



Using Two-dimensional Differential Transform to Solve Second Order Complex Partial Differential Equations

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Abstract. In this study, second order complex equations were solved by using two dimensional differential transform. Firstly these equations were separated to real and imaginary parts. Thus, two equalities were obtained. Later, real and imaginary parts of solution were obtained by using two dimensional differential transform method.

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1. Introduction

The concept of differential transform (one dimension) was first proposed and applied to solve linear and non linear initial value problems in electric circuit analysis by Zhou [6]. By using one dimensional differential transform method, nonlinear differential equations were solved in [5]. Solving partial differential equations by two dimensional differential transform method (DTM) was proposed by Cha'o Kuang Chen and Shing Huei Ho [3]. Partial differential equations was solved by using two dimensional DTM in [1, 3]. System of differential equation was solved using two dimensional DTM in [2]. Eigen value problems was solved by using this method in [4].

DTM consist of computing the coefficient of Taylor series of solution by using initial value. Moreover this method is an iterative method for obtain solution of Taylor series of differential equation.

Let $w = w(z, \bar{z})$ be a complex function. Here $z = x + iy$, $w(z, \bar{z}) = u(x, y) + iv(x, y)$. Derivative according to z and \bar{z} of $w(z, \bar{z})$ is defined as follows:

$$\frac{\partial w}{\partial z} = \frac{1}{2} \left(\frac{\partial w}{\partial x} - i \frac{\partial w}{\partial y} \right) \quad (1)$$

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right) \quad (2)$$

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$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad (3)$$

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \quad (4)$$

Similarly second order derivative of $w(z, \bar{z})$ are defined as following:

$$\frac{\partial^2 w}{\partial z^2} = \frac{1}{4} \left(\frac{\partial^2 w}{\partial x^2} - 2i \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right) \quad (5)$$

$$\frac{\partial^2 w}{\partial \bar{z}^2} = \frac{1}{4} \left(\frac{\partial^2 w}{\partial x^2} + 2i \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right) \quad (6)$$

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = \frac{1}{4} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{1}{4} \Delta w. \quad (7)$$

2. Two Dimensional Differential Transform

Definition 1. Two dimensional differential transform of function $f(x, y)$ is defined as follows

$$F(k, h) = \frac{1}{k!.h!} \left[\frac{\partial^{k+h} f(x, y)}{\partial x^k \partial y^h} \right]_{x=0, y=0} \quad (8)$$

In Equation (8), $f(x, y)$ is original function and $F(k, h)$ is transformed function, which is called T-function is brief.

Definition 2. Differential inverse transform of $F(k, h)$ is defined as follows

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} F(k, h) x^k y^h \quad (9)$$

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \frac{1}{k!.h!} \left[\frac{\partial^{k+h} f(x, y)}{\partial x^k \partial y^h} \right]_{x=0, y=0} x^k y^h \quad (10)$$

Equation (10) implies that the concept of two dimensional differential transform is derived from two dimensional Taylor series expansion.

Theorem 1 ([1, 3]). If $w(x, y) = u(x, y) \pm v(x, y)$ then $W(k, h) = U(k, h) \pm V(k, h)$.

Theorem 2 ([1, 3]). If $w(x, y) = \lambda u(x, y)$ then $W(k, h) = \lambda U(k, h)$.

Theorem 3 ([1, 3]). If $w(x, y) = \frac{\partial u(x, y)}{\partial x}$ then $W(k, h) = (k+1)U(k+1, h)$.

Theorem 4 ([1, 3]). If $w(x, y) = \frac{\partial u(x, y)}{\partial y}$ then $W(k, h) = (h+1)U(k, h+1)$.

Theorem 5 ([1, 3]). If $w(x, y) = \frac{\partial^{r+s} u(x, y)}{\partial x^r \partial y^s}$ then

$$W(k, h) = (k+1)(k+2)\dots(k+r)(h+1)(h+2)\dots(h+s)U(k+r, h+s).$$

Theorem 6 ([1, 3]). If $w(x, y) = u(x, y) \cdot v(x, y)$ then $W(k, h) = \sum_{r=0}^k \sum_{s=0}^h U(r, h-s) V(k-r, s)$.

