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|\Delta^r x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} \end{aligned}$$

which leads us to the desired results. □

4. Double Δ^r- Statistical Convergence

The concept of statistical convergence for single sequences was introduced by Fast [3] in 1951. Later, Mursaleen and Edely [7] defined the statistical analogue for double sequence $x = \left(x_{k,l}\right)$ as follows: A real double sequence $x = \left(x_{k,l}\right)$ is said to be *P-statistically convergent to L* provided that for each $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left\{ \text{ the number of } (k,l) : k < m, l < n; \left|x_{k,l} - L\right| \geq \varepsilon \right\} = 0.$$

In this case, we write $st_2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all P-*statistically convergent* double sequences by st_2.

Definition 3. *A real double sequence* $x = (x_{k,l})$ *is said to be* P-statistically Δ^r-convergent to L *provided that for each* $\varepsilon > 0$

$$P - \lim_{m,n} \frac{1}{mn} \left\{ \text{the number of } (k,l) : k < m, l < n; \left|\Delta^r x_{k,l} - L\right| \geq \varepsilon \right\} = 0.$$

In this case, we write $st_2(\Delta^r) - \lim_{k,l} x_{k,l} = L$ *and we denote the set of all* P-statistically Δ^r-convergent *double sequences by* $st_2(\Delta^r)$.

Theorem 8. *If* M *be an Orlicz function, then* $w^2[M](\Delta^r) \subset st_2(\Delta^r)$.

Proof. Suppose that $x = (x_{k,l}) \in w^2[M](\Delta^r)$ and $\varepsilon > 0$, then we obtain the following for every n and m

$$\begin{aligned} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right) &\geq \frac{1}{mn} \sum_{k,l=0,0 \,\&\, \left|\Delta^r x_{k,l} - L\right| \geq \varepsilon}^{m-1,n-1} M\left(\frac{\left|\Delta^r x_{k,l} - L\right|}{\rho}\right) \\ &\geq \frac{M(\varepsilon)}{mn} \left\{ \text{the number of } (k,l) : k < m, l < n; \left|\Delta^r x_{k,l} - L\right| \geq \varepsilon \right\}. \end{aligned}$$

Hence $x = (x_{k,l}) \in st_2(\Delta^r)$. □

Theorem 9. $st_2(\Delta^r) = w_o^2[M](\Delta^r)$ *if and only if the Orlicz function* M *is bounded.*

Proof. Suppose that M is bounded and $x = (x_{k,l}) \in st_2(\Delta^r)$. Since M is bounded then there exists an integer K such that $M(x) \leq K$, for all $x \geq 0$. Then for each m and n, we have

$$\begin{aligned} &\frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho}\right) \\ &= \frac{1}{mn} \sum_{k,l=0,0 \,\&\, \left|\Delta^r x_{k,l} - L\right| \geq \varepsilon}^{m-1,n-1} M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho}\right) + \frac{1}{mn} \sum_{k,l=0,0 \,\&\, \left|\Delta^r x_{k,l} - L\right| < \varepsilon}^{m-1,n-1} M\left(\frac{\left|\Delta^r x_{k,l}\right|}{\rho}\right) \\ &\leq \frac{K}{mn} \left\{ \text{the number of } (k,l) : k < m, l < n; \left|\Delta^r x_{k,l}\right| \geq \varepsilon \right\} + M(\varepsilon) \end{aligned}$$

and thus the Pringsheim's limit on m and n grant us the result.

Conversely, suppose that M is unbounded so that there is a positive double sequence (z_{mn}) with $M(z_{mn}) = (mn)^2$ for $m, n = 1, 2, \ldots$. Now the sequence $x = (x_{k,l})$ defined by $\Delta^r x_{k,l} = z_{mn}$ if $k, l = (mn)^2$ for $m, n = 1, 2, \ldots$ and $\Delta^r x_{k,l} = 0$, otherwise. Then we have

$$\frac{1}{mn} \left\{ \text{the number of } (k,l) :\ k < m, l < n; \left|\Delta^r x_{k,l}\right| \geq \varepsilon \right\} \leq \frac{\sqrt{mn}}{mn} \to 0, \text{ as } m, n \to \infty.$$

Hence $x_{k,l} \to 0 \left(st_2(\Delta^r)\right)$. But $x = (x_{k,l}) \notin w_o^2[M](\Delta^r)$, contradicting $st_2(\Delta^r) = w_o^2[M](\Delta^r)$. This completes the proof. □

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