Theorem 7 ([1, 3]). If $w(x, y) = x^m y^n$ then $W(k, h) = \delta(k-m, h-n)$.

3. Using Two-dimensional Differential Transform to Solve Second Order Complex Partial Differential Equations.

To demonstrate how to use two-dimensional transform to solve complex partial equations are solved in this section.

Example 1

Solve the following initial value problem

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = 4, \quad (11)$$

with the initial conditions

$$w(x, 0) = 5x^2 + 3x + 2 \quad (12)$$

$$\frac{\partial w}{\partial y}(x, 0) = i(2x - 1). \quad (13)$$

Since $w = u + iv$ and equation (11) we obtain that

$$\frac{1}{4} \left(\frac{\partial^2 u}{\partial x^2} + i \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i \frac{\partial^2 v}{\partial y^2} \right) = 4. \quad (14)$$

Therefore

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 16 \quad (15)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (16)$$

From differential transform of (15) and (16) we find following equality:

$$(k+1)(k+2)U(k+2, h) + (h+1)(h+2)U(k, h+2) = 16\delta(k, h) \quad (17)$$

$$(k+1)(k+2)V(k+2, h) + (h+1)(h+2)V(k, h+2) = 0 \quad (18)$$

From (12) equality is obtained that:

$$U(0, 0) = 2, U(1, 0) = 1, U(2, 0) = 5, U(i, 0) = 0 (i = 3, 4, 5, \dots), V(i, 0) = 0 (i = 0, 1, 2, \dots) \quad (19)$$

Similarly from (13) equality is obtained that:

$$V(0, 1) = -1, V(1, 1) = 2, V(i, 1) = 0 (i = 2, 3, 4, \dots), U(i, 1) = 0 (i = 0, 1, 2, \dots) \quad (20)$$

If we write $h = 0$ in equality (17) we get that

$$(k+1)(k+2)U(k+2, 0) + 2U(k, 2) = 16\delta(k, 0) \quad (21)$$

If we write $k = 0$ in equality (21) we get that

$$2U(2, 0) + 2U(0, 2) = 16 \quad (22)$$

From equalities (19) and (22) we get

$$U(0, 2) = 3 \quad (23)$$

If $k > 0$, then from in equality (21) $U(k + 2, 0) = 0$, so we have

$$U(k, 2) = 0 \quad (24)$$

If we write $h = 1$ in equality (17) we get that:

$$(k + 1)(k + 2)U(k + 2, 1) + 6U(k, 3) = 0 \quad (25)$$

From equalities (20) and (25) for every $k \in N$

$$U(k, 3) = 0 \quad (26)$$

Similarly if we write $h = 2$ in equality (17) we get that:

$$(k + 1)(k + 2)U(k + 2, 2) + 12U(k, 4) = 0 \quad (27)$$

From in equalities (24) and (27) for every $k \in N$

$$U(k, 4) = 0 \quad (28)$$

By continuing the operations it is seen that all the other components of U are zero.

If we write $h = 0$ in equality (18) we get that

$$(k + 1)(k + 2)V(k + 2, 0) + 2.V(k, 2) = 0. \quad (29)$$

From equalities (19) and (29) we have that for every $k \in N$

$$V(k, 2) = 0. \quad (30)$$

If we write $h = 1$ in equality (18) we get that

$$(k + 1)(k + 2)V(k + 2, 1) + 6.V(k, 3) = 0. \quad (31)$$

From equalities (20) and (31) we have that for every $k \in N$

$$V(k, 3) = 0. \quad (32)$$

If we write $h = 2$ in equality (18) we get that

$$(k + 1)(k + 2)V(k + 2, 2) + 12.V(k, 4) = 0. \quad (33)$$

From equality (30) we have that for every $k \in N$

$$V(k, 4) = 0. \quad (34)$$

It is easy to see that all other components of V are zero.

Thus, we find that

$$u(x, y) = 5x^2 + 3y^2 + x + 2 \quad (35)$$

and

$$v(x, y) = 2xy - y. \quad (36)$$

From (35) and (36) equalities we get that

$$\begin{aligned} w(x, y) &= u(x, y) + iv(x, y) \\ &= 5x^2 + 3y^2 + x + 2 + i(2xy - y) \\ &= x^2 - y^2 + 2ixy + 4x^2 + 4y^2 + x - iy + 2 \\ &= z^2 + 4z\bar{z} + \bar{z} + 2. \end{aligned} \quad (37)$$

Example 2

Solve the following initial value problem

$$\frac{\partial^2 w}{\partial z^2} + 3 \frac{\partial w}{\partial \bar{z}} = 12z + 18\bar{z} + 9, \quad (38)$$

with the initial conditions

$$w(x, 0) = 2x^3 + 3x^2 + 8x \quad (39)$$

$$\frac{\partial w}{\partial y}(x, 0) = i(6x^2 - 6x + 2). \quad (40)$$

From (38) equation we obtain following equation:

$$\frac{1}{4} \left(\frac{\partial^2 w}{\partial x^2} - 2i \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 w}{\partial y^2} \right) + \frac{3}{2} \left(\frac{\partial w}{\partial x} + i \frac{\partial w}{\partial y} \right) = 12(x + iy) + 18(x - iy) + 9 \quad (41)$$

Since $w = u + iv$, (41) equation equivalent following equation:

$$\begin{aligned} &\frac{1}{4} \left[\frac{\partial^2 u}{\partial x^2} + i \frac{\partial^2 v}{\partial x^2} - 2i \left(\frac{\partial^2 u}{\partial x \partial y} + i \frac{\partial^2 v}{\partial x \partial y} \right) - \frac{\partial^2 u}{\partial y^2} - i \frac{\partial^2 v}{\partial y^2} \right] \\ &+ \frac{3}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \right) \\ &= 30x + 9 - 6iy. \end{aligned} \quad (42)$$

From (42) equation we obtain following equations:

$$\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + 6 \frac{\partial u}{\partial x} - 6 \frac{\partial v}{\partial y} = 120x + 36 \quad (43)$$

$$\frac{\partial^2 v}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial y^2} + 6 \frac{\partial v}{\partial x} + 6 \frac{\partial u}{\partial y} = -24y \quad (44)$$

From differential transform of (43) and (44) equations we obtain that

$$(k+1)(k+2)U(k+2, h) + 2(k+1)(h+1)V(k+1, h+1) - (h+1)(h+2)U(k, h+2) + 6(k+1)U(k+1, h) - 6(h+1)V(k, h+1) = 120\delta(k-1, h) + 36\delta(k, h) \quad (45)$$

$$(k+1)(k+2)V(k+2, h) - 2(k+1)(h+1)U(k+1, h+1) - (h+1)(h+2)V(k, h+2) + 6(k+1)V(k+1, h) + 6(h+1)U(k, h+1) = -24\delta(k, h-1) \quad (46)$$

From (39) equation we obtain that

$$U(0, 0) = 0, U(1, 0) = 3, U(2, 0) = 2, U(3, 0) = 2, \\ U(i, 0) = 0 (i = 4, 5, 6, \dots), V(i, 0) = 0 (i = 0, 1, 2, \dots). \quad (47)$$

Clearly

$$\frac{\partial w}{\partial y} = \sum_{k=0}^m \sum_{h=1}^n h[U(k, h) + iV(k, h)]x^k y^{h-1} \quad (48)$$

From equalities (40) and (48) equation

$$V(0, 1) = 2, V(1, 1) = -6, V(2, 1) = 6, \\ V(i, 1) = 0 (i = 3, 4, 5, \dots), U(i, 1) = 0 (i = 0, 1, 2, \dots). \quad (49)$$

If we write $h = 0$ in equality (45) we get that:

$$(k+1)(k+2)U(k+2, 0) + 2(k+1)V(k+1, 1) - 2U(k, 2) + 6(k+1)U(k+1, 0) - 6V(k, 1) = 120\delta(k-1, 0) + 36\delta(k, 0). \quad (50)$$

If we write $k = 0$ in equality (50) we get

$$2U(2, 0) + 2V(1, 1) - 2U(0, 2) + 6U(1, 0) - 6V(0, 1) = 36 \quad (51)$$

By using equalities (47) and (49) from equality (51) we get

$$U(0, 2) = -3 \quad (52)$$

If we write $k = 1$ in equality (50) we get

$$6U(3, 1) + 4V(2, 1) - 2U(1, 2) + 12U(2, 0) - 6V(1, 1) = 120 \quad (53)$$

By using equalities (47) and (49) from equality (53) we get

$$U(1, 2) = 6. \quad (54)$$

If we write $k = 2$ in equality (50) we get

$$12U(4, 0) + 6V(3, 1) - 2U(2, 2) + 18U(3, 0) - 6V(2, 1) = 0. \quad (55)$$

By using equalities (47) and (49) from equality (55) we get

$$U(2, 2) = 0. \quad (56)$$

For $k \geq 3$ since $U(k+2, 0) = V(k+1, 1) = U(k+1, 0) = V(k, 1) = 0$ we have that

$$U(k, 2) = 0. \quad (57)$$

If we write $h = 0$ in equality (46) we get that:

$$\begin{aligned} & (k+1)(k+2)V(k+2, 0) - 2(k+1)U(k+1, 1) \\ & \quad - 2V(k, 2) + 6(k+1)V(k+1, 0) + 6U(k, 1) \\ & = 0. \end{aligned} \quad (58)$$

By using equalities (47) and (49) from equality (58) we get for every $k \in N$

$$V(k, 2) = 0. \quad (59)$$

If we write $h = 1$ in equality (46) we get that:

$$\begin{aligned} & (k+1)(k+2)V(k+2, 1) - 4(k+1)U(k+1, 2) \\ & \quad - 6V(k, 3) + 6(k+1)V(k+1, 1) + 12U(k, 2) \\ & = -24\delta(k, 0). \end{aligned} \quad (60)$$

If we write $k = 0$ in equality (60) we get

$$2V(2, 1) - 4U(1, 2) - 6V(0, 3) + 6V(1, 1) + 12U(0, 2) = -24. \quad (61)$$

By using equalities (49), (52) and (54) from equality (61) we get that:

$$V(0, 3) = -2. \quad (62)$$

If we write $k = 1$ in equality (60) we get

$$6V(3, 1) - 8U(2, 2) - 6V(1, 3) + 12V(2, 1) + 12U(1, 2) = 0. \quad (63)$$

By using equalities (49), (54) and (56) from equality (63) we get that:

$$V(1, 3) = 0. \quad (64)$$

For $k \geq 2$ since $V(k+2, 1) = U(k+1, 2) = V(k+1, 1) = U(k, 2) = 0$ we have that

$$V(k, 3) = 0. \quad (65)$$

If we write $h = 1$ in equality (45) we get

$$\begin{aligned} & (k+1)(k+2)U(k+2,1) + 4(k+1)V(k+1,2) \\ & \quad - 6U(k,3) + 6(k+1)U(k+1,1) - 12V(k,2) \\ & = 0 \end{aligned} \tag{66}$$

By using equalities (49), (59) for every $k \in N$ we get that

$$U(k,3) = 0 \tag{67}$$

If we write $h = 2$ in equality (45) we get

$$\begin{aligned} & (k+1)(k+2)U(k+2,2) + 6(k+1)V(k+1,3) \\ & \quad - 12U(k,4) + 6(k+1)U(k+1,2) - 18V(k,3) \\ & = 0 \end{aligned} \tag{68}$$

If we write $k = 0$ in equality (68) we get

$$2U(2,2) + 6V(1,3) - 12U(0,4) + 6U(1,2) - 18V(0,3) = 0 \tag{69}$$

By using equalities (54), (56), (62) and (64) from equality (69) we get that:

$$U(0,4) = 0 \tag{70}$$

For $k \geq 1$ since $U(k+2,2) = V(k+1,3) = U(k+1,2) = V(k,3) = 0$ we have that

$$U(k,4) = 0. \tag{71}$$

If we write $h = 2$ in equality (46) we get

$$\begin{aligned} & (k+1)(k+2)V(k+2,2) - 6(k+1)U(k+1,3) \\ & \quad - 12V(k,4) + 6(k+1)V(k+1,2) + 18U(k,3) \\ & = 0 \end{aligned} \tag{72}$$

For every $k \geq 0$, since $V(k+2,2) = U(k+1,3) = V(k+1,2) = U(k,3) = 0$ we get that:

$$V(k,4) = 0 \tag{73}$$

It is clear that all other components of U and V are zero.

Thus we find that

$$u(x,y) = 2x^3 + 3x^2 + 8x - 3y^2 - 6xy^2 \tag{74}$$

and

$$v(x,y) = 2y - 6xy + 6x^2y - 2y^3 \tag{75}$$

From (74) and (75) equalities we get that

$$\begin{aligned} w(x,y) &= u(x,y) + iv(x,y) \\ &= 2x^3 + 3x^2 + 8x - 3y^2 - 6xy^2 + i(2y - 6xy + 6x^2y - 2y^3) \\ &= 2(x^3 + 3ix^2y - 3xy^2 - iy^3) + 3(x^2 - 2ixy - y^2) + 5(x + iy) + 3(x - iy) \\ &= 2z^3 + 3(\bar{z})^2 + 5z + 3\bar{z}. \end{aligned}$$

Example 3

Solve the following initial value problem

$$\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0 \quad (76)$$

with the initial conditions

$$w(x, 0) = e^{3x} + e^x \quad (77)$$

$$\frac{\partial w}{\partial y}(x, 0) = i(3e^{3x} - e^x) \quad (78)$$

Since $w = u + iv$ and equation (9) we obtain that

$$\frac{1}{4} \left(\frac{\partial^2 u}{\partial x^2} + i \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i \frac{\partial^2 v}{\partial y^2} \right) = 0. \quad (79)$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (80)$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \quad (81)$$

From differential transforms of equalities (80) and (81) we get that:

$$(k+1)(k+2)U(k+2, h) + (h+1)(h+2)U(k, h+2) = 0 \quad (82)$$

$$(k+1)(k+2)V(k+2, h) + (h+1)(h+2)V(k, h+2) = 0. \quad (83)$$

We know that:

$$w(x, y) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h) x^k y^h$$

From (77) it is seen that

$$\sum_{k=0}^{\infty} W(k, 0) x^k = e^{3x} + e^x \quad (84)$$

From (84)

$$\sum_{k=0}^{\infty} [U(k, 0) + iV(k, 0)] x^k = \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} + \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \left(\frac{3^k + 1}{k!} \right) x^k \quad (85)$$

From (85) we get for every $k \in N$

$$U(k, 0) = \frac{3^k + 1}{k!}, V(k, 0) = 0. \quad (86)$$

From (78) it is seen that

$$\sum_{k=0}^{\infty} W(k, 1)x^k = i(e^{3x} - e^x) \quad (87)$$

From (87)

$$\sum_{k=0}^{\infty} [U(k, 1) + iV(k, 1)]x^k = i \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} - i \sum_{k=0}^{\infty} \frac{x^k}{k!} = i \sum_{k=0}^{\infty} \left(\frac{3^k - 1}{k!}\right)x^k \quad (88)$$

From (88) we get for every $k \in N$

$$U(k, 1) = 0, V(k, 1) = \frac{3^k - 1}{k!}. \quad (89)$$

If we write $h = 0$ in equality (82) we get

$$(k+1)(k+2)U(k+2, 0) + 2U(k, 2) = 0. \quad (90)$$

From (86) and (90) we see

$$U(k, 2) = -\frac{3^{k+2} + 1}{2!.k!} \quad (91)$$

If we write $h = 1$ in equality (82) we get

$$(k+1)(k+2)U(k+2, 1) + 6U(k, 3) = 0. \quad (92)$$

From (89) and (92) we see

$$U(k, 3) = 0. \quad (93)$$

If we write $h = 2$ in equality (82) we get

$$(k+1)(k+2)U(k+2, 2) + 12U(k, 4) = 0 \quad (94)$$

From (91) and (94) we have

$$U(k, 4) = \frac{3^{k+4} + 1}{4!.k!}. \quad (95)$$

It is easy to see that we obtained following equation for every $k, n \in N$,

$$U(k, 2n+1) = 0, U(k, 2n) = (-1)^n \frac{3^{k+2n} + 1}{(2n)!.k!} \quad (96)$$

Similarly, if we write $h = 0$ in equality (83) we get

$$(k+1)(k+2)V(k+2, 0) + 2V(k, 2) = 0. \quad (97)$$

From (86) and (97) we get for every $k \in N$

$$V(k, 2) = 0. \quad (98)$$

If we write $h = 1$ in (83) we get that

$$(k+1)(k+2)V(k+2,1)+6V(k,3)=0. \quad (99)$$

From (89) and (99) we get for every $k \in N$

$$V(k,3)=-\frac{3^{k+2}-1}{3!.k!} \quad (100)$$

If we write $h = 2$ in (83)

$$(k+1)(k+2)V(k+2,2)+12V(k,4)=0 \quad (101)$$

From (98) and (101) we get for every $k \in N$

$$V(k,4)=0 \quad (102)$$

It is clearly we obtain following equation for every $k, n \in N$,

$$V(k,2n)=0, V(k,2n+1)=(-1)^n \frac{3^{k+2n}-1}{k!(2n+1)!}. \quad (103)$$

From equalities (96) and (103) we get that

$$\begin{aligned} w(x,y) &= u(x,y) + iv(x,y) \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} [U(k,h) + iV(k,h)] x^k y^h \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} [U(k,2h) + iV(k,2h)] x^k y^{2h} + \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} [U(k,2h+1) + iV(k,2h+1)] x^k y^{2h+1} \\ &= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} (-1)^h \frac{3^{k+2h}+1}{(2h)!k!} x^k y^{2h} + i \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} (-1)^h \frac{3^{k+2h}-1}{k!(2h+1)!} x^k y^{2h+1} \\ &= \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} \left(\sum_{h=0}^{\infty} (-1)^h \frac{(3y)^{2h}}{(2h)!} + i \sum_{h=0}^{\infty} (-1)^{h+1} \frac{(3y)^{2h+1}}{(2h+1)!} \right) \\ &\quad + \sum_{k=0}^{\infty} \frac{x^k}{k!} \left(\sum_{h=0}^{\infty} (-1)^h \frac{y^{2h}}{(2h)!} - i \sum_{h=0}^{\infty} (-1)^{h+1} \frac{y^{2h+1}}{(2h+1)!} \right) \\ &= e^{3x} (\cos 3y + i \sin 3y) + e^x (\cos y - \sin y) = e^{3x} \cdot e^{3iy} + e^x \cdot e^{-iy} \\ &= e^{3(x+iy)} + e^{x-iy} \\ &= e^{3z} + e^{\bar{z}} \end{aligned}$$

References

- [1] F. Ayaz. On the two-dimensional differential transform method. *Applied Mathematics and Computation*. 143, 361-374. 2003.
- [2] F. Ayaz. "Solutions of the system of differential equations by differential transform method". *Applied Mathematics and Computation*. 147, 547-567. 2004.
- [3] C. K. Chen and S. H. Ho. Solving partial differential equations by two dimensional differential transform. *Applied Mathematics and Computation*. 106, 171-179. 1999.
- [4] C. K. Chen and S. H. Ho. Application of Differential Transformation to Eigenvalue Problems. *Applied Mathematics and Computation*. 79, 173-188. 1996.
- [5] Y. Keskin and G. Oturanç. The Differential Transform Methods For Nonlinear Functions And Its Applications. *Selcuk Journal of Applied Mathematics*. 9(1), pp 69-76. 2008.
- [6] J. K. Zhou. "Differential Transformation and Its Application for Electrical Circuits", Huazhang University Press, Wuhan, China. 1986.